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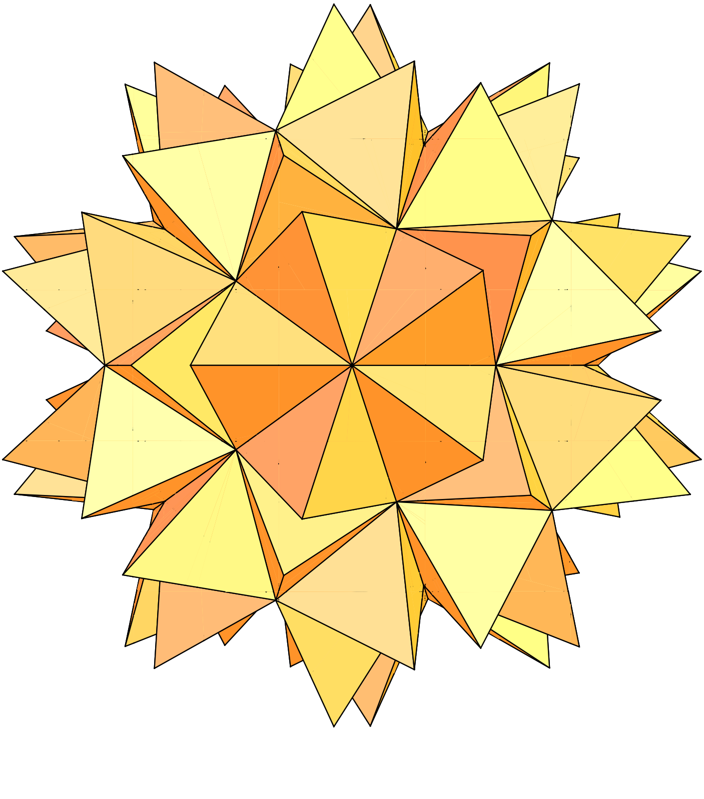
M500 197



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

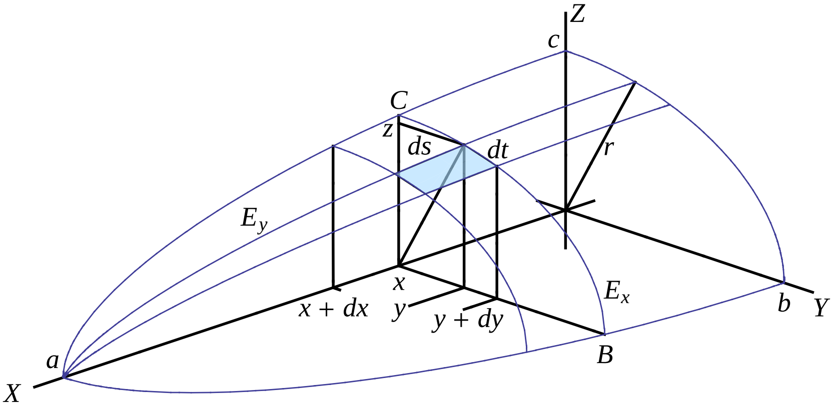
Solution 194.2 – Surface area of an ellipsoid

Obtain a formula for the surface area of the ellipsoid

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} + \frac{Z^2}{c^2} \;=\; 1.$$

Tony Forbes

Call the ellipsoid E. Consider a point (x, y, z) on the surface of the part of E defined by $X, Y, Z \geq 0$. Let $B = b/a\sqrt{a^2 - x^2}$ and $C = c/a\sqrt{a^2 - x^2}$.



Let E_x be the elliptical cross-section of E passing through (x, y, z) and parallel to the (Y, Z)-plane. Then E_x has radii B and C and equation

$$z \;=\; \frac{C}{B}\sqrt{B^2 - y^2}. \tag{1}$$

Let E_y be the ellipse which passes through (x, y, z), $(a, 0, 0)$ and $(-a, 0, 0)$. Let the radii of E_y be a and r. Then E_y has equation

$$w \;=\; \frac{r}{a}\sqrt{a^2 - x^2}. \tag{2}$$

But $w = \sqrt{y^2 + z^2}$; hence $r = a\sqrt{y^2 + z^2}/\sqrt{a^2 - x^2}$.

Consider[1] a small, approximately rectangular patch on E at (x, y, z) with side lengths ds and dt, where ds is in the direction of the X-axis. Then

$$ds \;=\; \sqrt{(dw)^2 + (dx)^2}, \qquad dt \;=\; \sqrt{(dz)^2 + (dy)^2} \tag{3}$$

[1] Footnote added 2014: This argument is rubbish when $b \neq c$; see M500 **259**, 15–16.

and the surface area of E is given by

$$S(a,b,c) \;=\; 8\int_{x=0}^{a}\int_{y=0}^{B} ds\, dt \;=\; 8\int_0^a\int_0^B f(x,y)\, dy\, dx, \qquad (4)$$

where, on substituting (1) and (2) in (3), we have

$$f(x,y) \;=\; \sqrt{\left(1+\frac{c^2y^2}{b^4\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right)}\right)\left(1+\frac{c^2x^2}{a^4\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{y^2}{c^2}\right)}\right)}.$$

I am now stuck. With or without help from computer software I am unable to carry out the double integration.

Henceforth let $c = b$. **Dick Boardman** shows that the problem can be solved completely. Sparing you the details, (4) becomes

$$\begin{aligned} S(a,b,b) \;&=\; \frac{8b}{a}\int_0^a\int_0^{b/a\sqrt{a^2-x^2}} \sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2b^2-a^2y^2-b^2x^2}}\, dy\, dx \\ &=\; \frac{4b\,\pi}{a^2}\int_0^a \sqrt{a^4+(b^2-a^2)x^2}\, dx \\ &=\; 2b^2\pi + \frac{2a^2b\,\pi}{\sqrt{b^2-a^2}}\;\log\frac{b+\sqrt{b^2-a^2}}{a}. \qquad (5) \end{aligned}$$

The formula holds for any a and b but if you are unhappy about taking logarithms of things involving imaginary quantities, there is a purely real alternative for $b < a$:

$$S(a,b,b) \;=\; 2b^2\pi + \frac{2a^2b\,\pi}{\sqrt{a^2-b^2}}\;\arctan\frac{\sqrt{a^2-b^2}}{b}.$$

Note that as $b \to a$, $\arctan(\sqrt{a^2-b^2}/b) \to \sqrt{a^2-b^2}/b$ and thus $S(a,a,a) = 2a^2\pi + 2a^2\pi = 4a^2\pi$, the correct formula for a sphere of radius a. Also by letting $a \to 0$ one can see that $S(0,b,b) = 2b^2\pi$, consistent with the thing degenerating into a two-sided disc of radius b. And if you put $c = 0$ in (4), then $f(x,y)$ reduces to 1 and you easily get $S(a,b,0) = 2ab\pi$, corresponding to an elliptical disc of radii a and b.

It is interesting to see how (5) compares with various simple formulae that might suggest themselves. First, let us put $b = a + \epsilon$ in (5), where ϵ is small. Then developing (5) as a power series in ϵ yields

$$S(a, a+\epsilon, a+\epsilon) \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{8\pi\epsilon^2}{5} + \frac{16\pi\epsilon^3}{105a} - \frac{4\pi\epsilon^4}{63a^2} + \ldots. \qquad (6)$$

David Kerr offers the formula

$$S_1(a, b, c) \;=\; \frac{4\pi}{3}(ab + bc + ca), \qquad (7)$$

obtained by painting the ellipsoid. The volume of E is $4\pi abc/3$. After applying a uniform coat of thickness δ, one might approximate the volume of the painted solid by $4\pi(a+\delta)(b+\delta)(c+\delta)/3$. Multiplying out, subtracting $4\pi abc/3$, dividing by δ and discarding terms containing δ gives (7). However, we can see that (7) is inconsistent with (6). Putting $b = c = a + \epsilon$ in (7), we have

$$S_1(a, a+\epsilon, a+\epsilon) \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{4\pi\epsilon^2}{3}, \qquad (8)$$

which deviates from (6) at the ϵ^2 term. Thus (8), which specializes (7) to the case of a nearly spherical ellipsoid, indicates that the general formula (7) is only approximate. Actually this is not unexpected. The same kind of reasoning gives, incorrectly, $\pi(a + b)$ for the circumference of an ellipse with radii a and b. Presumably the flaw in the argument is that the painted ellipsoid cannot be sufficiently well approximated by an ellipsoid. Perhaps someone can investigate why.

