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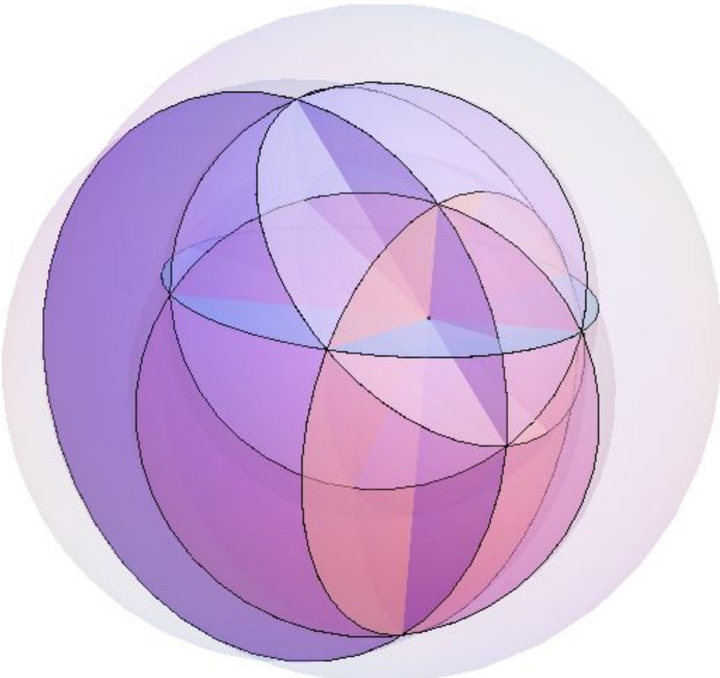
M500 262



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Problem 259.8 – Binomial ratio, revisited

Let r and s be positive integers. Suppose p is prime and that $(p^s - 1)/(p^r - 1)$ is an integer. Is it (i) obviously true, or (ii) true, or (iii) false that s/r is always an integer.

Tony Forbes

This is not a solution, just some remarks, further clarification and a new problem to solve. For brevity, write

$$F(s, r; x) \ = \ \frac{x^s - 1}{x^r - 1}.$$

Someone sent in a solution claiming that the restriction of p to primes was not necessary. Any integer $p \geq 2$ would work. His argument, involving the decomposition of $x^n - 1$ into cyclotomic polynomials $Q_\ell(x)$,

$$x^n - 1 \ = \ \prod_{\ell | n} Q_\ell(x),$$

and the irreducibility of $Q_\ell(x)$, proved that if s/r is not an integer, then $F(s, r; x)$ is not a polynomial in x with integer coefficients. However, I don't think this solves the problem. Even if $F(s, r; x)$ reduces to the ratio of two non-trivial polynomials, there is still the possibility of $F(s, r; p) \in \mathbb{Z}$ for some specific integer p.

Let us fix a prime p. We know from the cyclotomic polynomial decomposition of $x^n - 1$ that $r|s$ implies $F(s, r; p)$ is an integer. The problem is asking whether the converse is true. If $F(s, r; p)$ is an integer, must $r|s$?

Conceding that the restriction to primes might have been a red herring, we examine the possibility of extending the parameter p to an arbitrary integer m. Obviously $m = \pm 1$ causes trouble and $F(s, r; 0)$ is always equal to 1. Moreover, when s is odd $F(s, 2; -2) = ((-2)^s - 1)/3$ is an integer but of course $s/2$ isn't. If we exclude these exceptional values of m, I cannot find any further examples where exactly one of $F(s, r; m)$ and s/r is an integer. So let's scrap 259.8 and replace it with Problem 262.1, below.

I am now of the opinion that (i) is not the answer to the original problem!

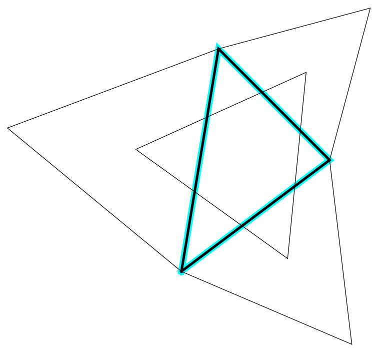
Problem 262.1 – Binomial ratio

Let r and s be positive integers and suppose m is an integer other than -2, -1, 0, 1. Show that $\dfrac{m^s - 1}{m^r - 1}$ and $\dfrac{s}{r}$ are either both integers or both non-integers. Or find a counter-example.

