M500 NEWSLETTER No. 10 February 1974

Editor: (Mrs) Marion Stubbs, Southampton, ...

TRANSCENDENTAL - Ronald Davidson, M201, MDT241 (Romsey)

I was interested in Peter Weir's 'Unprovable Facts' (M500/9). There is still a vast amount of work to be done.in studying "our own back yard" as he puts it in his letter, indeed our understanding of natural numbers or the number continuum is quite sketchy.

My own interest is with transcendental numbers. A number that is not the root of any algebraic equation with rational coefficients is called Transcendental. The problem of deciding whether a number is rational, algebraic or transcendental is still in general unsolved. J. Liouville was the first mathematician to construct real numbers that are transcendental. Georg Cantor, who developed set theory, showed also that 'most' numbers are not algebraic. In other words, there are rather a lot of transcendentals, yet only two of these numbers are really well known. These two numbers have a very important role to play in Science, Technology, and in many other subjects. The numbers I'm referring to are, of course, 3.141 59 26535 89793… and 2.71828 18285 90452…, pi and e. I won't attempt to count up just how many formulae include these two little numbers - I'm sure readers can recall a few for themselves - but just consider for a moment if these two transecendentals were not available for our use…

One wonders if any other such transcendentals could have as much value to our civilisation?

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Bob Eacolme - M100 (Merstham)

Peter Weir, writing on unprovable facts in Newsletter No. 9, ends by saying: "we do not know our own back yard (the natural numbers) well enough yet", implying perhaps that one day we will know it well enough not to have unprovable facts in arithmetic.

In 1931 Kurt Godel proved that we never will. I am unlikely ever to read or understand his paper, but the following, from a layman's guide to it, is clear enough:

> "...it was tacitly assumed that each sector of mathematical thought can be supplied with a set of axioms sufficient for developing systematically the endless totality of true propositions about the given area of inquiry.
>
> Godel's paper showed that this assumption is untenable. He presented mathematicians with the astounding...conclusion that the axiomatic method has certain inherent limitations, which rule out the possibility that even the ordinary arithmetic of the integers can ever be fully axiomatized.
>
> ...Godel's conclusions...show...that there is an endless number of true arithmetical statements which cannot be formally deduced from any given set of axioms by a closed set of rules of inference..."

The extracts are from ‘Godel’s Proof’ by Ernest Nagel and James E. Newman, published by Routledge & Kegan Paul, 1958. It outlines Godel’s method of proof and his conclusions. Although I found this layman’s guide difficult, it should be emphasised that it is a layman’s guide. For anyone who wants a mind-bending and staggering experience, it is well worth the time spent reading it, and it is quite short.

Mark Kac and Stanislaw Ulam devote **some** pages to Godel’s 1931 paper, and to some of his later results, in the Pelican paperback 'Mathematics and Logic'. Again, this is well worth reading.

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Anon 2nd-level

In a recent “Horizon” TV programme I noticed that unqualified A100 or D100 students had a 1 in 2 chance of passing, but M100 unqualified students only 1 in 6 chances. I am comparing unqualified students because I believe that their levels of intelligence and ability will be randomly distributed with respect to whatever courses they are taking, while qualified students would already have been through a process of training and selection, preventing us from making this assumption.

The differences in pass rates give rise to some interesting speculation:

a) By a statistical freak, all idiots elect to do maths.
b) The M100 Course team can’t put it across like the A100 or D100 teams.
c) The intellectual demands of the courses very enormously.