Colin Davies suggests an even simpler formula,

$$S_2(a, b, c) \;=\; 4\pi(abc)^{2/3}.$$

Doing the same sort of thing as before, we get

$$S_2(a, a+\epsilon, a+\epsilon) \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{8\pi\epsilon^2}{9} + \ldots,$$

again, agreeing with (6) up to the ϵ term but not beyond.

Curiously, a better (but still only approximate) formula results if instead we use the arithmetic mean of the radii:

$$S_3(a, b, c) \;=\; 4\pi\left(\frac{a+b+c}{3}\right)^2.$$

Now we have

$$S_3(a, a+\epsilon, a+\epsilon) \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{16\pi\epsilon^2}{9},$$

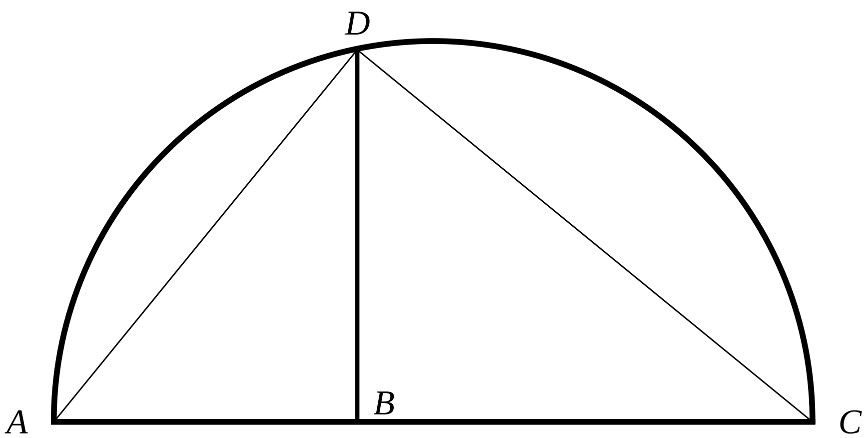
and the ϵ^2 term nearly agrees with $16\pi\epsilon^2/10$, the ϵ^2 term in (6).

The arithmetic/geometric mean inequality

John Spencer

Recent issues of M500 have carried several problems involving the arithmetic/geometric mean inequality. In one form or another this inequality has been a staple of mathematical puzzles and problems for more than two millennia. Like most of the great mathematical theorems, it never seems to lose its capacity to surprise.

Euclid (*Elements*, Book VI, Prop. 13) gives a construction for finding the geometric mean of two given straight lines AB and BC. The construction involves placing AB and BC in a straight line ABC which is then made the diameter of a semicircle. A line from B at right angles to ABC meets the semicircle at D.



Clearly, BD is a mean proportional to AB and BC; that is to say, AB bears the same proportion to BD as BD does to BC, i.e. $AB : BD = BD : BC$. So BD, the geometric mean, is $\sqrt{AB \cdot BC}$. Another way of looking at it is that $\triangle ABD$ and $\triangle BCD$ (and $\triangle ACD$) are similar triangles. The AGM inequality expresses the fact that BD cannot be longer than the radius of the semicircle, $(AB + BC)/2$, which is not hard to see from the diagram.

The inequality can be derived algebraically from the observation that the square of any number is always either positive or zero. Then

$$(\sqrt{a} - \sqrt{b})^2 \ \geq \ 0 \quad \rightarrow \quad a + b - 2\sqrt{ab} \ \geq \ 0.$$

Adding $2\sqrt{ab}$ to both sides and dividing by two gives the standard form of the inequality, with the arithmetic mean on the left and the geometric on the right:

$$\frac{(a+b)}{2} \;\geq\; \sqrt{ab}.$$

Problem 193.4 asked for proof that

$$n! \;\leq\; \left(\frac{n(n+1)^3}{8}\right)^{n/4}.$$

Most readers will accept, I imagine, that $n! \leq n^n$. The more difficult part of the problem is to establish that

$$n! \;\leq\; \left(\frac{n+1}{2}\right)^{n}.$$

An elegant proof of this assertion, due to Gauss, is given in *Invitation to Discrete Mathematics* by Jiří Matoušek and Jaroslav Nešetřil (Oxford, 1998), a book I found entertaining and useful when I was doing M203 and MT365.

Each of the numbers $i \in \{1, 2, \ldots, i, \ldots, (n-1), n\}$ is multiplied by the number at the same place in the reverse sequence $\{n, (n-1), \ldots, n+1-i, \ldots, 2, 1\}$ giving $i(n+1-i)$, so that the product of all these pairs

$$\prod_{i=1}^{n} i(n+1-i) \;=\; (n!)^2$$

and

$$n! \;=\; \prod_{i=1}^{n} \sqrt{i(n+1-i)}.$$

Now replace a by i and b by $n+1-i$ in the arithmetic/geometric mean inequality to give

$$\sqrt{i(n+1-i)} \;\leq\; \frac{i+n+1-i}{2} \;=\; \frac{n+1}{2}$$

so that

$$n! \;=\; \prod_{i=1}^{n} \sqrt{i(n+1-i)} \;\leq\; \prod_{i=1}^{n} \frac{n+1}{2} \;=\; \left(\frac{n+1}{2}\right)^{n}.$$

So we can write

$$(n!)^4 \;\le\; n^n \left(\frac{n+1}{2}\right)^{3n},$$

and taking positive, real, fourth roots on each side,

$$n! \;\le\; \left(\frac{n(n+1)^3}{8}\right)^{n/4}.$$

An earlier issue of M500 contained Problem 191.7, which asks for proof that

$$(a_1 + a_2 + a_3 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ... + \frac{1}{a_n}\right) \;\ge\; n^2.$$

Let us assume this is the case for $n = k$. Let $S_k = a_1 + a_2 + a_3 + ... + a_k$ and $R_k = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ... + \frac{1}{a_k}$. Then

$$S_{k+1}R_{k+1} \;=\; (S_k + a_{k+1})\left(R_k + \frac{1}{a_{k+1}}\right) \;=\; S_k R_k + 1 + \frac{S_k}{a_{k+1}} + R_k a_{k+1}$$

and

$$\frac{S_k}{a_{k+1}} + R_k a_{k+1} = \frac{a_1}{a_{k+1}} + \frac{a_2}{a_{k+1}} + \cdots + \frac{a_k}{a_{k+1}} + \frac{a_{k+1}}{a_1} + \frac{a_{k+1}}{a_2} + \cdots + \frac{a_{k+1}}{a_k}$$

in which each fraction is matched by its reciprocal. But by the AGM inequality

$$\frac{a_n}{a_{k+1}} + \frac{a_{k+1}}{a_n} \;\ge\; 2\sqrt{\frac{a_n}{a_{k+1}}\frac{a_{k+1}}{a_n}} \;=\; 2.$$

So, since there are k pairs of fractions, $S_k/a_{k+1} + R_k a_{k+1} \ge 2k$ and by our assumption

$$S_{k+1}R_{k+1} \;\ge\; k^2 + 2k + 1 \;=\; (k+1)^2.$$

If the proposition is true for $n = k$, it is true for $n = k + 1$. By induction, then, since $S_1 R_1 = 1^2$,

$$(a_1 + a_2 + a_3 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ... + \frac{1}{a_n}\right) \;\ge\; n^2$$

for all n.

