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ISSN 1350-8539

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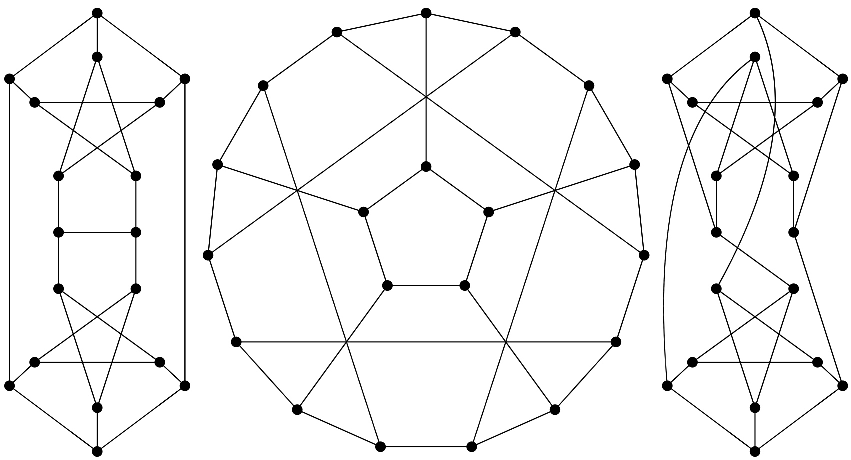

The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

The Four Card Problem

Sebastian Hayes

As stated in *The Improbable Machine* by Jeremy Campbell (Touchstone 1989), the problem is as follows.

> Suppose you are shown four cards, laid out face-down in a row. On the visible side of the cards are printed these symbols.
>
> F 4 7 E
>
> You are told that each card has a letter on one side and a number on the other. You are then given a rule: *If there is a vowel on one side of the card, then there is an even number on the other side.* You must decide whether this rule is true or false by turning over those cards, and only those cards, that show the rule to be true or false.
>
> *The Improbable Machine*, p. 80

The Four Card Problem was devised by Peter Wason, who discusses it in his article 'The Importance of Cognitive Illusions' which appeared in *Behavioural and Brain Sciences* **4**" in 1981. The accepted 'correct' solution is

turn over cards E and 7 only.

Why is this? Because turning over these cards may conceivably disprove the proposition—for example, if the card with E on one side had 9 on the other side. Whatever is on the back of the other two cards is, allegedly, neither here nor there since the proposed rule has nothing to say about permitted combinations of consonants and numbers. Whatever is on the back of F is thus deemed irrelevant. Also, the rule does not say that an even number *must* have a vowel on the other side, so the card marked 4 is also irrelevant.

This solution is very counter-intuitive as is shown by the results of tests given to university students—less than 10 percent made the right choices. One might conclude from this, as Wason and Campbell do, that even quite intelligent people use faulty rules of logic all the time, and, moreover, as the ensuing discussion showed, are very reluctant to admit that they were mistaken. But one might equally well draw the opposite conclusion: namely, that there is something dubious about the problem itself and the approved 'solution'. All I personally would conclude, were I a potential employer of candidates who were set this question as a test, would be that those who answered 'correctly' had received a fairly intensive training in mathematics.

Why is it that we have the bizarre situation that a single case can apparently knock out a proposition whilst no amount of positive cases can establish it? One reason, of course, is the prestige of Popper, one of the most irritating thinkers of the twentieth century, who put science—or rather the theory of scientific discovery since he can hardly be counted as a scientist himself—on a thoroughly negative track. Popper considered that all 'scientific' propositions should be capable of disproof and in his more extreme moments, goes so far as to suggest that statements not susceptible of disproof are 'meaningless'. Actually, few propositions which apply to the universe as a whole, or a large part of it, can be definitively 'disproved' (or for that matter proved) and history is littered with examples of theories such as the particle theory of light which would seem to have been 'disproved', only to rear their heads again unexpectedly. In any case, inventors' and discoverers' minds simply do not function in negative mode: it is much more interesting to work out how something can be done than to work out why and how it cannot be done (though in pure mathematics the reverse is often the case).

But a more serious reason for throwing out a proposition on the strength of a single counter-example is that it is often the only viable strategy in pure mathematics, especially the theory of numbers. Since the number of cases to be considered usually goes to infinity, there can be no question of testing every possible case, and the number of 'exceptions' to a particular rule may all too often turn out to be itself 'infinite'—or, as I prefer to put it, 'indefinitely extendable'. Of course, if we actually know the exact number of 'exceptions', we can rephrase the theorem to exclude these, but usually we do not.

Most branches of mathematics in the West have been cast into the Euclidian model as closed systems developed from a handful of axioms, where the vast majority of propositions (though not necessarily all if Gödel is to be believed) can be established by referring back to the axioms while using uncontroversial deductive rules. The classical scientists had at the back of their minds the image of a mighty brain constructing the universe on the basis of a handful of formulae, some of which were already known thanks to people like Galileo and Newton. But today we are a good deal less confident and statements which were once classed as 'laws of Nature' are more often than not viewed as observed regularities which remain vulnerable to falsification. The discoveries of twentieth century subatomic physics have confounded much that was once treated as gospel: it is even sometimes suggested by physicists today that there is an inherent lawlessness and randomness built into the universe just as there seems to be a degree of 'error-proneness' deliberately built into biological systems (to make them more flexible to unexpected changes).