Equilateral triangles and the trisection of angles

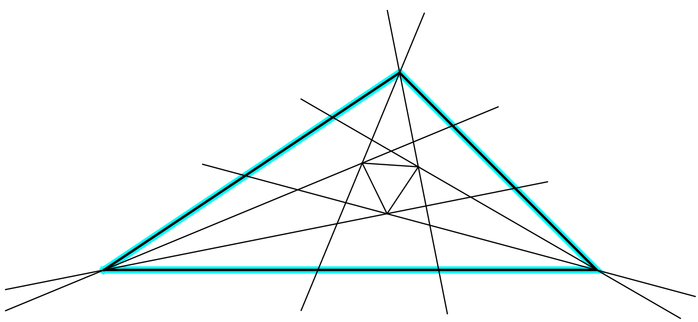
Bryan Orman

The mathematical literature contains many interesting geometrical results concerning triangles, and one very unusual theorem, said to be due to Napoleon, is stated thus: For a given triangle, if three equilateral triangles are constructed *externally* on its three sides, then the centres of these three equilateral triangles form another equilateral triangle, as shown in the following figure.



Alternatively, if the three equilateral triangles are constructed *internally* on the three sides of the given triangle then their centres also form an equilateral triangle.

The trisection of an angle can produce an equilateral triangle, and the classic construction, due to Morley, is stated thus. Take any triangle, and if its *internal* angles are trisected then three of the points where the trisectors meet form an equilateral triangle, as shown in the following figure.

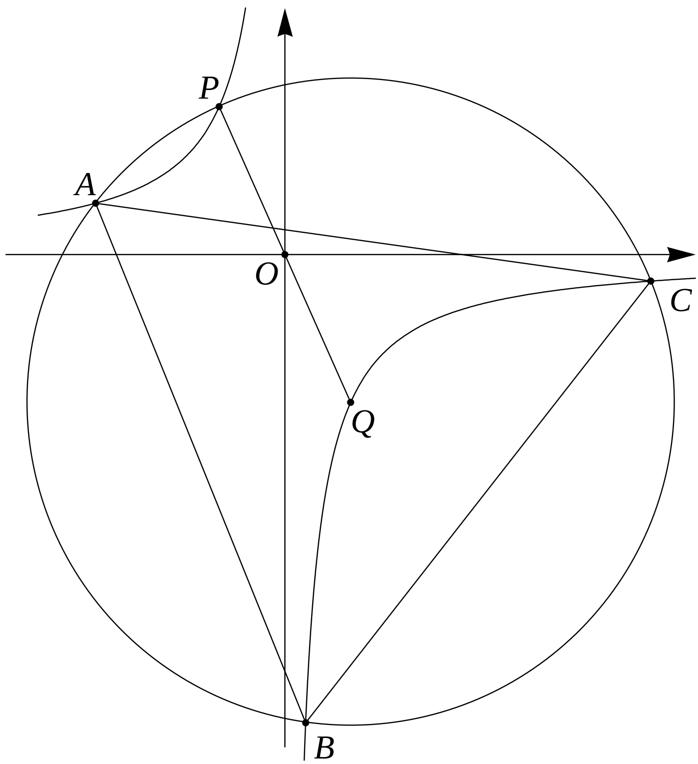


Alternatively, if the three *external* angles of the given triangle are trisected then another three equilateral triangles are formed. (Constructing the second set of equilateral triangles in both of the above figures is left as an exercise!) So here the trisection of angles has produced an equilateral triangle.

Is it possible to trisect an angle from the construction of an equilateral triangle?

Consider the following.