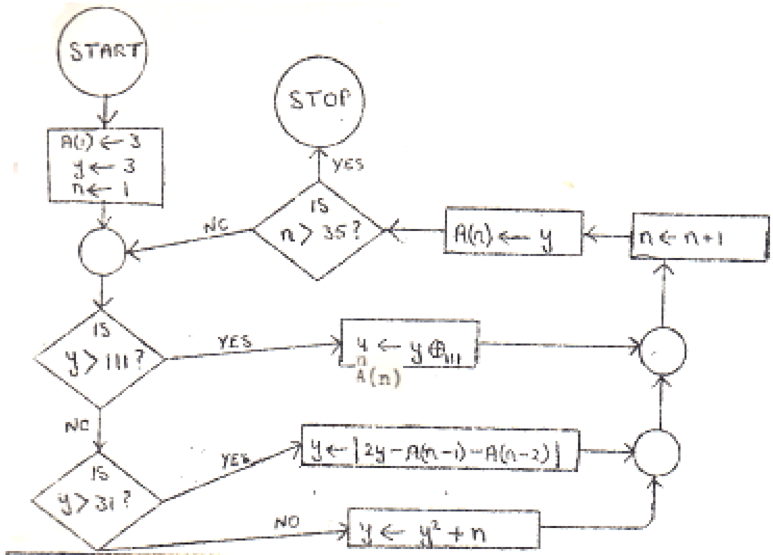
Now I think we can forget item (a), and with some reluctance, item (b), which leaves us stuck with item (c). Here we are up against a fundamental difficulty, because although we have measuring units for practically everything - e.g. a MilliHelen is that unit of beauty that will launch exactly one ship - we have no way of defining a unit of mathematical work, let alone measuring a task in terms of this unit.

The result is that our courses appear to have been constructed on a custom and usage basis, where some Professor says: “In my experience that is about right for M100, or A100, or whatever.” One couldn’t quibble with this empirical method if it gave rise to equitable results from course to course, but it obviously doesn’t. Until such time as the educational technologists get down to devising a qualitative method of comparing the intellectual effort needed to cope with various courses, I thing we shuld apply the empirical method fairly, and adjust the course contents until they all return equal pass rates. This is justifiable on the grounds that in all probability equally intelligent people work equally hard trying to pass all the various courses.

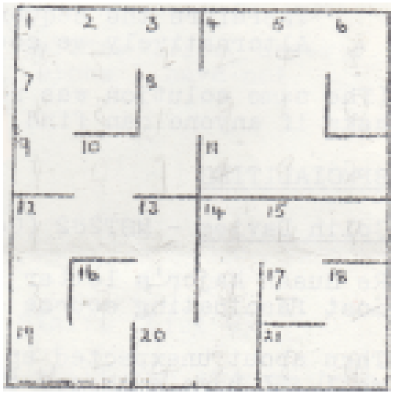
Alternatively, keeping the present maths courses, we should recognise that they are more difficult to pass, so whole and half credits should be re-rated at 1.2 and 0.6 credits respectively, so that a student electing to take all maths subjects (or science or technology) could obtain his degree with five whole courses, not six. This would not only redress an obvious unfairness, but would make maths a much more attractive subject, which would surely please the OU.

CROSSNUMBER 2 - Michael Gregory.

A sequence of numbers (A(n), n=1,2,....,36;A(1)=3) is defined by the diagram, in which || is the modulus function, and $y \oplus_{111}$ is the remainder of y on dividion by 111. A(18)=25 and A(36)=649 are given as checks. Solutions to the clues are to be entered in the second diagram, <u>all solutions are values of A(n)</u> as above e.g. 15 down means A(n)=2A(n-2)+✓n. No A(n) appears more than once in the diagram.



n	1	2	3	4	5	6	7	8	9	10	11	12
A(n)	3											
n	13	14	15	16	17	18	19	20	21	22	23	24
A(n)						25						
n	25	26	27	28	29	30	31	32	33	34	35	36
A(n)												649



ACROSS

1. nn^*+2 such that $A(n^*)=n$
4. A(n–2)–6
7. 4(n+n/10)
8. A(n+1)+A(n–1)+2n+1
9. 11ac. –1
11. see 9ac. and 4down
12. 17n(n–1)
14. 14down+6 it's divisible by 11
16. 2n–1–√[A(n–1)]
17. $(n–1)^2/2$
19. n+1
20. a prime, (21ac.)/2
21. see 20ac.