The rift between pure mathematics and physical reality has, paradoxically, widened as we have learned more about the universe. The truths of mathematics are 'timeless' which is at once their appeal and their limitation. Apart from a few minor additions and reformulations, plane geometry remains more or less as it was more than two thousand years ago, and propositions about (whole) numbers made in Plato's time remain just as true today. Numbers are either prime or composite, and prime numbers do not, and cannot, become composite, nor can composite ones ever become prime. However, if we pass over to biology, we might in our naivety think that the distinction between what is 'living' and what is 'dead' is something akin to the distinction between prime and composite. Nothing could be farther from the case. Biologists are still arguing over whether viruses should be classed as independent living bodies, and some people, such as Kevin Kelly, the author of *Out of Control*, argue that creatures such as internet viruses fulfil the normal criteria for being classed as alive. Mitochondria in human and chloroplasts in plant cells were once independent living entities but have been reduced to the role of subsidiary organelles, whilst, at the other end of the spectrum, certain spores which have been to all intents and purposes 'dead' for thousands, even millions, of years, locked in peat bogs for example, can 'come alive' given the right conditions. The clarity and elegance of mathematical systems is entirely a product of human intelligence and serves our purposes not those of Nature which accepts anything just so long as it works. How Plato, who considered that all classification should be by dichotomy (either/or) would have hated contemporary biology!

But, of course, the main reason why real life situations are so very different from mathematics is that we have limited time and energy at our disposal: there is no question of getting out a complete proof or disproof of a proposition since by that time we might well be dead. As a general rule any answer to a problem is better than no answer at all, and we generally make our decisions on the basis of experience, not abstract reasoning.

The 'rules' we employ in real life differ from the approved rules of mathematical logic in at least the following particulars.

1. We are concerned with likelihood and unlikelihood rather than proof or disproof.

2. We are usually more concerned with establishing a proposition than rejecting it.

3. A single case, even several, does not automatically lead us to abandon a proposition.

4. We often do take into account cases which have no direct bearing on the matter since situations may arise in which they can become relevant.

If a detective worked according to the principles of the Four Card Problem as understood by Wason, I doubt if many criminals would be caught at all. To show what I mean, I propose to recast the Four Card Problem as follows. We have as before four cards laid out face down. On one side of these cards is the name of a European city and on the other either the word 'AIR' or ' TRAIN'—to keep things simple I shall not list the different aircraft lines or passenger train companies which would correspond to odd and even numbers. The cards, which have been designed on the basis of personal documents and information gathered from travel agencies, are

PARIS AIR VIENNA TRAIN

A crime has been committed on a Eurostar train bound for Paris and it is known that the person who committed it got off the train at Paris and did not go any further. I am a suspect though there is no definite proof that I was a passenger on that particular day. I claim that it cannot be me because

I never travel to a French city possessing an airport by train.

According to Wason's style of reasoning, we should turn over the card PARIS and the card TRAIN but neither of the other two cards. The 'rule' says nothing about my mode of travel to other European capitals so there is, allegedly, no point in turning over the card VIENNA. Likewise, the card AIR is irrelevant since it cannot actually disprove or prove my claim: I may well have travelled by air to Paris on one occasion but still have taken the train on the day when the crime was committed.

Now, suppose the card PARIS does have AIR on the back. The police have a copy of a ticket or other document showing I have, on at least one occasion in the past, travelled by Air France to Paris. This contradicts my sworn statement and, though it would not be enough to convict me, would be circumstantial evidence against me that a prosecuting counsel would be sure to use. I am dumbfounded and ask for the evidence to be produced. I then say, "Ah, yes, that's true, I forgot that case. On that day I couldn't take the train because there was a strike, so I had to take a plane." This shows the danger in real life of basing proof or disproof on a single instance. Generally, when we say 'always' we mean something more like 'in normal circumstances'.

Secondly, what if the card VIENNA has TRAIN on the back, i.e. the police have evidence that I have on at least one occasion travelled to Vienna by train. If I were a detective I would consider this rather odd since it is hard to believe that someone would have chosen rail for the much longer and more inconvenient journey to Vienna whereas he always chooses to travel by air to Paris (it actually takes longer to get to the centre of Paris travelling by air if you include the time spent on coaches to and from airports). Once again, the evidence would only be circumstantial but nonetheless .... And if we were not limited to the absurdly small sample of four choices, i.e. the police had drawerfulls of records of train tickets to other European capitals and no records of air trips, this would make my claim even more implausible.

Of course, the original problem is hopelessly vague because the total number of possible cards (trips) is not specified. But supposing we assume that we have a full set of alphabet cards (without repeats) each of which has a different number on the back. We also assume, for the moment, that the numbers on the back are equally divided into evens and odds. We thus have four classes, V (vowels), C (consonants), E (evens), O (odds) and there are eight possible combinations

$$\text{V/E}, \quad \text{V/O}, \quad \text{C/E}, \quad \text{C/O}, \quad \text{E/V}, \quad \text{E/C}, \quad \text{O/V}, \quad \text{O/C}.$$