Draw the rectangular hyperbola, $xy = -ab$, where $a > 0$ and $b > 0$. Draw a line through the origin O with slope $-\arctan(b/a)$ to intersect the hyperbola in two points, namely $P = (-a, b)$ and $Q = (a, -b)$. Then, with centre Q draw a circle of radius $|PQ|$ to intersect the hyperbola at P, and at three further points A, B, and C. Finally construct the triangle ABC, as shown in the following figure.



We now show that the triangle ABC is an equilateral triangle.

The equation of the hyperbola is $xy = -ab$ and the equation of the circle is $(x-a)^2 + (y+b)^2 = 4(a^2+b^2) = 4c^2$, where $2c$ is the radius of the circle. If this circle is parametrized by writing

$$x = a + 2c\cos\theta, \qquad y = -b + 2c\sin\theta,$$

then the polar angles of the points of intersection, relative to the centre of the circle, are given by

$$(a + 2c\cos\theta)(-b + 2c\sin\theta) = -ab.$$

Putting $\gamma = \arctan(b/a)$, this reduces to $\sin(2\theta) = \sin(\gamma - \theta)$, with general solution

$$\theta = \frac{\gamma}{3} + \frac{2\pi k}{3}, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Specifically, the polar angles of A, B and C are respectively

$$\frac{\gamma}{3} + \frac{2\pi}{3}, \quad \frac{\gamma}{3} - \frac{2\pi}{3}, \quad \text{and} \quad \frac{\gamma}{3},$$

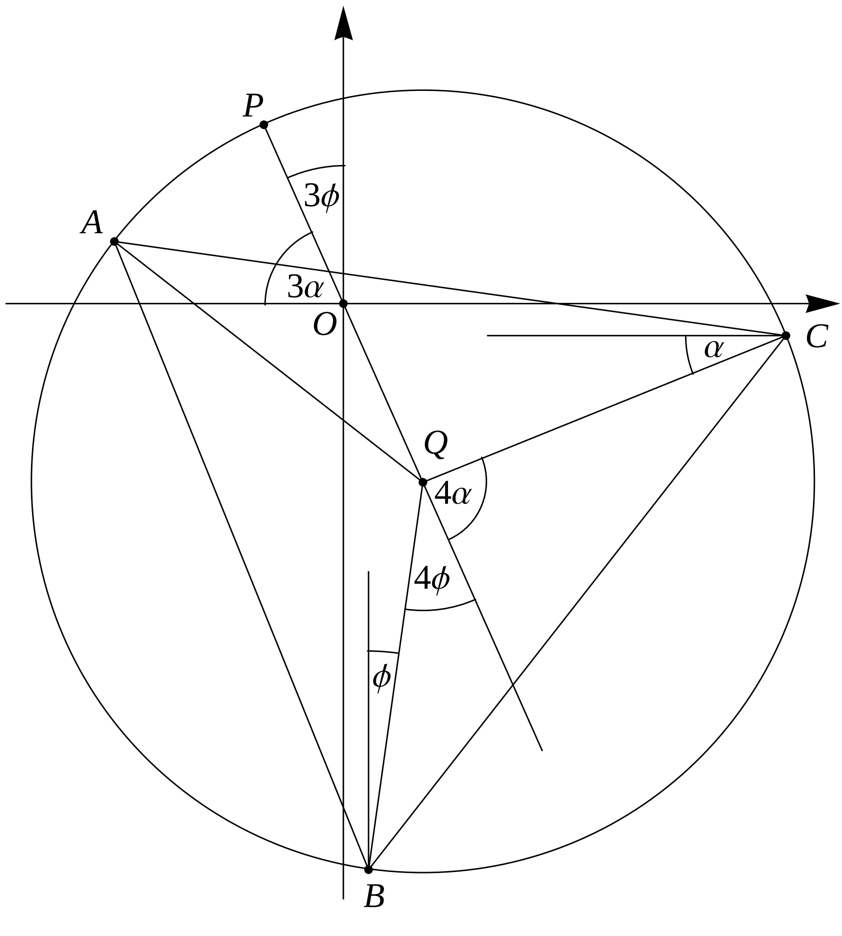
with the point P having polar angle $\pi - \gamma$. The three chords of the circle (the sides of the triangle) subtend equal angles at the centre of the circle so the triangle is indeed equilateral.