Solution 193.3 – Thirteen tarts

> There are 13 tarts. All weigh the same, with one exception. Either (i) devise a strategy involving three weighings to determine the odd tart and whether it is lighter or heavier than the others; or (ii) prove that (i) is impossible.
>
> As usual, a weighing means selecting two sets of tarts and determining which set (if any) is lighter.
>
> What about $(3^n - 1)/2$ tarts?

John Seldon

This is a perennial problem and was presented to me as an example in information theory. This allows one to dispense with long lists of tables and weighings by rote.

The theory says that an experiment, in this case a weighing, yields the optimum information if all outcomes are equally likely. With 12 tarts there are 24 possibilities. Weighings can have one of 3 results. Set up an experiment aiming to reduce the possibilities to 8. Easy—weigh 4 against 4. Eight possibilities are left and the next experiment can be set up accordingly.

What about 13 tarts with 26 possibilities? Put 1, 2, 3, 4, 5 or 6 tarts on each side of the scales. The respective possible results are {2 – left up, 2 – left down, 22 – unmoved}, {4, 4, 18}, {6, 6, 14}, {8, 8, 10}, {10, 10, 6}, {12, 12, 2}. In each case 10 or more possibilities remain; 10 cannot be resolved in a further two weighings, so 13 tarts cannot be resolved in three weighings.

What about 40 tarts with 80 possibilities? The most informative weighings are 13 or 14 tarts against each other yielding {26, 26, 28} and {28, 28, 24}. Thus 28 possibilities cannot be resolved in a further three weighings.

As a simple statement rather than a proof for larger n, note that the results of weighings always yield an even number of possibilities. But $3^n - 1$ divided into three even number partitions always includes one partition greater that 3^{n-1} which cannot be resolved in $n - 1$ further weighings. QED.

Police in Xiangzhou, China, who jailed the members of two prostitution rings, said that all of the women were over 70 years old. 'This is not just a moral issue,' a spokesman from the vice squad explained. 'More seriously, these people are clearly contravening the terms of their pension by continuing to work.' The oldest was 93. [Sent by EK.]

Products of digits

David Singmaster

Here are some results on a problem I saw some time ago, but don't remember the source of. The original question began with an example for two digits. Starting with 38, form the product of the digits, which is 24. Repeat, getting 8. The process now loops and we saw the iteration lasts for two steps before this ending. Question – what two digit number has the longest iteration?

There are are two ways to attack this. One method is to start at the end and work backward. For example, if we end at 2, we must have come from 12 or 21. But 12 can come from 26, 34, 43, 62, while 21 can come from 37, 73. Though the factoring and branching is awkward, it is small enough to get the longest path fairly easily. The other method is to just repeat forming the product of the digits for each number 0, 1, ..., 99.

$K = 2$ 10 32 34 23 1 77
$K = 3$ 10 247 340 310 84 9 679, 688
$K = 4$ 10 2759 3258 2590 1152 219 12 6788
$K = 5$ 10 33821 35922 18354 9087 2409 377 20 68889
$K = 6$ 10 402539 375227 123860 66772 24654 4488 2450 68889, 168889, 238889, 246889, 266688, 267799, 336888, 344889, 346688, 347799, 366779, 377779, 377889, 444689, 446668, 467789, 666778
$K = 7$ 10 4619788 3658522 907620 503573 224610 41196 39452 5229 2677889, 2799999, 3477889, 3667788, 3679999, 4467789, 4666778, 6888999, 6999999

I had the calculations for the first method for two digit numbers on my desk for some time as I had thought of posing this on *Puzzle Panel* in August 2003 but decided it was too much work for it. But I wondered if I could extend this to three digits. But now 2 can arise from 12, 21, 112, 121, 211, etc. And one has to factor three-digit numbers. Hence this becomes a formidable task to do by hand and rather messy to do by computer.

So I decided to use the second method and just let the computer iterate the function: $F(N)$ = product of the digits of N, until the result was < 10. I could have much shortened the program and the output by observing that one only needs to consider integers N whose digits are in non-decreasing order. There does not seem to be any pattern, so I simply tabulate the results, opposite, for K-digit numbers, $K = 2, \ldots, 7$. Note that this really means numbers with at most K digits.

The first row gives the number of values whose iteration has length L, $L = 0, 1, 2, \ldots$. Note that $L = 0$ if and only if $N = 0, 1, \ldots, 9$. The total of the entries in this row is 10^K. Below this row, I list the values, with digits in non-decreasing order, which give the maximum length.

Note that the maximum L fails to increase from $K = 5$ to $K = 6$.

Is there someone out there who is willing to continue the calculations?

London Mathematical Society
Popular Lectures 2004

This year's LMS popular lectures will take place on the following dates.

Manchester – 6 May 2004

London – 11 June 2004

The speakers are

Professor Ken Binmore (University College London),
Big money mathematics,

Professor Helen Byrne (University of Nottingham),
Just a spoonful of maths helps the medicine.

For further information contact the local organizers: Manchester: Nige Ray (nige@ma.man.ac.uk); London: Susan Oakes (oakes@lms.ac.uk), London Mathematical Society, De Morgan House, 57–58 Russell Square, London WC1B 4HS (tel: 020 7637 3686). Full details are available on the LMS web site www.lms.ac.uk.

Solution 194.3 – Sixteen lamps

> There are 16 lamps of 1 ohm resistance wired up to a 1 volt battery as in the circuit diagram on page 11, opposite. What are the currents flowing through the bulbs when the switches are closed as indicated?

ADF

I must admit that originally I had no idea how to deal with this kind of problem other than by actually building the circuit and going round it with a meter. But now I am enlightened.

We use Kirchhoff's First Law. If you have never heard of Kirchhoff, or his First Law, it doesn't matter—common sense works just as well. Take any point, P, on an electrical circuit. Then the total current flowing into P equals the total current flowing out of P.

Let the voltages on the vertical wires be a, b, 1, c. Let the voltages on the horizontal wires be d, e, 0, f. Let the current out of the battery be t. Then the currents through the lamps are

$$\begin{bmatrix} a-d & b-d & 1-d & c-d \\ a-e & b-e & 1-e & c-e \\ a & b & 1 & c \\ a-f & b-f & 1-f & c-f \end{bmatrix}.$$

For each of the eight distribution wires, choose a point on the wire and equate current in to current out:

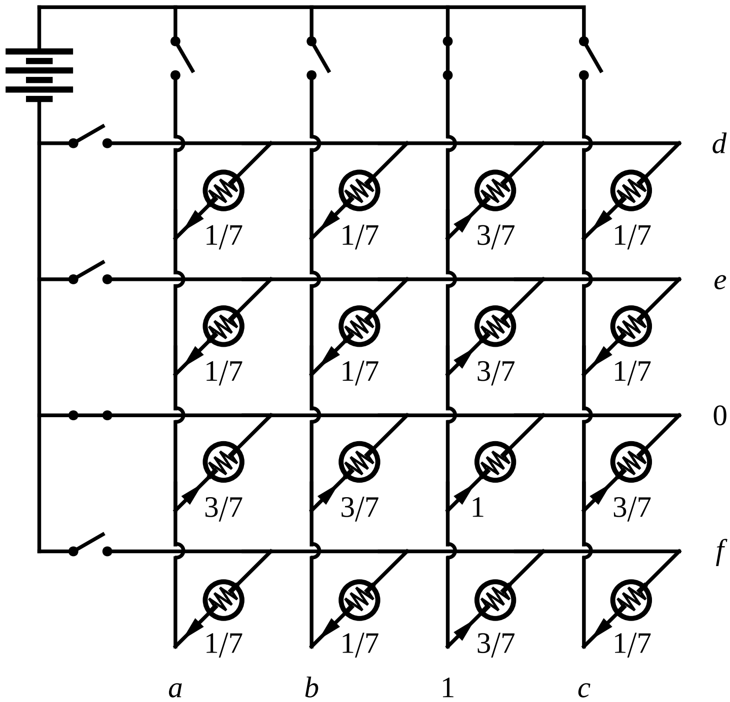
$$\begin{aligned} 4a-d-e-f &= 0, & a+b+1+c-4d &= 0,\\ 4b-d-e-f &= 0, & a+b+1+c-4e &= 0,\\ 4-d-e-f &= t, & a+b+1+c &= t,\\ 4c-d-e-f &= 0, & a+b+1+c-4f &= 0. \end{aligned}$$

Solving yields

$$a=\frac{3}{7},\quad b=\frac{3}{7},\quad c=\frac{3}{7},\quad d=\frac{4}{7},\quad e=\frac{4}{7},\quad f=\frac{4}{7},\quad t=\frac{16}{7},$$

which give the currents shown in the diagram.