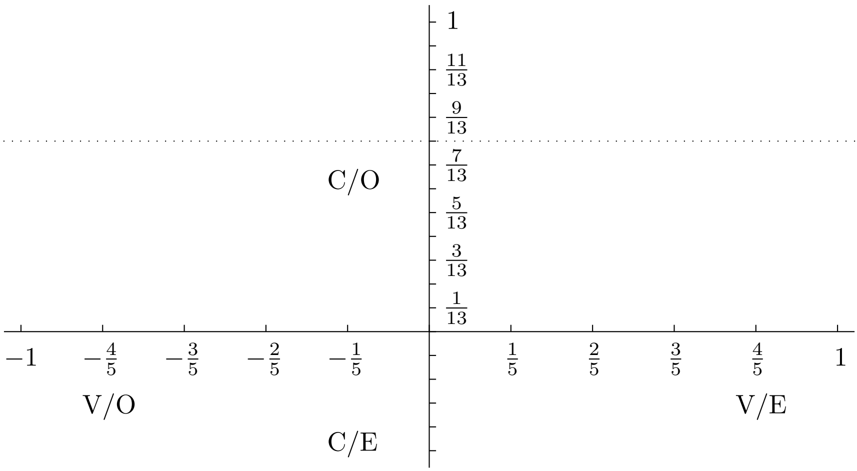
The rule does not specify the order of appearance of a letter or number, i.e. which of the two is visible, so V/E is equivalent to E/V and so on, reducing the classes to four only, say V/E, V/O, C/E and C/O.

If $n(v)$ indicates the number of vowels, $n(e)$ the number of evens and so on, the values for $N = 26$ using a single pack of Letter/Number cards are

$$n(v) = 5; \quad n(c) = 21; \quad n(e) = 13; \quad n(o) = 13.$$

The proposition is *If there is a vowel on one side of the card, then there is an even number on the other side.* We now attribute 'proof values' to the various choices of cards that we are able to turn over. The official solution in effect attributes the value -1 to any case which is found to violate the rule, the value 0 to all other cases, and assumes that a (total) proof value of -1 disproves the proposition completely. Is this a reasonable way of proceeding? Clearly it is not unless N is very large indeed and the total number of permitted choices c very small. The ratio c/N turns out to be crucial.

This sort of question can be modelled using a coordinate system where occurrences which contribute to directly establishing or disproving the rule are marked along the x axis. We need first of all to determine the minimum number of events required to completely demolish or completely establish the proposed rule. In this case min = 5. Any combination such as V/E deserves to be rated $+1/5$, since it establishes the proposition *to that extent*, and any combination such as V/O deserves to be rated $-1/5$ for the same reason.



Assume m choices of cards to turn over, where $m < N$. If the visible faces of these m cards include all the vowels, we simply turn over the vowel cards and make a mark on the x axis according to whether there is an odd or an even on the other side. We take the count and the value will lie between $+1$ and -1. The usefulness of the rule can be determined by its deviation from the origin: a total near zero shows the rule to be worthless since we could have done as well with no rule at all.

Cards with odd numbers showing can make no *positive* contribution to establishing the rule, but they can make a *negative* contribution since if one of them has a vowel on the other side, it shows the rule is invalid in this case and we make a mark on the negative side of the x axis.

What about combinations which have no *direct* bearing on the truth or falsity of the proposition, cards such as C/E and C/O? Are cards with a consonant showing worth turning over at all? Accepted logical wisdom says no. But if we have a deck of thirteen cards in front of us, nine cards of the type C/O show that the rule is valid in at least one case out of the five, while thirteen consecutive C/O cards show that the rule is completely correct because there are only evens left and the vowels have yet to appear. Similarly, nine cards of the type C/E show the rule to be false in at least one case and thirteen that it is completely false.

We can mark occurrences that are *compatible* with the truth or falsity of the proposition on the y axis, while making provisions for them to be transferred to the x axis (the 'Proof Axis') when and if they provide definite information leading to the eventual establishment or rejection of the supposition. Combinations C/O will be marked above the x axis, y positive, because, if sufficiently numerous, they contribute to establishing the truth (+) of the proposition. Likewise C/E combinations will be marked below the origin because they can at best make a negative contribution.

It is now imperative to establish 'threshold values' for the positive and negative parts of the y axis, i.e. values beyond which occurrences of the kind in question will get transferred to the x axis. If no vowel cards have been turned up, and we have eight cards of the type C/O, any further card of the type C/O means that the rule cannot be true in every case since at least one vowel is bound to appear with an even number on the back. The threshold value is given by the number of odds, 13, minus the number of vowels, 5, namely 8.

$$\mathrm{thr} \;=\; n(\mathrm{odd}) - \mathrm{min}.$$

The 'proof value' of a card such as C/O or C/E is ± 1 once the threshold value has been exceeded. If we mark the y axis in multiples of $1/13$ on each side of the origin with the threshold value in this case being $\pm 8/13$ we can see how close we are to establishing or disproving the proposition. Note, however, that this 'threshold value' changes according to the number of points marked on the x axis. For example, if we have already turned up a V/E card, we will need to have ten C/O cards on the y axis (instead of nine) to contribute $+1/5$ to the positive x axis.

Clearly, with the values of the original Wason problem, where $m = 4$ and $N = 26$, no cards marked on the y axis are going to be relevant, so he is, strictly speaking, in the right. But suppose we consider the rule

> *Every card with a vowel on one side has a multiple of four on the other side.*

This time $n(\text{vowel}) = 5$; $n(\text{consonant}) = 21$ as before but $n(\text{multiple of four}) = 6$; $n(\text{non-multiple of four}) = 20$. In such a case it only needs two cards such as L/4 and Q/8 to show that the rule cannot be completely correct since there are not enough multiples of four left to fit with the vowels.