And what about the trisection of an angle? With the information already obtained, the relevant angles in the previous figure can be readily evaluated, as shown in the figure on the next page (with the now redundant hyperbola removed). Here, for simplicity, the given angle γ has been changed to 3α and a further angle 3ϕ has been identified in the figure. Note that $\alpha + \phi = \pi/6$. The figure shows that the *given* angle 3α has been trisected, with the angle α given by the slope of the radial line QC. Furthermore, the complement 3ϕ has been trisected, the angle ϕ given by the slope of the radial line BQ. Quite simply, the point B 'trisects' the angle 3ϕ and the point C 'trisects' the angle 3α.

Footnote: The trisection of an angle is one of the three classical geometrical problems of antiquity, the other two being the duplication of a cube, and the squaring of a circle. The duplication of a cube is equivalent to finding the cube root of 2, and the squaring of a circle to finding π. Each of these three problems is to be solved by a geometrical construction involving straight lines and circles, that is, by Euclidean geometry. Specifically by 'rulers and compasses'. To trisect a given angle 3α we could start with the sine of the angle and solve the equation

$$\sin 3\alpha \;=\; 3\sin\alpha - 4\sin^3\alpha$$

for $\sin\alpha$, and this is a cubic equation. A construction using straight lines $(ax + by = c)$ and circles $(x^2 + y^2 + dx + ey + f = 0)$ cannot be equivalent to the solution of a cubic. However, if we allow conic sections, as in the method given here with the hyperbola, then the trisection of an angle can be achieved.



Hyperplanes that meet at a common point

Tony Forbes

Suppose C_1, C_2, …, C_n are n points in $(n-1)$-dimensional space, and let s_1, s_2, …, s_n be any real numbers. Define the $n \times n$ matrix $\mathcal{T}$ by

$$\mathcal{T} \;=\; \begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,n-1} & 1 \\ c_{2,1} & c_{2,2} & \dots & c_{2,n-1} & 1 \\ & & \dots & & \\ c_{n,1} & c_{n,2} & \dots & c_{n,n-1} & 1 \end{bmatrix},$$

where $c_{i,j}$ are the coordinates of C_i:

$$C_i \;=\; (c_{i,1}, c_{i,2}, \dots, c_{i,n-1}), \qquad i = 1,\ 2,\ \dots,\ n.$$

Let $\mathcal{T}_j$ denote the matrix obtained from $\mathcal{T}$ be replacing the entry in row i, column j with s_i for $i = 1$, 2, …, n. Thus

$$\mathcal{T}_1 \;=\; \begin{bmatrix} s_1 & c_{1,2} & \dots & c_{1,n-1} & 1 \\ s_2 & c_{2,2} & \dots & c_{2,n-1} & 1 \\ & & \dots & & \\ s_n & c_{n,2} & \dots & c_{n,n-1} & 1 \end{bmatrix}, \quad \mathcal{T}_2 \;=\; \begin{bmatrix} c_{1,1} & s_1 & \dots & c_{1,n-1} & 1 \\ c_{2,1} & s_2 & \dots & c_{2,n-1} & 1 \\ & & \dots & & \\ c_{n,1} & s_n & \dots & c_{n,n-1} & 1 \end{bmatrix}, \quad \dots.$$

Theorem 1 Suppose C_1, C_2, …, C_n, s_1, s_2, …, s_n, $\mathcal{T}_1$, $\mathcal{T}_2$, …, $\mathcal{T}_{n-1}$ and $\mathcal{T}$ are defined as above. Suppose also that $\det \mathcal{T} \neq 0$. For $1 \le i < j \le n$, let $L_{i,j}$ denote the hyperplane of dimension $n-2$ defined by

$$L_{i,j}: \qquad (x_1, x_2, \dots, x_{n-1}) \cdot (C_i - C_j) \;=\; s_i - s_j.$$

Then (i) $L_{i,j}$ is orthogonal to the line C_iC_j. Moreover, (ii) the $n(n-1)/2$ hyperplanes $L_{i,j}$ meet at a common point,

$$Q \;=\; \left(\frac{\det \mathcal{T}_1}{\det \mathcal{T}},\ \frac{\det \mathcal{T}_2}{\det \mathcal{T}},\ \dots,\ \frac{\det \mathcal{T}_{n-1}}{\det \mathcal{T}} \right).$$

Proof (i) As is well known, if A is a vector in $\mathbb{R}^{n-1}$, the set of vectors X such that the scalar product $(C_i - C_j) \cdot (X - A) = 0$ defines the $(n-2)$-dimensional hyperplane through A that is orthogonal to the vector $C_i - C_j$. In our case $s_i - s_j = (C_i - C_j) \cdot A$.