Clearly, it is possible to do the same kind of thing with any combination of switch settings. For instance, as **Keith Drever** points out, if all the switches are on, the circuit behaves as if all the lamps are wired in parallel. The current is 1 amp through each lamp. The case where all the switches are off is even easier.



Problem 197.1 – Consecutive integers

Roger Winstanley

Show that every even number which is not a power of two can be represented as a sum of three or more consecutive integers. Thus $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, $12 = 3 + 4 + 5$, $14 = 2 + 3 + 4 + 5$, etc. Which odd numbers can be so represented?

ADF—We did something similar before (M500 **114**, **117**). If you want have a go at something completely different, try the following.

Problem 197.2 – Consecutive cubes

Which numbers can be expressed as a sum of two or more consecutive cubes? For example, $9 = 1 + 8$, $35 = 8 + 27$, $36 = 1 + 8 + 27$, etc.

Solution 193.1 – Smallest square

Given a convex quadrilateral Q with area A and diagonals r, s, show that the smallest square containing Q has area at least

$$\frac{r^2s^2 - 4A^2}{r^2 + s^2 - 4A}.$$

[As we shall see, the wording of the problem should not be taken too literally.]

David Porter

There is a problem with this problem, namely that what we are asked to prove is not true for all convex quadrilaterals but only for some.

First a basic result. If α is the acute angle between the diagonals r and s of a convex quadrilateral then the area A is given by

$$A = \frac{rs}{2}\sin\alpha.$$

This shows that given r, s and A we have defined the angle between the diagonals but have not defined the point where the diagonals cross.

Let B be the area of the minimum circumscribing square so what we are being asked to prove is that

$$B \geq \frac{r^2s^2 - 4A^2}{r^2 + s^2 - 4A}.$$

In the special case when $r = s$ this becomes

$$B \geq \frac{r^4 - 4A^2}{2r^2 - 4A} = \frac{r^2}{2} + A,$$

which is the sum of the area of the square with diagonal r and area of the quadrilateral.

Now consider what happens if we make our quadrilateral a nearly square rectangle. The above relationship would predict that the smallest square containing Q would have an area of at least twice that of Q when it is perfectly obvious that square obtained by extending the shorter sides of the rectangle slightly would 'do the business' and have a much smaller area.

In fact what the expression has given is the area of a circumscribing square touching all four corners of the rectangle and with the diagonals of the square coinciding with the two twofold axes of symmetry of the rectangle. For our nearly square rectangle (α close to a right angle) this is obviously the wrong orientation but for a long thin rectangle (α close to zero) it would be perfect. Somewhere between $\alpha = 0$ and $\alpha = \pi/2$ the orientation of a rectangle that gives the minimum square will flip from one of these to the other. A fairly simple bit of trig shows the two areas to be the same when $\alpha = \pi/4$.

To try and see what is going on let's consider the slightly more general set of quadrilaterals where the only restriction is that the diagonals bisect each other, i.e. the parallelograms. With no further loss of generality we can assume $r \geq s$.

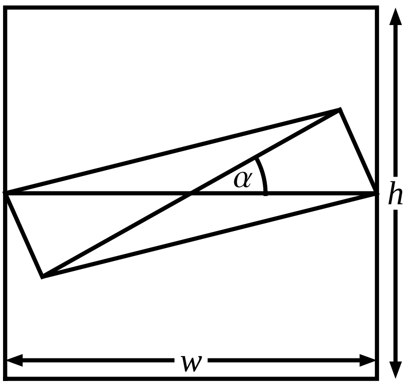


Diagram 1

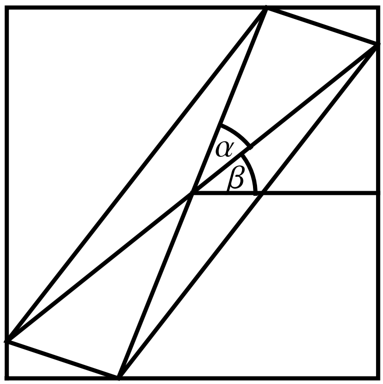


Diagram 2

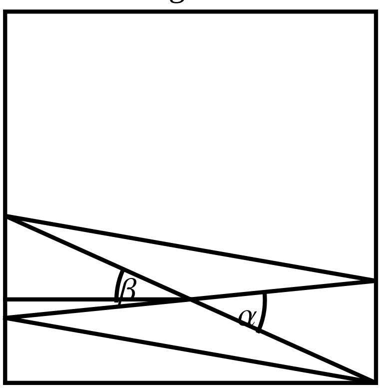


Diagram 3

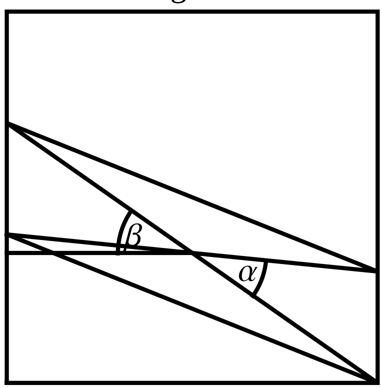


Diagram 4

Now if we start from the position in Diagram 1 and rotate the parallelogram anticlockwise about its centre and at the same time adjust the size of the circumscribing square so that the ends of the long diagonal continue to lie on the vertical edges, the size of the square will reduce and one of two things will happen. Either at some stage the horizontal edges of the square will encounter the ends of the short diagonal, or the parallelogram will have been turned through an angle of $\pi/4$ and the long diagonal will coincide with a diagonal of the square. Whichever happens, any further anticlockwise rotation will result in an increase in the size of the circumscribing square, i.e. we have reached a local minimum for the size of the circumscribing square. Any circumscribing square must have a diagonal at least as long as the longer diagonal of the parallelogram (i.e. must have a minimum area of $B_0 = r^2/2$); so there is no need for further analysis of the latter case. For the former case the position is now as shown in Diagram 2 and we have that

$$w = r\cos\beta, \qquad h = s\sin(\alpha+\beta);$$

but $h = w$, so on expanding the sin we get $r\cos\beta = s(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$. Hence $\tan\beta = (r - s\sin\alpha)/(s\cos\alpha)$ and, since $\cos^2\beta = 1/(1+\tan^2\beta)$, we have

$$\cos^2\beta = \frac{s^2\cos^2\alpha}{r^2 - 2rs\sin\alpha + s^2\sin^2\alpha + s\cos^2\alpha} = \frac{s^2(1-\sin^2\alpha)}{r^2+s^2-2rs\sin\alpha}.$$

So, since the area of the square (B_1) is $w^2 = r^2\cos^2\beta$ and remembering that $2A = rs\sin\alpha$ we have that

$$B_1 = \frac{r^2s^2 - 4A^2}{r^2+s^2-4A}.$$

Now if we could say that $B = B_1$ for all parallelograms, then we might argue as follows. The parallelogram is the most 'compact' convex quadrilateral with area A and diagonals r and s so any other convex quadrilateral with this specification will require the smallest square containing it to be at least this size and hence $B \geq (r^2s^2 - 4A^2)/(r^2+s^2-4A)$ for all convex quadrilaterals. Unfortunately, we have seen that this is not true.