By changing the values of the four variables involved, we get very different situations and further analysis along these lines rapidly becomes quite complicated. In the original Four Card Problem with the values given, I suggested marking out the x axis in multiples of 1/5 but in another situation the multiple chosen would be the smaller of the two parameters that appear in the rule. Also, the third and fourth variables, here the odds and evens would not normally be equal so the y axis would not be marked out symmetrically. One should in principle be able to work out the probability of a particular combination of 'compatible' cards becoming significant and thus contributing to the proof or disproof of the rule. If one starts investigating rules which involve more than four variables the problem rapidly becomes intractable except for a specialist.

My basic point, however, is that Wason, and others like him, are misleading the public with such 'trick questions': although he is technically correct in the case he gives, the public, relying on common sense and not academic logic, is right to be suspicious of such 'solutions'. I am not sure that, if I were an employer, I would not prefer someone who actually failed the Wason Four Card Test by turning over the card marked 4. I was told recently by someone working in computers that his firm did not give a preference to potential candidates with advanced mathematical training, rather the reverse. The reason is doubtless that the 'ideal' conditions that mathematicians are familiar with do not exist in the real world, not even in physics and electronics, and assuming that they do exist can lead to error. Although mathematicians, notably von Neumann, have contributed to the development of cybernetics and computer theory, the actual technological advances have been made by more pragmatic people. Such has always been the case: at about the time the Wright Brothers were building the first air craft, a mathematician at John Hopkins University proved that heavier than air flight was impossible. In mathematics all depends on the initial assumptions: get these wrong and the conclusions with their subsequent applications, though not necessarily without interest, will not be valid in the real world. Von Neumann himself 'proved' that no 'hidden variable' theory could account for the known results of Quantum Mechanics but subsequently Bohm and Wiley actually produced one, and so have certain other people. The author of the book from which I got this concluded by saying, "No, the great Hungarian mathematician didn't get his sums wrong, he simply slipped in an assumption which Bohm and others did not require".

Solution 231.2 – 45 degrees

Show that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\left(\frac{6}{9^n}+\frac{7}{49^n}\right) = \frac{17}{21} - \frac{713}{27783} + \frac{33857}{20420505} - \cdots.$$

Steve Moon

Clearly,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\left(\frac{6}{9^n}+\frac{7}{49^n}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\left(\frac{2}{3^{2n-1}}+\frac{1}{7^{2n-1}}\right)\\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\frac{1}{3^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\frac{1}{7^{2n-1}}.\end{aligned}$$

Now expanding $1/(1+x^2)$ with $|x|<1$, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots,$$

which on integrating becomes

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

So we have, putting $x = 1/3$ and $x = 1/7$,

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\frac{1}{3^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\frac{1}{7^{2n-1}} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}.$$

Using the identity $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, we have

$$\tan\left(2\tan^{-1}\frac{1}{3}\right) = \frac{3}{4} \qquad \text{and} \qquad \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}\right) = 1.$$

Hence $\tan\left(2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}\right) = 1$ and therefore

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\left(\frac{6}{9^n}+\frac{7}{49^n}\right) = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}.$$

Problem 241.1 – Three locks

Tony Forbes

You wish to open a combination lock which has four wheels, each with numbers $0, 1, \ldots, 9$, and you don't know the key. However, you are patient; so you plan to work your way systematically through every combination, with the wheels set initially to 0–0–0–0. Devise a way of doing this with only 9999 moves. A move is the rotation, clockwise or anticlockwise, of a single wheel by one unit. In other words, you are allowed to change one digit by $\pm 1 \pmod{10}$. In real life, of course, you might if you are lucky discover the key long before you reach 9–9–9–9.

Later in the day you want to break into another building, also guarded by a 4-digit combination lock. But this time there is a 10-digit key-pad (as on a calculator or a mobile telephone). The door will open as soon as the lock recognizes the correct sequence of four digits, regardless of any previous input. Now see if you can devise a way to try out all possible combinations with the smallest number of key presses. Can you do it with the theoretical minimum, 10003?

And for your third break-in, the door is guarded by a lock similar to the key-pad operated device of the previous building except that the key is known to consist of *distinct* decimal digits. As before, the door opens as soon as the lock recognizes the correct sequence of four digits. This time the theoretical minimum is $10 \cdot 9 \cdot 8 \cdot 7 + 3 = 5043$.

Actually I am really looking for a general method in each case, one that will work for a key of length n with q symbols, say. If you solve the three problems as written, I expect the generalizations will be straightforward. I chose $n = 4$ and $q = 10$ with practical application in mind. Locks with those parameters seem to be quite common, and one particular example is the nice brass padlock that I bought at a car boot sale for only 20 pence. Unfortunately it's too well made for the usual method of breaking into such locks to work.

Problem 241.2 – Irrational numbers

If πe is irrational, prove that at most one of $\pi + e$, $\pi - e$, $\pi^2 + e^2$, $\pi^2 - e^2$ is rational. (As usual, $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.)

Thanks to Robin Whitty for communicating this to me (TF).

Do the rationals form a group under addition? No. For example, $3 \in \mathbb{Q}$ and $\frac{1}{7} \in \mathbb{Q}$ but $3 + \frac{1}{7} = \frac{22}{7} = \pi$, and π is irrational.