(ii) Consider the system of linear equations

$$\begin{aligned} c_{1,1}\,x_1 + c_{1,2}\,x_2 + \dots + c_{1,n-1}\,x_{n-1} + x_n &= s_1, \\ c_{2,1}\,x_1 + c_{2,2}\,x_2 + \dots + c_{2,n-1}\,x_{n-1} + x_n &= s_2, \\ &\dots, \\ c_{n,1}\,x_1 + c_{n,2}\,x_2 + \dots + c_{n,n-1}\,x_{n-1} + x_n &= s_n. \end{aligned} \tag{1}$$

The matrix of coefficients is $\mathcal{T}$, and since $\mathcal{T}$ is non-singular the system has a unique solution given by Cramer's rule (Gabriel Cramer, 1750):

$$x_i = \frac{\det \mathcal{T}_i}{\det \mathcal{T}}, \quad i = 1, 2, \ldots, n.$$

So $(x_1, x_2, \ldots, x_{n-1}) = Q$ and $x_n = (\det \mathcal{T}_n)/(\det \mathcal{T})$ satisfies (1). Since $L_{i,j}$ is obtained from (1) by subtracting equation j from equation i and thus eliminating variable x_n, it must be that Q lies on $L_{i,j}$. □

In what follows we show that Theorem 1, a fairly straightforward result in linear algebra, provides us with a universal tool for proving things about things intersecting in a common point. I expect you remember from your high-school days all those constructions where you take a triangle, choose three lines according to some prescription and then prove that they must meet in a single point, which might or might not have a fancy name associated with it. Well, we shall see that some of these proofs are covered by Theorem 1 with $n = 3$. Also we can use Theorem 1 with $n = 4$ to get similar results about a tetrahedron and six planes. And then we could continue with a 4-simplex and ten hyperplanes but by this time the mind is starting to boggle. Instead we just proceed to consider some simple cases.

Example 1 Suppose $s_1 = s_2 = \cdots = s_n$. Then clearly $Q = (0, 0, \ldots, 0)$ and the hyperplanes all pass through the origin.

Example 2 Hopefully more exciting than Example 1, we look at one of the triangle theorems alluded to above. Suppose $n = 3$, and

$$s_1 = -C_2 \cdot C_3, \quad s_2 = -C_1 \cdot C_3, \quad s_3 = -C_1 \cdot C_2.$$

Then the $L_{i,j}$ are the altitudes of triangle $C_1C_2C_3$. For instance, we know from Theorem 1 that $L_{2,3}$ is perpendicular to C_2C_3. Also $L_{2,3}$ passes through C_1 since $C_1 \cdot (C_2 - C_3) = s_2 - s_3$. Thus Q, the common point guaranteed by Theorem 1, is the *orthocentre* of $\triangle C_1C_2C_3$.

Example 3 Again suppose $n = 3$ but this time let

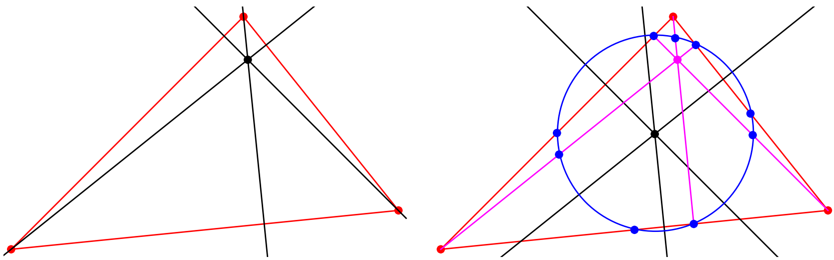
$$s_1 = \frac{C_1 \cdot C_1}{4} - \frac{C_2 \cdot C_3}{2}, \ s_2 = \frac{C_2 \cdot C_2}{4} - \frac{C_1 \cdot C_3}{2}, \ s_3 = \frac{C_3 \cdot C_3}{4} - \frac{C_1 \cdot C_2}{2}.$$