Let's now step back to Diagram 1 and consider what happens if we rotate our parallelogram clockwise instead of anticlockwise. Again there are two possibilities. The first is that we again turn through $\pi/4$ without the shorter diagonal encountering an edge of the circumscribing square. The second possibility is that we reach a point where the short sides of the parallelogram lie on the vertical sides of the square; if in doing this we have turned through an angle of less than α, we will again have reached a position of local minimum in square size (Diagram 3). In this case we have that

$$w = r\cos\beta, \qquad w = s\cos(\alpha-\beta);$$

so on equating and expanding the cos we get $r\cos\beta = s(\cos\alpha\cos\beta + \sin\beta\sin\alpha)$; hence $\tan\beta = (r - s\cos\alpha)/s\sin\alpha$ and

$$\cos^2\beta = \frac{s^2\sin^2\alpha}{r^2 - 2rs\cos\alpha + s^2\cos^2\alpha + s\sin^2\alpha} = \frac{s^2\sin^2\alpha}{r^2+s^2-2rs\cos\alpha}.$$

So since the area of the square (B_2) is $w^2 = r^2\cos^2\beta$ and remembering that $2A = rs\sin\alpha$ we have that

$$B_2 = \frac{4A^2}{r^2+s^2-2\sqrt{r^2s^2-4A^2}}.$$

Not quite such an elegant relationship as before.

Finally, if $\beta \geq \alpha$ (Diagram 4) then a local minimum has not been reached and further rotation will continue to reduce the size of the square until eventually the short diagonal will span the vertical sides of the square and the long diagonal the horizontal sides. The situation will then be as if Diagram 2 had been rotated by $\pi/2$ clockwise and so this is not a new local minimum.

What, if anything, can be salvaged from all this? I suspect that the best we can do for a lower limit on B is

$$B \geq \max\{B_0, \min\{B_1, B_2\}\},$$

and even this will be an increasingly bad estimate of B as the crossing point of the diagonals moves away from their centres.

Solution 193.5 – Dissect a triangle

> Dissect an equilateral triangle into three triangles whose areas are in the ratio 3:3:2 and all sides of all triangles are integers.

David Porter

Let the triangle have sides of length 8 units with A the apex and B and C the corners of the base. Let D be the mid-point of BC and let E and F be the points on BC one unit from D. Then $AD = 4\sqrt{3}$ and $ED = 1$. So $AE = \sqrt{AD^2 + ED^2} = \sqrt{48+1} = 7$. Thus the three triangles ABE, AEF and AFC all have integer sides and since they all have the same height (AD) their areas are proportional to their bases $BE = FC = 3$ and $EF = 2$.

Prime prime

Tony Forbes

In M500 **194** Eddie Kent asks 'Can you find a prime number from which successive right-hand digits can be stripped off, leaving a prime number at each step?' However, there is (in at least my opinion) a much more interesting problem: Find long chains of primes, $p_1, p_2, \ldots, p_n$, linked by the condition $p_{i-1} = [p_i/10]$, $i = 2, 3, \ldots, n$. This is the same as Eddie's problem except we don't require the sequence to start at a single-digit prime.

The best I can manage is this sequence of 12 primes:

1457011	14570117399	145701173999399
14570117	145701173999	1457011739993993
145701173	1457011739993	14570117399939939
1457011739	14570117399939	145701173999399393

Can anyone do better? Notice that once you hit a prime which is $\equiv 2 \pmod 3$ the numbers thereafter must end in 3 or 9.

Now consider changing the number base from 10 to 2. Unlike the base-10 case, interest in the binary version of the problem seems to extend beyond the realm of mathematical curiosities. A sequence of n primes, p, $2p + 1$, $4p + 3$, $\ldots$, $2^{n-1}p + 2^{n-1} - 1$ is known in the literature as a *Cunningham chain of order n*. The longest to date is a 16-chain, starting with 810433818265726529159, discovered by Paul Jobling in 2002.

There is a connection with primality testing. For instance, one way of proving that P is prime involves factorizing $(P - 1)/2$. If $Q = (P - 1)/2$ turns out to be prime, the problem has been reduced by half. Conversely, once the primality of the smallest element of a Cunningham chain has been established, one can deal with the other members by working up the chain.

It is also worth mentioning that Cunningham chains of order 2 feature in Sophie Germain's proof of a special case of Fermat's Last Theorem. If p and $2p + 1$ are prime and if xyz is not a multiple of p, then $x^p + y^p = z^p$ has no non-trivial solutions.

Finally, to answer to EK's original question you can verify that these eight numbers,

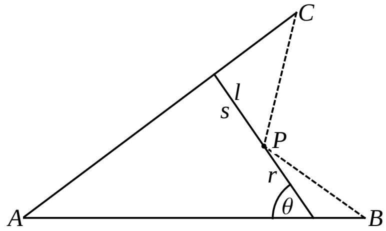
$$3, 37, 373, 3733, 37337, 373379, 3733799, 37337999,$$

are prime and that 373379991, 373379993, 373379997, 373379999 are not.

Correction. M500 **195**, page 28, line 4. Delete the words 'do not'.

Solution 193.2 – Concave to convex

Start with a non-convex quadrilateral. By removing some bits of it you can end up with a convex polygon. What is the minimum area you have to remove?



David Singmaster

As shown, one wants the line l through P which trims off the least area. It is easier to look at maximizing the triangle at A and this simplifies the problem to the following. Given an angle A and a point P inside it, find the line l through P which forms the largest triangle at A. The effects of the limiting positions determined by points B and C are considered afterward.

Consider a line l through P meeting the sides of the angle. Let r, s be the distances along l from P to the sides of the angle and let θ be the angle between l and the lower side of the angle. A small turn of the line l corresponding to a small increase $\Delta\theta$ in the angle causes the area of the triangle to increase by an approximate amount $\Delta A = r^2\Delta\theta/2 - s^2\Delta\theta/2$. From this, we see that if $r \geq s$, then we want to increase θ; otherwise, we want to decrease θ. Taking these to the extremes, we get infinite area when the line l is parallel to either side of the angle. However, one may wonder if $r = s$ can occur.

Let d, e be the perpendicular distances from P to the sides of the angle and let the size of angle A be α. Then $r = d\csc\theta$, $s = e\csc(\theta - \alpha)$. These are equal iff $d\sin(\theta - \alpha) = e\sin\theta$. A little inspection of such curves shows that these curves cross just once. Hence our area A has one minimal value for some θ and increases in both directions from this minimum. So for the quadrilateral problem, the maximum area A must occur at one of the limiting situations where l passes through B or C, and one has only to compare these two cases.

Problem 197.3 – Cot series

Sebastian Hayes

Show that for positive integer n,

$$\cot\frac{\pi}{4n} - \cot\frac{3\pi}{4n} + \cot\frac{5\pi}{4n} - \cdots \pm \cot\frac{(2n-1)\pi}{4n} \;=\; n.$$

Solution 194.4 – Getting dressed

> Your wardrobe consists of h hats, b bras, p panties, d dresses, s pairs of socks and f pairs of shoes. In how many ways is it possible for you to get properly dressed? Assume (i) you wear one of each type of clothing; (ii) underwear goes on before dress, and sock before shoe; (iii) socks and shoes are paired; (iv) chirality is relevant for shoes but not socks.

ADF

The answer is $6720\,h\,b\,p\,d\,s\,f$.

First assume that you have chosen each item of clothing. We want to determine the number of ways of putting them on. Split them into three independent subsets: head-wear = {hat}, body-wear = {bra, panties, dress}, footwear = {left sock, right sock, left shoe, right shoe}. Assume that you have already chosen orderings for body-wear and footwear, say the sequences just given. Assume also that you have designated which sock is to go on the left foot.

We now ask: In how many ways can you splice (bra, panties, dress) into the sequence (left sock, right sock, left shoe, right shoe)? There are three items and five available slots. We could choose to put the bra, panties and dress into three different slots, say, for example, (bra, left sock, right sock, panties, left shoe, dress, right shoe). There are $\binom{5}{3} = 10$ ways of doing this. Or we could put the bra into one slot and the panties and dress into a second slot; that's $\binom{5}{2} = 10$. Or we could put the bra and panties into one slot and the dress into another; that's also $\binom{5}{2}$. Or we could put all three into the same slot; 5 ways. Adding them together makes 35.