Solution 217.6 – Triangle

Take any triangle and label its vertices A, B and C. Let a, b, c denote the lengths of the sides opposite A, B, C respectively. Show that

$$\log c \;=\; \log a - \frac{b}{a}\,\cos C - \frac{b^2}{2a^2}\,\cos 2C - \frac{b^3}{3a^3}\,\cos 3C - \ldots .$$

Steve Moon

Consider the expression $\log\left(1 - \dfrac{b}{a}e^{iC}\right)$, where in the triangle ABC we have $0 < C < \pi$. Since we want to expand this as a Taylor series we need $\left|\dfrac{b}{a}e^{iC}\right| = \dfrac{b}{a} \le 1$. So we choose $b \le a$ without loss of generality. Hence

$$\log\left(1 - \frac{b}{a}e^{iC}\right) \;=\; -\frac{b}{a}e^{iC} - \frac{b^2}{2a^2}e^{2iC} - \frac{b^3}{3a^3}e^{3iC} - \ldots$$

and since $e^{in\theta} = \cos n\theta + i\sin n\theta$ we take the real part of the expansion:

$$\Re\log\left(1 - \frac{b}{a}e^{iC}\right) \;=\; -\frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \ldots .$$

Also from complex analysis we have $\operatorname{Log} z = \log|z| + i\operatorname{Arg} z$. However, since we are working with z such that $\operatorname{Arg} z \in (-\pi/2, \pi/2)$, we have $\operatorname{Log} z = \log z$ and thus $\Re\log z = \log|z|$.

$$\begin{aligned}
\Re\log\left(1 - \frac{b}{a}e^{iC}\right) &= \log\left|1 - \frac{b}{a}e^{iC}\right| \\
&= \log\sqrt{\left(1 - \frac{b}{a}\cos C\right)^2 + \left(\frac{b}{a}\sin C\right)^2} \\
&= \frac{1}{2}\log\frac{a^2 + b^2 - 2ab\cos C}{a^2} \\
&= \frac{1}{2}\log\frac{c^2}{a^2} \;=\; \log c - \log a,
\end{aligned}$$

where we have used the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos C$.

Squaring numbers

Eddie Kent

How do you square a number in your head? I was explaining a simple method for squaring two-digit numbers to some people when one of them said "Why don't you write that down?"

There's no point, surely, one might explain. It will be on the Net somewhere. Now while that is certainly true, it turns out that you need uncommon perseverance to find something worthwhile. Many ingenious methods are given and one must admire the dedication shown. But they tend to be sadly limited in scope. One site has a technique that works for numbers between 41 and 59 for instance.

To square 43 using this you say that $43 - 25 = 18$, giving the first part of the answer, and $50 - 43 = 7$, which squared is 49, giving the second part. Thus $43^2 = 1849$. But why? Since it is true there will be a proof, and proofs are always worth finding, but not by me on this one.

Another site tells you how to square numbers near to 50; and so it goes on, page after page all quite admirable, but hardly useful. However, one method that at first sight seems just as limited—it is for numbers ending in 5—in fact gives a hint of a true universal method.

It was at a Sociology Summer School many years ago when you couldn't get a degree entirely from one faculty; a statistician I was talking to said he knew a quick method of squaring numbers ending in a half. This is a very useful technique in statistics as you know. He demonstrated by example: seven and a half squared is equal to 7 times 8 plus a quarter: $56\frac{1}{4}$. He was right, but again why?

Back to the Internet. A long hunt finally turned up the useful method, but not clearly set out. The example given is

$$23^2 \;=\; 10 \times 2(23 + 3) + 32 \;=\; 10 \times 2 \times 26 + 9 \;=\; 520 + 9 \;=\; 529.$$

The method depends on $(a + b)^2 = a^2 + b^2 + 2ab$ which might suggest, if we let $a = 20$ and $b = 3$, that $23^2 = 20^2 + 3^2 + 2 \cdot 20 \cdot 3$; but really that is so clumsy you might as well have done it on paper from the start.

However, a little manipulation gives, with a and b as before,

$$(a + b)^2 \;=\; a^2 + b^2 + 2ab \;=\; a(a + 2b) + b^2.$$

Using this on the above 43^2 gives $40 \times 46 + 9$ which is really not hard to do. And the 'half' technique easily falls out from this formulation.

Note also that $(a-b)^2 = a^2 + b^2 - 2ab = a(a-2b) + b^2$ and so $28^2 = 26 \times 30 + 4 = 784$, for instance. And of course you need not use the closest power of ten: to be perverse $28^2 = 20 \times 36 + 64$.

While this method is primarily useful for two-digit numbers, and with a little practise you should beat a calculator, it is of course universal. Trivially $8^2 = 6 \times 10 + 4$ and, harder, $128^2 = 126 \times 130 + 4 = 16384$. Still, 999^2 is easy to do and impresses the natives no end.

Letters

Composite numbers and sums of squares

Dear Tony,

Let C_1 be n. Let C_2 represent the composites containing 2 or more prime factors. Let C_3 represent the composites containing 3 or more prime factors, so that C_m represents the composites containing m or more prime factors. It follows that $C_1 - C_2$ represents the number of primes. Gauss's formula can therefore be expressed as $C_1/\log C_1$ approximately equals $C_1 - C_2$.

Likewise $C_2 - C_3$ approximately equals $C_2/\log C_2$, and $C_m - C_{m+1}$ approximately equals $C_m/\log C_m$.