Now the $L_{i,j}$ meet at the centre of the 9-point circle of $\triangle C_1C_2C_3$. This is the circle which passes through the feet of the altitudes of the triangle as well as the midpoints of the sides and the midpoints of the lines joining the orthocentre to the vertices—nine points altogether.

To see this, observe that the 9-point circle is actually the circumcircle of the midpoints of the sides of $\triangle C_1C_2C_3$, and therefore its centre is on the line M_1, say, that is perpendicular to C_2C_3 and passes through the midpoint of the line joining $(C_1+C_2)/2$ to $(C_1+C_3)/2$. So M_1 has equation

$$M_1: \quad (x,y)\cdot(C_2-C_3) \;=\; \left(\frac{C_1+C_2}{4}+\frac{C_1+C_3}{4}\right)\cdot(C_2-C_3).$$

But the right-hand side simplifies to $s_2 - s_3$ by a straightforward calculation and hence M_1 is $L_{2,3}$. So the meeting point of $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$ determined by Theorem 1 is the centre of the 9-point circle. And with some extra work (which I leave to the reader) one can prove that this circle really does go through the other six points.



Example 4 Suppose $n = 3$ and

$$s_1 \;=\; \frac{C_1\cdot C_1 - C_2\cdot C_3}{3}, \quad s_2 \;=\; \frac{C_2\cdot C_2 - C_1\cdot C_3}{3}, \quad s_3 \;=\; \frac{C_3\cdot C_3 - C_1\cdot C_2}{3}.$$

Then the $L_{i,j}$ meet at the the point $(C_1+C_2+C_3)/3$, the *centroid* of $\triangle C_1C_2C_3$. For instance, we have

$$\frac{C_1+C_2+C_3}{3}\cdot(C_1-C_2) \;=\; \frac{C_1\cdot C_1 - C_2\cdot C_2 + C_1\cdot C_3 - C_2\cdot C_3}{3} \;=\; s_1-s_2,$$

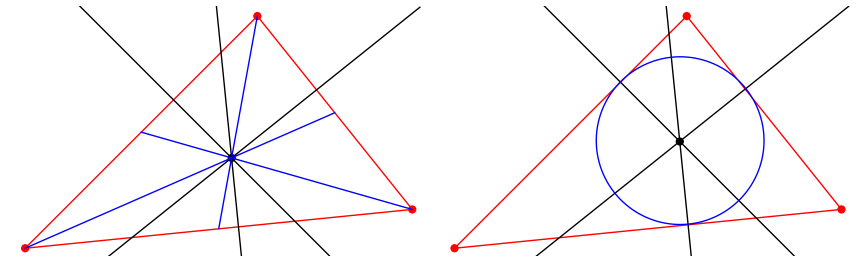
confirming that $L_{1,2}$ passes through the centroid.

Example 5 Suppose $n = 3$ and

$$s_1 \;=\; \frac{C_1\cdot C_1 - |C_1-C_2||C_1-C_3|}{2}, \quad s_2 \;=\; \frac{C_2\cdot C_2 - |C_1-C_2||C_2-C_3|}{2},$$

$$s_3 \;=\; \frac{C_3\cdot C_3 - |C_1-C_3||C_2-C_3|}{2}.$$

Then the $L_{i,j}$ meet at the incentre of $\triangle C_1C_2C_3$. However, my proof is rather messy and unfit for presentation here; so I will leave it for the reader.