There are two valid orderings of bra, panties, dress, namely (bra, panties, dress) and (panties, bra, dress). And, as you can easily verify, there are six ways to sequence the footwear. Hence there are $420 = 35 \cdot 2 \cdot 6$ ways of dealing with the seven non-hat items of clothing.

Since you can now place the hat into any one of eight slots, the number of ways of getting dressed from a given set of clothes is $8 \cdot 420 = 3360$.

To get the final answer we must multiply by the numbers of ways of choosing the items of clothing. For the hat, bra, panties and dress, these are clearly h, b, p and d, respectively. For the first sock it is $2s$ but for the first shoe it is f. For the second sock and the second shoe it is 1. (Here we are assuming that your footwear comes in pairs which you would never consider splitting.) After doing the multiplication we get the stated result.

Now that we have explained how to do this kind of combinatorial enumeration, have a go at solving a similar problem for your own wardrobe. Make up your own rules if you want. And you don't have to adhere to our sequencing restrictions. Indeed, at a mathematics summer school I witnessed someone giving a live demonstration that that it is possible to remove your bra before taking off your dress. Presumably, therefore, the inverse is possible.

If you get any interesting results, do send them in.

Problem 197.4 – Travels

Estimate how far you have travelled during the last twelve months: (i) relative to your home; (ii) relative to the Sun; (iii) relative to the centre of the Milky Way; (iv) through space-time, relative to your home twelve months ago.

Estimate (ii) or (iii) can be useful as a suitable riposte to those people who insist on trying to impress you with tales of their exotic holidays in far-off places. On the same scale that six-week adventure trek through the rain forests of South America, or whatever, should generate about the same level of excitement as your last trip to the local supermarket!

Problem 197.5 – Toilet paper

Find a formula that relates the radius of a toilet roll (R), the total length (L), the paper thickness (t) and the radius of the cylindrical cardboard thing at the centre (r).

Mathematics in the kitchen – III

For this experiment you will need an egg, some water, a sheet of paper and a smooth, level tabletop.

Spread the paper on the tabletop, dip the egg in the water so that it is completely covered and roll it along the table. You should then see a wavy line, being the trace of the point of contact between the egg and the paper as the egg wobbles its way across the table. We ask: What is the function that describes the trace?

As the lopsidedness of a typical hen's egg introduces an extra level of complication, you might prefer to try the experiment with a more regular ovoid object such as a football, rugby or American.

As with all scientific experiments, please take care. *Do not perform the experiment if you are unwilling to take responsibility for accidents.*

Under the skin

Colin Davies

I read an article about a married couple who lived in England in the seventeenth century and who had several children. The article stated that after an interval of 300 years, almost everybody in Britain must be descended from them. That statement struck me as unlikely to be correct.

Many years ago, I used to know a fellow who claimed that he was descended from William the Conqueror. This claim struck me as very likely to be correct. I decided to investigate three problems:

1. If you take two British resident people at random (but exclude obvious relatives and recent immigrants), how far back in time would you have to go to be 99 per cent certain that they have a common ancestor?

2. If you are of apparent British ancestry, what in the probability of having any particular historical person as an ancestor?

3. How closely is any particular person in Britain likely to be related to anybody else? (But again excluding close relatives and recent immigrants.)

Amongst the more obvious criteria, these problems depend on the length of a generation and the past population of Britain. I will take thirty years as an average generation.

The population of Britain is very uncertain before the first census in 1841, but I have taken figures from *A History of England* by K. Feiling, and *A Concise Economic History of Britain* by J. Clapham, and interpolated the figures a bit. The table at the end shows the results.

Problem 1. Consider a person born in 1980. Going back ten generations (= 300 years), to 1680, that person will have $2^{10} = 1024$ great^8 grandparents. The population of Britain was six million in 1680, so the probability of two people in Britain having a common ancestor in 1680 is $1 - ((6 \cdot 10^6 - 1024)/(6 \cdot 10^6))^{1024} = 0.16$. By this, I mean that 16 people in every hundred alive in 1980 will have had a common ancestor in 1680.

Before 1680, the probability of having had ancestors in common increases rapidly. The table at the end of this article has been worked out using the probability formula above, and it shows that it is almost inevitable that two people taken at random in Britain will have had a common ancestor 12 or 13 generations ago, around 1600.

There are objections to this simple model. Parents in other countries are not considered and one person's great^8 grandparent is likely to be another person's great^9 or great^7 grandparent, etc. The possibility of foreign parents suggests that the simple-model probability is too high and the simultaneous existence of different generations down different lines suggests it is too low. Furthermore, prior to about 1250 the required number of ancestors exceeds the population of the country. Obviously the same ancestor is cropping up several times down different lines of descent. I don't know what effect this has. It would be useful to have a better mathematical model, if there is one.

Problem 2. This seems to be equivalent to an 'urn model', in which an urn holds s balls, so that s represents the population, and a ball is taken at random and replaced. After n takings and replacements, the probability of any particular ball being picked at least once is $1 - ((s-1)/s)^n$.

The table shows that between 1350 and 1320, call it 1340, the number of ancestors will be the same as the population of 2.5 million. So if we say that in 1340 there were 2.5 million people, and each of us now has 2.5 million ancestors, then in the formula $1-((s-1)/s)^n$ we can put $s = n = 2.5$ million, and calculate that the probability of Edward III being your ancestor (down one or more lines) works out at 0.632. Going back another 300 years, the table shows that the probability of anybody now living not being descended from William the First is less than 10^{-99}.

For practical purposes, everybody in the country must be descended from William I.

I can see two objections to this model. Firstly it does not take the lack of mobility of population into account. I suspect that in olden times, people tended to stay in much the same place for generations, and one would not get a good mix of population. This is equivalent to the balls in the urn not being properly mixed during sampling. The second objection is that not everyone had children.

Problem 3. First cousins have grandparents in common; second cousins have great grandparents in common. The table shows that everybody has an ancestor in common after 12 or 13 generations, so again excluding recent immigrants, everybody in Britain is at least an 11th cousin of everybody else, and most people are more closely related than that.

The table shows this data together with the number of ancestors and likely British population from 1800 to 1050. The probability of two random people having a common ancestor is very low after 1740, but a near certainty before 1590. The probability of being descended from a particular person living after 1500 is very low, but before 1260 is a near certainty.

I would be interested to have comments on the population figures I have given, and suggestions for better mathematical models of populations and samplings.

Generation	Ancestors	Year	Population millions	Probability of two people in 1980 having at least one common ancestor
6	64	1800	11	0.00037
7	128	1770	8	0.0002
8	256	1740	8	0.00082
9	512	1710	7	0.037
10	1024	1680	6	0.016
11	2048	1650	5.5	0.53
12	4096	1620	5.5	0.95
13	8192	1590	5	0.999998