There are an infinite number of examples of the sum of two squares equalling a square. These can be constructed by taking any odd square say 49 and separating this odd square into two parts with a difference of 1 between these two parts, that is 25 and 24, with $25 + 24 = 7^2$. Then amazingly this can be expressed as $(25+24)(25-24) = 49$, or $25^2 - 24^2 = 7^2$. This can be done for any odd square, but not for even squares which cannot be split into two parts with a difference of 1. This can only apply for the power of 2 and not for any higher powers which confirms (but does not prove) Fermat's Last Theorem.

Peter Griffiths

Parity

If a number is expressed in modulo 2, we call it parity. Is there a term for numbers expressed in modulo N, where $N > 2$? Perhaps M500 readers could invent a few.

Ken Greatrix

Mathematics in the kitchen – VIII

The fundamental problem of recycling glass bottles was posed by Ken Greatrix in M500 **239**. How much water do you put in a bottle to achieve the most efficient rinsing?

Ken Greatrix

I was hoping you wouldn't pass it back because I'm not very good at modelling. However, I've given it some thought and I think the following describes it in outline.

Assume the bottle is an idealized cylinder, closed at both ends. Ignore friction, gravity, etc. The bottle is shaken along its longitudinal axis and has a sinusoidal motion (SHM). This motion can be described simply as

$$y = a \sin bt.$$

If a survey is initiated as the oscillation crosses the axis—define time as $t = 0$ at this point—the bottle's velocity is a maximum;

$$v = \frac{dy}{dt} = ab \cos bt = ab.$$

It is at this velocity that the water continues to move. Let the motion of the 'open' surface of the water be w; then

$$w = abt - \frac{h}{2} + d,$$

where d is the depth of the water in the bottle and h is the height of the bottle

The bottle now slows but the water continues under its momentum until the 'open' surface of the water impacts with the other end of the bottle. We want this impact to occur when the bottle is at maximum negative velocity. This occurs when $bt = \pi$. At this time $w = h/2 = a\pi - h/2 + d$; that is, $d = h - a\pi$.

I reach a bit of a full-stop here because the next few steps would define the impact force by calculating the change in momentum per change in time. Also, since this relationship is linear, it's no use trying to define the maximum depth from it. But I did notice the following.

Let M be momentum of the water; then $M = mv$, where m is the mass of the water. Thus

$$M = kdab = kab(h - a\pi),$$

where k is a constant for the area of the water and its density. If we now differentiate M with respect to a,

$$\frac{dM}{da} \;=\; hkb - 2akb\pi \;=\; 0$$

when this is a maximum. So $hkb = 2akb\pi$ and hence $h = 2a\pi$. But $d = h - a\pi$ (see above) and so $d = h/2$. That is, the bottle is half full, as I suspected. But is the maths correct?

Problem 241.3 – Four integrals

Tamsin Forbes

Let r be a real number. Show that

$$\int_0^{\log(1+\sqrt{2})} \cosh^r x\, dx \;=\; \int_0^{\pi/4} \frac{dx}{\cos^{r+1} x}.$$

Let s be a non-negative real number. Show that

$$\int_0^{\log(2+\sqrt{3})} \sinh^s x\, dx \;=\; \int_0^{\pi/3} \frac{\sin^s x}{\cos^{s+1} x}\, dx.$$

Inequalities

Tony Forbes

This came up in a discussion. Undergraduates at a London university were puzzled by the seemingly ad-hoc method of solving an elementary inequality such as

$$x^2 \;<\; 4.$$

We begin by square-rooting both sides. Thus we get $\pm x < 2$ and then we can solve each case separately: $x < 2$ and $-x < 2$. Indeed, the solution really is $x \in (-2, 2)$.

But why can't you choose to place the $\pm$ sign on the other side; that is, on the 2 rather than the x? Now we have $x < \pm 2$, leading to $x < -2$, a completely wrong answer.

Well?

A quotient group of $\mathbb{Q}^*$

Tommy Moorhouse

A simple example $\mathbb{Q}^*$ is the multiplicative group of positive rational numbers. In this article we will consider some quotient groups $\mathbb{Q}^*/G$ where G is a subgroup of $\mathbb{Q}^*$. We will first consider the integer logarithm function defined for any positive integer $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ by the rule

$$L\big(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}\big) \;=\; k_1 + k_2 + \cdots + k_r.$$

It is easy to see that this function is a logarithm with values in the positive integers, so that $L(1) = 0$ and $L(nm) = L(n) + L(m)$. We now extend the range of L to $\mathbb{Q}^*$ by the rule

$$L(n/m) \;=\; L(n) - L(m).$$

Clearly L is a well defined map from $\mathbb{Q}^*$ to $(\mathbb{Z}, +)$.

The subgroup of $\mathbb{Q}^*$ we will consider is the set of positive rational numbers q such that $L(q) = 0$ with standard multiplication. We will denote this subgroup $L^{-1}(0)$. It is a subgroup because it contains the identity element 1: $L(1) = 0$, and if $q_1 \in L^{-1}(0)$, $q_2 \in L^{-1}(0)$ we have $L(q_1 q_2) = L(q_1) + L(q_2) = 0$ so that $q_1 q_1 \in L^{-1}(0)$. If $q_1 \in L^{-1}(0)$ then $L(q_1^{-1}) = -L(q_1) = 0$ so every element of $L^{-1}(0)$ has an inverse in $L^{-1}(0)$. Finally, the associativity of multiplication is inherited from $\mathbb{Q}^*$.