1. $n^2–2n+1$
2. 2n
3. (n–1)A(n–1)+1
4. 11ac. reversed
5. (n–1)A(n–1) –4
6. $n(n–1)^2$
10. 10n/3
12. (n+1)(3n+1)
13. A(n+2)+6
14. 14ac. –6
15. 2A(n–2)+ √n
16. 3n+1
18. n–7

PROBLEM CORNER No. 10 - Diane Miles, D100 (Utoxeter)

Mastermind

The game of Mastermind involves deducing the positions and colours of a set of coloured pegs. The Challenger sets out 4 pegs in a row, chosen from a set of 6 coloured pegs. The Mastermind makes a guess - a permutation of 4 pegs out of the 6 possible colours. The Challenger indicates how many of these pegs are neither the right colour nor in the right position (totalling 4 indications.) After making several guesses the Mastermind is able to guess the particular permutation of pegs which the Challenger has chosen.

Assumptions: The pegs are all different colours and the Mastermind is as brilliant but as unlucky as possible.

Problem: What is the minimum number of possible attempts which the Mastermind must make to solve the problem?
(Answer unknown.)

Square root of Wonderful

$(\text{OODDF})^2 = \text{WONDERFUL}$

Each letter represents a digit from 0−9 (neither O nor W is zero).
Find the digit which each letter represents.
(Solution not supplied.)

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SOLUTION TO M500/9 Problem 1

R. Seton-Browne

Consider $(n+1)!$
Let $(n+1)! = N$
Then $N+2$ is divisible by 2
$N+3$ is divisible by 3
and generally $N+m$ is divisible by m, for $2 \le m \le n+1$
Therefore the sequence $N+2, N+3, \ldots, N+n+1$ contains no primes.
Alternatively we could consider $N-2$, $N-3$, etc.

(The same solution was received from Eddie Kent who asks if anyone can find a smaller set.)

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SPECIALITIES

Colin Davies - MST282 (Gerrards Cross)

Re Susan Major's letter in M500/9. Personally I found M202 the most fascinating course of all. Particurlarly groups and Galois.

Then about unexpected specialisations. I can speak Finnish quite well. I have a theory that this language was invented by a mathematician. It is so regular, logical, predictable and self-consistent that it is easy to learn. The trouble is that, as a language, it is not much use outside Finland. However, I submit it as an unexpected specialisation.

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Cynthia Griffiths - M100 (Teddington)

Since you want to know about unususl specislisations, and in order to keep the librarians' end up, perhaps I should mention that I am an unusual librarian - I work free-lance with my partner on bibliographical research, company information, S.D.I., etc. Having been librarian of no less than three professional institutions (amongst others) I was always concerned about the non-members who had a genuine need for information, perhaps only once, so that it was not worth their joining the relevant organisation.

My partner had been thinking along similar lines, and when I had to give up full-time work recently to start a family, it seemed an ideal opportunity to test out our theories. Although **we** have been in operation for three months, we are extremely encouraged by the response we are getting from authors, private consultants, small firms (and not a few large ones), who do not have library facilities to call their own, or whose existing library staff are overloaded.

If any readers are interested, I would be pleased to send them a list of the services we are able to offer.

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ON MOUTHS - extracts from notes enclosed with the sae's and 3p stamps

Anonymity is always strictly preserved in the OU. For three years I have been beating my head against a solid wall of anonymity. Perhaps you mark the breakthrough

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I find that living in a rural area where 3 regional boundaries meet my problems of contacting other students are great. There could, in fact, be students ¾ miles from me in either of two Regions and, as I discovered last year, trying to penetrate the OU defences of anonymity in other regions, even for self-help purposes, is no mean task. A scheme like MOUTHS could help enormously.

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I should very much like to be on the M500 mailing list. However, I don't want to appear on the MOUTHS list, since I have not yet enough knowledge to assist anyone.