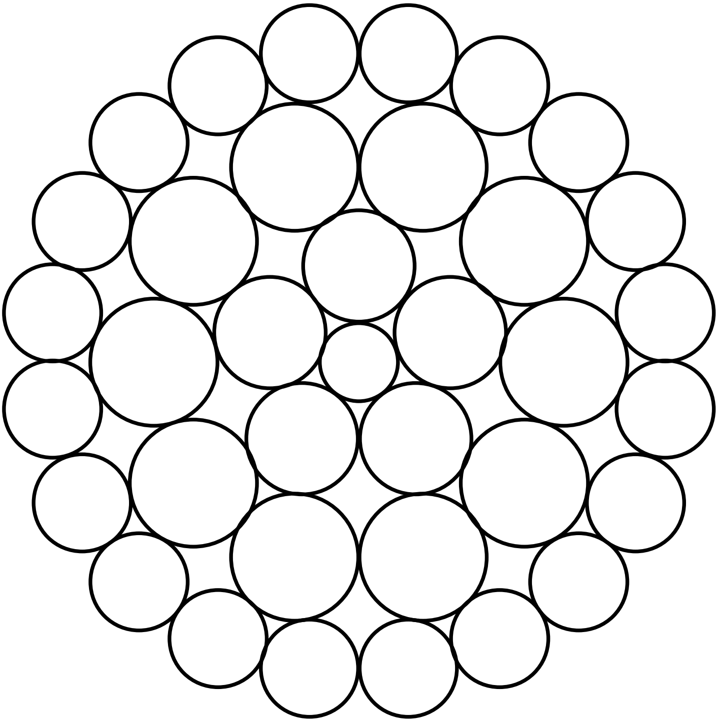
Generation	Ancestors	Year	Population millions	Probability of a particular historical individual being an ancestor of a person alive in 1980
14	16384	1560	4.5	0.0036
15	32678	1530	4	0.0081
16	65536	1500	3.5	0.019
17	131072	1470	3	0.043
18	262144	1440	2.5	0.1
19	524288	1410	2.5	0.19
20	1048578	1380	2	0.41
21	2097152	1350	2.5	0.57
22	4194304	1320	2.5	0.81
23	8388608	1290	3	0.94
24	16777216	1260	2.5	0.9988
25	33554432	1230	2.5	0.999998
26	67108864	1200	2	1
...	...	...	...	...
31	2147483648	1050	2	1

This article is an extensively rewritten version of an essay that originally appeared in M500 **87** (April 1984).

Problem 197.6 – 36 circles

Tony Forbes

The circle in the middle has radius 1. Each of the five circles adjacent to it has radius a. The ten circles in the next ring have radius b. The twenty circles in the outer ring have radius c. What are a, b and c?



Sheila Bear

With great sadness we have to inform you that Sheila Bear, who was an officer of the M500 Society from 1986 to 1994, died at the Churchill Hospital, Oxford, on 11 March 2004 after a long illness. Sheila helped to organize the 1986 and 1987 Revision Weekends and she occupied various secretarial posts from 1987 to 1994. She was a great Committee member, and we always got on with her very well. Her contribution to the Society was much appreciated. We offer our sympathy to her family and friends.

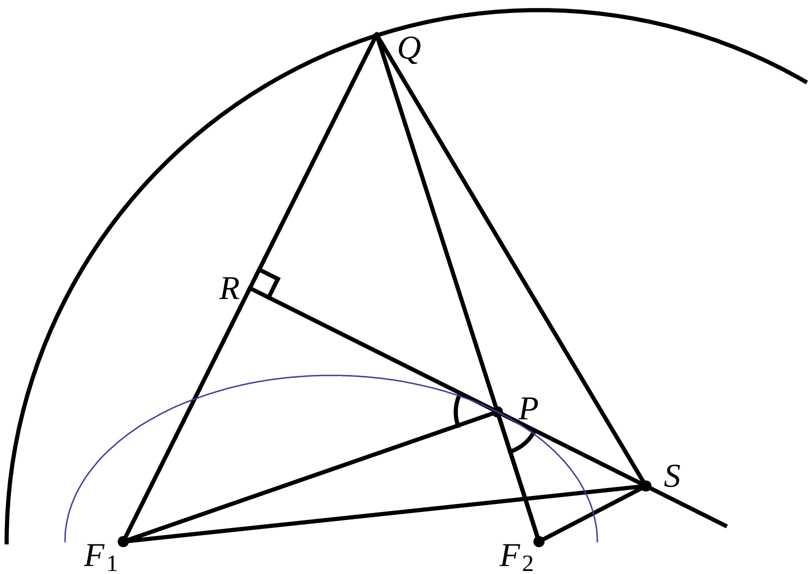
Some maths from even longer ago

Dick Boardman

Bob Escolme ('Some maths from long ago', M500 **194**) uses co-ordinate geometry and calculus to find the tangents and normals to conics. However, some of the results predate both these techniques. The following proof of the focal reflection/focal distance property of an ellipse is adapted from the book *Feynman's Lost Lecture*, by Goodstein and Goodstein, who attribute it to Isaac Newton and say that a diagram similar to the one below appears in his *Principia Mathematica*.

Statement Consider the curve which is the locus of a point P, such that the sum of its distances from two fixed points is a constant. The lines joining P to the fixed points make equal angles to the tangent to the curve at P.

Construction Given: two points F_1 and F_2 and a point P. Draw a circle, centre F_2, radius $PF_1 + PF_2$. Extend the line F_2P to meet this circle at Q. Draw the line QF_1. Draw a line through P perpendicular to QF_1 and meeting it at R.



Proof Consider triangles PQR and PF_1R. Then $PQ = PF_1$ (by construction), $\angle PRQ = \angle PRF_1 = 90^\circ$ (also by construction) and PR is common. Hence these triangles are congruent and therefore angles QPR and F_1PR are equal. Also $\angle QPR$ is equal to the angle between PF_2 and RP extended.

It remains only to prove that RP is a tangent to the curve to complete the proof. Choose any point S on PR. Then triangles QSR and F_1SR are congruent, as before. So $SF_1 = SQ$. However, $QS + SF_2 > QF_2$ by the triangle inequality. Thus the line RP meets the curve only at P and is therefore a tangent to the curve. QED.

The normal to the curve at P is parallel to QF_1.

The book referred to above is fascinating in its own right. It gives a proof of Newton's famous result, proving Kepler's laws of planetary motion. This proof uses only the methods available to Newton, excluding calculus. Richard Feynman, the author of the original lecture, a world class mathematician and Nobel prize winner, says he can't follow the proof in Newton's *Principia Mathematica* and has had to supply a crucial section himself! If today's best mathematicians cannot follow Newton's proof, how could the mathematicians in Newton's time?

Chocolate math

Try this it works, weird!! It only takes a minute. This is pretty neat how it works out. DON'T CHEAT BY SCROLLING DOWN FIRST It takes less than a minute....... Work this out as you read. Be sure you don't read the bottom until you've worked it out! This is not one of those waste of time things, it's funny.

1. First of all, pick the number of times a week that you would like to have chocolate. (try for more than once but less than 10)

2. Multiply this number by 2 (Just to be bold)

3. Add 5. (for Sunday)

4. Multiply it by 50 – I'll wait while you get the calculator................

5. If you have already had your birthday this year add 1753.... If you haven't, add 1752..........

6. Now subtract the four digit year that you were born.

You should have a three digit number.... The first digit of this was your original number (I.e., how many times you want to have chocolate each week). The next two numbers are......... YOUR AGE! (Oh YES, IT IS!!!!!) THIS IS THE ONLY YEAR (2003) IT WILL EVER WORK, SO SPREAD IT AROUND WHILE IT LASTS. IMPRESSIVE, ISN'T IT?

[*Sic.* Reproduced from something that appeared in **Colin Davies**'s in-box.]

Solution 191.4 – What's next?

Here are the first few terms of an infinite sequence, S_n:

$$10, 10, 11, 12, 15, 16, 21, 21, 23, 27, 27, 2\mathrm{B}, \ldots .$$

(i) What's the next term? (ii) What's the rule? (iii) Prove that the terms always have two digits. (iv) What can you say about the behaviour of the first digit of S_n as $n \to \infty$?

Tony Forbes

We have not had anything on this. However, it looks too interesting to ignore and, besides, I need to fill this page to complete the task of editing M500 **197**.

It is the sequence of primes as represented in different number bases; S_n is the nth prime in base $n+1$. Thus $2 = 2+0$, $3 = 3+0$, $5 = 4+1$, $7 = 5+2$, $11 = 6+5$, $13 = 7+6$, $17 = 2 \cdot 8+1$, $19 = 2 \cdot 9+1$, $23 = 2 \cdot 10+3$, $29 = 2 \cdot 11+7$, $31 = 2 \cdot 12+1$, $37 = 2 \cdot 13+11$, and so on. Obviously we have to use symbols other than numerals to encode values greater than 9: A = 10, B = 11, etc. What happens after Z = 35 I leave to your imagination.