We will show that $\mathbb{Q}^*/L^{-1}(0)$ is isomorphic to the free abelian group on one generator, that is, to the infinite cyclic group. First, consider the quotient group equivalence

$$q_1 \sim q_2 \;\text{ if }\; q_1 = \alpha q_2 \;\text{ where }\; \alpha \in \mathbb{Q}^* \;\text{ and }\; L(\alpha) = 0.$$

An alternative way of expressing this is to say that two elements are equivalent if $L(q_1) = L(q_2)$. Now note that we can take a power of 2 as a representative of each class. The class of rational numbers such that $L(q) = r$ contains 2^r, which can therefore be used as a representative of the class. The product of any two classes is given by $[2^a][2^b] = [2^{a+b}]$, and clearly these classes form a group. We could in fact use any prime p instead of 2, and we see that the quotient group is isomorphic to the (multiplicative) free abelian group $\{x^r : r \in \mathbb{Z}\}$ with no relations. We will denote this group simply by $\langle x \rangle$.

Another quotient group? Much of the above works for more general integer logarithms. One which gives an isomorphic quotient group is

$$\kappa\big(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}\big) \;=\; k_1 p_1 + k_2 p_2 + \cdots + k_r p_r.$$

Once again the logarithm properties are easily verified, and once again the fact that $\kappa^{-1}(0)$ is a subgroup of $\mathbb{Q}^*$ is a straightforward check. Again the quotient group $\mathbb{Q}^*/\kappa^{-1}(0)$ is isomorphic to the group $\langle x \rangle$. To see this note the quotient group equivalence

$$q_1 \sim q_2 \text{ if } q_1 = \alpha q_2 \text{ where } \alpha \in \mathbb{Q}^* \text{ and } \kappa(\alpha) = 0.$$

For any $q \in \mathbb{Q}^*$, $\kappa(q)$ is an integer, say m, which uniquely defines the class to which q belongs. We can pick representatives for each class of the form $(3/2)^r$, for example, because $\kappa((3/2)^r) = r$. In this case we cannot choose arbitrary primes to represent the classes, but any rational number n such that $\kappa(n) = 1$ gives a generator of the infinite cyclic group, and we can define an isomorphism taking $n \to x$.

The general case can be deduced from the above results. Consider a logarithm defined by an integer function ζ (that is, $\zeta(n)$ is an integer for all n)

$$L_\zeta\left(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}\right) = k_1\zeta(p_1) + k_2\zeta(p_2) + \cdots + k_r\zeta(p_r)$$

extended to $\mathbb{Q}^*$. If there are two primes p_1 and p_2 such that the integers $\zeta(p_1)$ and $\zeta(p_2)$ are relatively prime then there exists a rational number of the form $k = p_1^a p_2^b$ such that $L_\zeta(k) = 1$ and k generates the quotient group $\mathbb{Q}^*/L_\zeta^{-1}(0)$. To see this note that if s and t are relatively prime then there are integers a and b such that $as + bt = 1$. Now take $s = \zeta(p_1), t = \zeta(p_2)$ and confirm that $L_\zeta(k) = 1$.

As a corollary we note that for any non-trivial logarithm L_f there is an isomorphism between $\mathbb{Q}^*$ and $L_f^{-1}(0) \times \langle x \rangle$. The reader is left to deduce the details.

Conclusion We have shown that the quotient groups $\mathbb{Q}^*/L^{-1}(0)$ for non-trivial L are all isomorphic to the free abelian group with one generator. In a way this is quite a strange result, with the structure of the group $L^{-1}(0)$ apparently playing essentially no part.

References Some interesting results on free and finite abelian groups can be found in S. Lang, *Algebra* (Springer, 2002, pp 38–39) and W. Ledermann, *Introduction to Group Theory* (Longman, 1977).

Problem 241.4 – Product

Obtain an expression (as a function of n) for the product $\displaystyle\prod_{k=2}^{n} \frac{k^2}{k^2-1}$.

Solution 237.2 – Arctan sum

Show that

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} \;=\; \frac{3\pi}{4}.$$

Tommy Moorhouse

My approach to this problem is to recast the sum in terms of an infinite product and use results from the theory of the gamma function. We write $\epsilon_n = 2/n^2$ and $y_n = \arctan \epsilon_n$. Then

$$\epsilon_n \;=\; \tan y_n \;=\; \frac{e^{iy_n} - e^{-iy_n}}{i(e^{iy_n} + e^{-iy_n})}.$$

Solving for y_n in terms of ϵ_n we have

$$y_n \;=\; \frac{1}{2i} \log \frac{1 + i\epsilon_n}{1 - i\epsilon_n}.$$

The sum then becomes

$$\sum_{n=1}^{\infty} y_n \;=\; \frac{1}{2i} \log \prod_{n=1}^{\infty} \frac{1 + i\epsilon_n}{1 - i\epsilon_n} \;=\; \frac{1}{2i} \log \prod_{n=1}^{\infty} \frac{n^2 + 2i}{n^2 - 2i}.$$

The terms of the product can be written as

$$\frac{(n + \bar{a})(n - \bar{a})}{(n + a)(n - a)},$$

where $a = 1+i$. Now we use a result proved in paragraph 12.13 of Whittaker and Watson's *A Course of Modern Analysis*, namely

$$\prod_{n=1}^{\infty} \frac{(n - a_1)(n - a_2) \cdots (n - a_k)}{(n - b_1)(n - b_2) \cdots (n - b_k)} \;=\; \prod_{m=1}^{k} \frac{\Gamma(1 - b_m)}{\Gamma(1 - a_m)}.$$