(Ed: Tell him, somebody?!)

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LETTERS

Peter Foster, Lichfield

If anyone wants help with M251 - or better still can give me help - l'd love to hear from them. With a year off from hard study I shall keep up with M251, re-read the units, watch TV progammes, etc.

What about house-swapping for the summer? Anyone like a C.H. 4-bedroom house with all mod cons for a couple of weeks during the summer in this historic city, in exchange for same by the seaside?

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Eddie Kent - M231, MST282, SDT286 (Farnham)

I can't resist anything to do with mathematics or philosophy. I will be happy to talk to anyone at any time. I mostly sit here all day by myself trying to do OU work amongst other things. I can't ring friends because they're usually at work, and I have this Victorian concept of a boss fuming away in the corner while an underling fritters away the time in nonprofitmaking fripperies.

(Ed: Some of us happen to be the boss, of course! Anyone wanting contact during 9−5 hours please contact M500. I'll amend your MOUTHS entries accordingly.)

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Judy Murfitt - M202 (Crawley)

Could you please explain why you call yourselves M500? I am sure there must be some significance in the name.

(Ed: Reasons given by Peter Weir were: (1) Why not (2) A top-level course in communications. Full credit. (3) It's an overview of OU maths (4) Why not (6) I thought of it.
These seemed good enough at the time, and still serve well. I am now toying with 'Open Interval' or 'Open Set', but would prefer these to be very erudite subsets of M500, published occasionally. Ideas on this welcomed.)

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David Wilkinson - M202 et al (Hertford)

M500/MOUTHS seems to be just what I need, as my main problem is one of isolation. That may sound surprising from one living in the Home Counties, but I am a GPO night telephone operator, doubling up as 'housewife' during the day while my wife works. That means that I am usually working on study centre nights, or, should they chance to coincide with my night off (which only happens once every couple of months or so) then I am too busy trying to catch up on sleep! I think last year I managed to get to the study centre twice - and that was for two foundation courses! I think contact with other students is important - I found Summer Schools marvellous last year mainly for that reason - and I think there must be many others like myself for whom M500 and MOUTHS will fill this need. If it stops us feeling like Eddystone Lighthouse keepers it will be doing a good job. Keep up the good work!

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Bonita Thomson - MST281 (Bromley)

My husband, who has been very impressed with the OU, has applied to become a student next year, electing to take M100. It will be interesting to compare the two courses.

I look forward to hearing from, or meeting, members of the group and will contact some of the far-flung ones during our holidays planned in Scotland and the Lakes.

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<u>Hugh Tassell</u> - M201, M202 (Marden)

Although I have joined MOUTHS, please let the members know that I am hard of hearing, and would they kindly raise their voices!

I am a farmer, and as a result times are a reverse mapping in suitability to those of the OU. In late winter we are not very busy on the farm and there is no OU. Last year, instead of harvesting corn and stocking straw, I was struggling with probability theory and differential equations at Stirling (M100). When I should have been picking apples I was hung up in test tubes, rocks and optics at Loughborough (S100). And finally, while the sowing of corn was accomplished by my long-suffering parent, I managed to finish overdue TMAs and CMAs on time, to find I had 10 days to revise for 2 consecutive days of exams.

Determined not to be caught out again, I purchased 2nd-hand the set of M202 correspondence texts. However, I found myself in a mental stupor, and succumbed to the lure of TV and Christmas. So although I received the texts last November, I didn't really start until the New Year celebrations were over. Is it really necessary to slog through a course in a non-stop 10-month stretch? I would like to hear from people who have a heavily-loaded summer programme on top of their OU courses, and if they have any ideas about 'spreading' things a bit.