Denote the nth prime by P_n and the number of primes up to n by $\pi(n)$. (They are almost inverse functions: $\pi(P_n) = n$ for all n, $P_{\pi(n)} = n$ for prime n but $P_{\pi(n)} < n$ for composite n.) Then (iii) asserts that

$$\text{there exists } a, b,\ 0 \le a, b < n+1, \text{ such that } P_n = a(n+1) + b. \qquad (1)$$

The prime number theorem states that $\pi(n) \sim n/\log n$ and hence (1) should follow easily. However, the PNT is only an asymptotic result (the symbol '$\sim$' is loosely interpreted as 'tends to') whereas what we really want is an inequality involving P_n.

Instead we use a form of Tchebychev's theorem:

$$\pi(n) \ > \ \frac{1}{6} \frac{n}{\log n} \quad \text{for } n > 1. \qquad (2)$$

The proof of (2) is actually quite tricky; see, for example, George Andrews, *Number Theory* (Dover). But (2) implies $\pi(n) > \sqrt{n}$ for $n \ge 5$. Indeed, $n/(6 \log n) > \sqrt{n}$ for $n \ge 2109$ and $\pi(n) > \sqrt{n}$ for $n \in [5, 2108]$ by trial. Hence $P_n < (n+1)^2$ (for all $n \ge 1$) and (1) follows immediately.

Replacing n in (2) by P_n and using $P_n < (n+1)^2$, we obtain $P_n < 12(n+1)\log(n+1)$. Therefore in (1) we have a significantly tighter bound for the first digit: $a < 12 \log(n+1)$. Furthermore, from the prime number theorem it follows that $a/\log n \to 1$ as $n \to \infty$. This answers (iv).

Letters to the Editor

Ellipsoids

Dear Tony,

Firstly, Sebastian Hayes, in his article in M500 **193**, seems unhappy about being a sceptical, non-believing mathematician. In my humble opinion, this is excellent thing to be! When you use mathematics to solve a problem, you need an underlying theory and you must make simplifications, assumptions and measurements that have a limited accuracy. Deficiencies in any of these will render the mathematical solution valueless. Hence, any solution suggested by mathematics must always be checked very carefully against the original problem and every effort should always be made to find checkable cases. The situation with regard to computer-produced solutions is even worse. All the above caveats apply plus the additional possibility that the computer is not actually calculating what was intended. Bugs in computer programs are very common! So *long live sceptical mathematicians and computer users.*

Problem 194.2 – Surface area of an ellipsoid. My copy of Schaum's *Mathematical Tables* gives volumes and surface areas for a number of different solids; however, for an ellipsoid the surface area is left blank. No reason is given, but I suspect that no simple solution exists. I set up the integral in MATHEMATICA but this program could not find a symbolic solution. However, if I set two of the axes equal, so that the ellipsoid became a solid of revolution, I could integrate it and produce a somewhat unpleasant expression. [See page 1.]

For individual cases, you can evaluate the (general) integral numerically, or you can replace the surface with a mesh of triangular facets and add up all the areas of the triangles. Alternatively you could use the sort of reasoning that makes mathematicians shudder whenever they talk to engineers.

Suppose you actually need an answer; for example, you might wish to roof over an elliptical sports stadium with an ellipsoidal roof and want to have an approximate figure for the area of sheeting and the weight that the foundations need to support. You could work out the volume of the ellipsoidal solid and then reduce the major axes by the skin thickness and work out the volume of the smaller ellipsoidal solid. Subtracting the two will give you the volume of the roof. Dividing by the volume of a sheet will then give you a very rough estimate of the number of sheets. Even if you had an accurate figure for the surface area, you would still have to make allowances for joints and supports; so why bother? Ugh!!

Dick Boardman

What's next?

Dear Eddie and Tony,

Many thanks for M500 **195**. Surprised to see that I had submitted the only solution for '50 pence', however nastily it was got.

I took the first question in 'What's next' seriously, and asked the advice of my classical linguist friend Mindaugas Strockis. On the basis of what he says (with selective use of sources and a little fudging) the sequence of multiplets should be: 2 duplets, 3 triplets, 4 quadruplets, 5 quintuplets, 6 sextuplets, 7 septuplets, 8 octuplets, 9 novemplets, 10 decemplets, 11 undecimplets, 12 duodecimplets, 13 tredecimplets, 14 quattuordecimplets, 15 quindecimplets, 16 sedecimplets, 17 septendecimplets, 18 octodecimplets (or duodevigintiplets), 19 novendecimplets (or undevigintiplets), 20 vigintiplets. (Note the change from decem- to -decim- in 11+.)

And here is my continuation of the π poem. It seems sensible to represent a 0-letter word by π itself, a non-member of the roman alphabet.

Sir, I seek a rhyme excelling	*3.14159*
In mystic force and magic spelling.	*265358*
Celestial spheres elucidate,	*979*
But my own feelings can't relate	*323846*
To digits that are ill disposed	*264338*
And in eternal regiments posed.	*32795*
π*, an unending rambling line,*	*02884*
A reckoning lacking a design,	*19716*
Ineffable and bothering—	*939*
Evaporate, you ghostly thing!	*9375*
O π*! Cease worrying me,* π*—*	*105820*
Irregular inanity!	*97*

Best wishes,

Ralph Hancock

ADF writes—Excellent!

We also had responses from a number of people who provided various alternative offerings that they found on the WWW. There is no point reproducing them here because all you have to do is enter 'pi mnemonics', or something similar, in your favourite search engine. I also tried 'celestial spheres elucidate' and I was quite amazed at the number of hits, mainly from astrological sites. I suppose I shouldn't have been.

I am specifically interested in *Sir, I seek* because lines 1–4, which I have known since childhood, are permanently engraved in my brain. It is the poem that I use if ever I feel the urge to rattle off the first 21 digits of π. I must have read it in a book somewhere but the precise reference is lost in the mists of time. It is absent from the WWW—although there is an example starting almost identically with 'Sir, I have a rhyme excelling' but deviating significantly at the fourth line. Anyway, I am extremely grateful to Ralph for providing eight original new lines—and his device for representing zero is surely a stroke of genius!

Generally, π mnemonics do not exceed 32 words; so they don't have to worry about encoding zero. There is a notable exception.

Robin Marks writes—But why stop at silly poems? Let's go for a really silly story. It comes from *The Mathematical Intelligencer* **8**, No. 3, 56–57. The story—which is a story about itself—contains the first 402 decimals of π. A punctuation mark other than a full stop represents a zero. Words which are longer than 9 letters represent two consecutive digits (for example, a ten-letter word represents the two digits 1, 0). A numeral stands for the same digit in the expansion (surely this is cheating).

The story begins: 'For a time I stood pondering on circle sizes. The large computer mainframe quietly processed all of its assembly code. Inside my entire hope lay for figuring out an elusive expansion. Value: pi. Decimals expected soon. I nervously entered a format procedure. The mainframe processed the request. Error. I, again entering it, carefully retyped. . . . '

And concludes: '. . . There a narrative will I trust immediately appear, producing fame. THE END.'

At least the author didn't reach decimal place 601, where we get 000. Or decimal place 772, where we encounter 9999998!

ADF again—After consulting my copy of the *Intelligencer* volume quoted by Robin, I can confirm the existence of 'Circle Digits: A Self-Referential Story' by Michael Keith. I noticed that in the printed version of the story it is written: '... First number slowly displayed on the the flat screen – 3. Good. ...' (*sic*), and for a time I wondered if the repeated definite article was actually represented in the π sequence. Fortunately it's only a misprint; the correct digits for that part of the story are indeed 56692346034.

Scorer – Mervyn, you require two. Commentator 1 – Which way will he go? Commentator 2 – Double-one, I should think.

[World Darts commentary, BBC2, 8/1/2004. Spotted by JRH.]

Contents

M500 197 – April 2004