This immediately gives

$$\sum_{n=1}^{\infty} y_n \;=\; \frac{1}{2i} \log \frac{\Gamma(1 - a)\Gamma(1 + a)}{\Gamma(1 - \bar{a})\Gamma(1 + \bar{a})}.$$

We reduce this further using $\Gamma(1+z) = z\Gamma(z)$ and $1-a = -i, 1+a = 2+i$ and so on to get

$$\sum_{n=1}^{\infty} y_n \;=\; \frac{1}{2i} \log \frac{\Gamma(-i)\, i\,(1+i)\,\Gamma(i)}{\Gamma(i)\,(1-i)\, i\,\Gamma(-i)} \;=\; \frac{1}{2i} \log(-i).$$

Using the standard convention for the angle in the complex plane, $-i = e^{3\pi i/2}$ and so, finally

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} \;=\; \frac{1}{2i} \log\left(e^{3\pi i/2}\right) \;=\; \frac{3\pi}{4}.$$

Problem 241.5 – Diagonal elements

Tony Forbes

Let a, b, c and d be integers. Draw up an $\infty \times \infty$ table with a, b, c and d in the top left corner, as shown.

a	c	$\dots$
b	d	$\dots$
$\dots$	$\dots$	$\dots$

Then fill the rest of the table according to the rules: (i) if three consecutive rows in the same column contain x, y and z in that order, then $z = x + y$; (ii) if three consecutive columns in the same row contain x, y and z in that order, then $z = x + y$. Obtain a formula for the diagonal elements of the table.

In particular, if $a = b = c = 0$ and $d = 1$, you should get the squares of the Fibonacci numbers: 0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, 7921, ...

Problem 241.6 – Flagpole

Tony Forbes

Denote the radius of the (perfectly spherical) Earth by R. A flagpole of height 1 is observed at a time chosen at random on a sunny day. What is the expected length of its shadow? Assume that this takes place near the Equator on a day when the sun is directly overhead at midday.

Solution 198.4 – Determinant

$$\text{Find} \quad |M| \;=\; \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-2} & 0 \\ 0 & 1 & x & x^2 & \dots & x^{n-2} \\ x^{n-2} & 0 & 1 & x & x^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x^2 & \dots & x^{n-2} & 0 & 1 & x \\ x & x^2 & \dots & x^{n-2} & 0 & 1 \end{vmatrix}.$$

Steve Moon

For $n \geq 2$ the value of the determinant is given by

$$|M| \;=\; 1 + x^n + x^{2n} + \cdots + x^{(n-2)n}.$$

The method involves a series of $n-1$ two-step iterations, each reducing the order of the determinant which remains to be evaluated by one until a 1×1 determinant remains. Recall that value of a determinant is unchanged if you add a multiple of a row or column to another row or column respectively. We [SM and TF] suggest that you work through the procedure with $n = 6$, which is manageable and sufficiently non-trivial to show the principles.

Iteration 1, step 1. Subtract x times row 2 from row 1 to generate row 1 with 1 in column 1, $-x^{n-1}$ in column n and zero elsewhere.

Step 2. Add x^{n-1} times column 1 to column n. Then row 1 is zero except for a 1 in column 1 and the problem is reduced to evaluating the $n-1 \times n-1$ determinant comprising elements $M_{i,j}$, $2 \leq i \leq n$, $2 \leq j \leq n$ but with an extra x^{2n-i} added to the row i, column n. The element in row n, column n is now $1 + x^n$.

For iteration k, $2 \leq k \leq n-1$, step 1 (subtracting x times row $k+1$ from row k) yields $-x^{k(n-1)}$ in row k, column n, and step 2 (adding $x^{k(n-1)}$ times column k to column n) converts the top of the smaller determinant to $[1\ 0\ 0\ \dots\ 0]$. The entry in row n, column n is $1 + x^n + x^{2n} + \cdots + x^{kn}$.

After $n-1$ iterations we are left with

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -x^{n-1} & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -x^{n-1} & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -x^{n-1} & 1 & 0 \\ x & x^2 & x^3 & x^4 & \dots & x^{n-2} & 0 & 1 + x^n + x^{2n} + \dots x^{(n-2)n} \end{vmatrix},$$

which clearly gives the stated result.

Problem 241.7 – Multiplicative function

Let f be a function that maps positive integers to positive integers. Suppose also that f is multiplicative; in other words, if $\gcd(x, y) = 1$ then $f(xy) = f(x)f(y)$. Suppose moreover that f is increasing; in other words, if $y > x$ then $f(y) > f(x)$. Suppose furthermore that $f(2) = 2$. Show that f must be the identity function.

Fork handles

Try this experiment next time you host a birthday party, preferably in a darkened room. Take four candles. Place two on the cake spaced apart so that their flames won't interfere with each other. Place the remaining two candles next to each other and bring their wicks together so that when lit they produce a single flame. Does this arrangement give out more light than the combined output of the two candles burning separately? (Thanks to Judith Furner's grand-niece Lily for suggesting this problem.)

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See the inside of the front cover for her address.

We will have the usual extras. On Friday we will be running a pub quiz with Valuable Prizes, and for the sing-song on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.

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Front cover: Snarks. Bridgeless 3-regular graphs which do not admit a 3-colouring of edges where two edges with the same colour are not allowed to meet at a vertex.