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<u>EDITORIAL</u>

This issue is being typed and sent out at the same time as requests are still coming through by the daily dozen for No. 9. You can guess that the M500 office (i.e. the kitchen) is somewhat chaotic - but one wishes it were more so. Where are all our North Country students? What about Scotland - and Wales? As usual, M500 seems to be back to the person-to-person means of publicity, and I hope all members will be telling their friends at tutorials (? What tutorials for M231?) or pinning samples up on notice boards. No. 9 will remain available throughout 1974, so it is never too late for people to apply. Extra copies can be sent to anyone wishing to pin it on a study centre board - not Wales, though, as the Regional Office is doing just that for us, for which we are grateful. If anyone <u>honestly</u> hasn't a quid to spare, this publication is sufficiently non-grasping to accept less in such cases, though one would like at least some sae's.

For the first tiime, there is a small surplus of written items and these have been held over for No.11, with apologies to the enthusiastic authors. However, we have to fill 10 issues, and it is almost certain that most of you will relapse into calligraphic silence with a bang during the next few weeks, never to be heard of again until November, apart from spasmodic cries for help down a telephone. Compulsive writers <u>please</u> carry on sending in your contributions. I need a lot <u>now</u>, even for No.11. Deadline dates do not strictly operate, but in general, I start typing arount the first of the month, earlier if possible.

The following two items will probably be unintelligible to all except past and present M202 students, but may give M100 and M201 a foretaste of the delightful things they may be able to do with M202 under their belts.

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SOME THOUGHTS ON HOOPS (ref.M500/9 Problem 2) - Bob Margolis, Staff Tutor

What do the axioms mean? I seem to think in terms of Cayley tables for finite sets - so xox=x means that the table headings will appear down the leading diagonal.
The cancellation laws (xoy=xoz implies y=z, and xoy=zoy implies x=z) are interesting because they mean that each element can appear once only in each row and column of the table - so we seem to be looking for latin squares of a special form.
The rather curious distributive laws are also interesting as they say that 'multiplication' on the left (or right) by a fixed element is a morphism of the hoop to itself, i.e. an automorphism.

Never mind hoops of order 10 - are there any hoops? I'm happy that:

({x}, o)

o	x
x	x

defines a hoop!

First real result: There can be no hoop of order 2.

Proof:

o	x	y
x	x	
y		y

cannot be completed to a latin square.

(This cost more thought than I dare admit!!)

There is a hoop of order 3 (You check it!)

o	x	y	z
x	x	z	y
y	z	y	x
z	y	x	z

What is more, it is commutative and unique.

At about this stage I indulged in a few flights of fancy. If hoops behave anything like groups, then a hoop of order 10 would have to have a subhoop of order 2 which is impossible.....A search for some results about subhoops suggests:

Definition: A subhoop of a hoop H is a non-empty subset of H which is closed under the hoop operation. (The other axioms will then work in the subset and it will be a hoop withim a hoop.)

Definition: A coset of a subhoop S of a hoop H is defined as the set: xoS = {xos: s E S}

Surprise (at least if you have read anything about groups, e.g.M100/33)
Every coset of S is also a subhoop.

Proof: if xos and xos′ are two elements of xoS then
(xos)o(xos′) = xo(sos′) Axiom 2
= xos″ where s″ E S because S is closed and thus xos″ is an element of xoS.

We have proved that xoS is closed, i.e. a subhoop.

There really are some subhoops. If s is any element of the hoop H then S={s} is a subhoop. Also any coset xoS = {xos} is also a subhoop, which is comforting if not very thrilling.
More to the point of the attempt to follow in Lagrange's footsteps:

Lemma: All cosets of a given subhoop have the same number of elements.

Proof: Since we do (order of S) 'sums' to calculate the elements of xoS we cannot get more than order of S elements in the coset. Now suppose that the two results were the same: xos = xos′ for different s, s′ in S.

The cancellation axiom gives s=s′, contradiction.
Thus order of xoS = order of S.

It's easier to write O(S) than order of S, so I will! There are a string of minor techniques which I've found useful in trying to prove that different cosets don't intersect.

The latin square property of the hoop table means that if we have two given elements x and y of a hoop we can always find a unique element z of the hoop such that xoz = y, and a z′ such that z′ox = y.

Lemma: Let S be a subhoop of the hoop H, and x an element of H.

(i) $x \notin S$ implies $x \notin xoS$, and $S \cap (xoS) = \emptyset$.

(ii) if T is another subhoop of H and $S \cap T = \emptyset$ then $(xoS) \cap (xoT) = \emptyset$

(iii) if T is as in (ii) and t E T then S,T, toS are pairwise disjoint.

Proofs:

(i) Suppose that x E xoS, then we would have x=xos for some s in S. Axiom 1 gives xox=x so we get xox=xos
cancelling: x=s, contradicting the fact that $x \notin S$.
Suppose now that $S \cap (xoS) \neq \emptyset$. Then we would have s=xos′ for some s, s′ in S.
By the property referred to above, applied inside S, there is an s′′ in S such that s=s′′os′.
Combining s′′os′ = xos′, i.e. s′′=x, contradiction.

(ii) If the result were not true we would have xos = xot for some s in S and t in T. Hence s=t and $S \cap T \neq \emptyset$, contradiction.

(iii) Because $t \notin S$, toS doesn't intersect S, the rest of the proof is like the others above.

Putting this lot together in one way gives:

Theorem: If S is a proper subhoop of H then $O(S) \leq 1/3\ O(H)$

Proof: because S is a proper subhoop there is an x E H, $x \notin S$. Put T = xos and apply the above results. We end up with S,T, toS; all with the same order, no two intersecting. Thus O(H) is at least three times O(S).

I have now gone on far too long but I would like to mention that Alex Graham has proved:

Theorem: If a hoop has a subhoop of order 3 then the order of the hoop is divisible by 3. (The proof is not too bad once you've seen it!!!)

Using this result Richard Ahrens has proved:

Theorem: There is no hoop of order 10. This proof is hard and uses a substantial amount of group theory.

Finally a note of thanks to Richard for having the time to sit and talk.

Hoop morphisms (hopomorphisms??) later if anyone still cares!

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(Ed: Yes, please. The above theorems will be henceforth be referenced as the Margolis, Graham and Ahrens theorems respectively.)

<u>NON-SOLUTION TO M500/9 PROBLEM 2</u> - John Bennett, M231, A303

I don't know if there is a hoop of order 10, although every attempt by hand or computer to construct one has failed. The question is tied to a more profound one: is hoop theory part of field theory?

The archetypal hoop is the set of real numbers with the arithmetic mean as operation, and obviously:

$$\frac{x+x}{2} = x, \quad \left(\frac{x+y}{2} + z\right)/2 = \left(\frac{x+z}{2} + \frac{y+z}{2}\right)/2,$$

suggesting the self-distributive laws. Also if $(x+z)/2 = (x+y)/2$ then $y=z$ which suggests the cancellation rules. Abstracting from the sysrem (and dropping the commutative laws) we get the hoop.

Examples abound; the geometric and harmonic means, the operation of taking the mid-point in n-dimensional space. If $x o y = px + (1-p)y$, where $p \neq 0$ or 1, any field may yield a hoop, and in particular, finite fields may give finite hoops. It turns out that every finite field except that of order 2 does give a hoop because, except in that case, there are elements p and $1-p$, neither being 0. This gives hoops of order p^n, where p is prime and $p^n \neq 2$, i.e. 3,4,5,7,8,9,11,13 etc. Since the direct product of two hoops is a hoop it is easy to show (really) that there are hoops of all odd numbers and all multiples of 4. If all finite hoops come from fields then there are no hoops of any order $4n+2$, that if there is such a hoop then hoop theory is an independent subject. There is no hoop of any order 2 (almost trivial) or 6 (tedious but not profound), thus the question - is there a hoop of order 10.

Stop Press: Richard Ahrens writes from Walton Hall giving the outline of a proof that says there are no hoops of orders 10 or 14.

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