Example 6 Suppose $n = 3$ and

$$s_1 \;=\; \frac{C_1 \cdot C_1}{2}, \quad s_2 \;=\; \frac{C_2 \cdot C_2}{2}, \quad s_3 \;=\; \frac{C_3 \cdot C_3}{2}.$$

Observe that the line through $(C_i + C_j)/2$ perpendicular to C_iC_j has equation

$$\left((x, y) - \frac{C_i + C_j}{2}\right) \cdot (C_i - C_j) \;=\; 0.$$

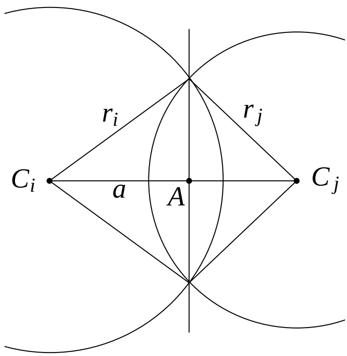
On rearranging, this becomes

$$(x, y) \cdot (C_i - C_j) \;=\; \frac{C_i + C_j}{2} \cdot (C_i - C_j) \;=\; \frac{C_i \cdot C_i - C_j \cdot C_j}{2} \;=\; s_i - s_j.$$

So $L_{i,j}$ is the perpendicular bisector of C_iC_j and Q, the meeting point given by Theorem 1, is the circumcentre of triangle $C_1C_2C_3$.

Example 7 More generally, suppose $n = 3$, let r_1, r_2 and r_3 be non-negative numbers, and imagine that there is a circle S_i of radius r_i centred on C_i, $i = 1, 2, 3$. Suppose also that

$$s_i \;=\; \frac{C_i \cdot C_i - r_i^2}{2}, \qquad i = 1, 2, 3.$$



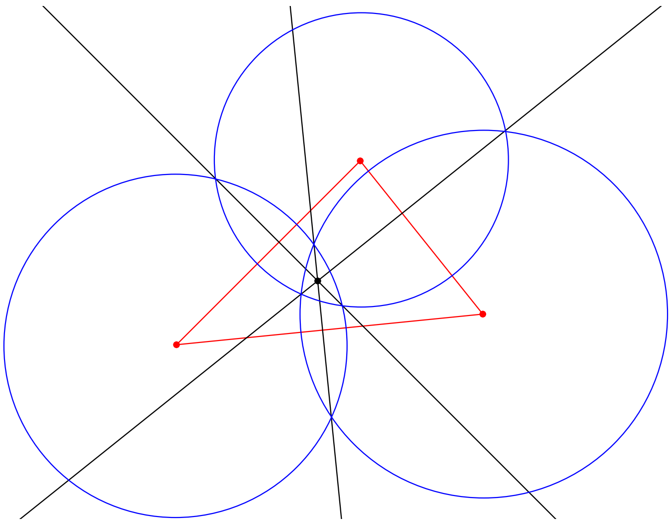
If S_i and S_j intersect in two points, then $L_{i,j}$ is their common chord, i.e. the line that passes through the two points of intersection. In the special case where there are three such chords the result is sometimes known as *Ollerenshaw's theorem*. However, it doesn't matter if S_i and S_j do not intersect in two points. The lines $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$ are always well defined and indeed meet at the point given by Theorem 1.

In the diagram on the previous page we consider the line through A perpendicular to C_iC_j. Let $d = |C_i - C_j|$ and $a = |C_i - A|$. After some triangle-bashing we get $a/d = (d^2 + r_i^2 - r_j^2)/(2d^2)$ and hence

$$A = C_i + \frac{a}{d}(C_j - C_i) = \frac{C_i + C_j}{2} + \frac{r_i^2 - r_j^2}{2d^2}(C_j - C_i).$$

Therefore the constant on the right of the definition of $L_{i,j}$ is

$$A \cdot (C_i - C_j) = \frac{C_i \cdot C_i - C_j \cdot C_j}{2} - \frac{r_i^2 - r_j^2}{2} = s_i - s_j.$$



Thanks to Robin Whitty for drawing my attention to an item in *The Times* of 3 October 2014, where Des MacHale of University College, Cork refers to this result, at least in the case where the circles truly pairwise intersect in exactly two points, as Ollerenshaw's Theorem, after Dame Kathleen Ollerenshaw, who died on 10 August 2014, aged 101.

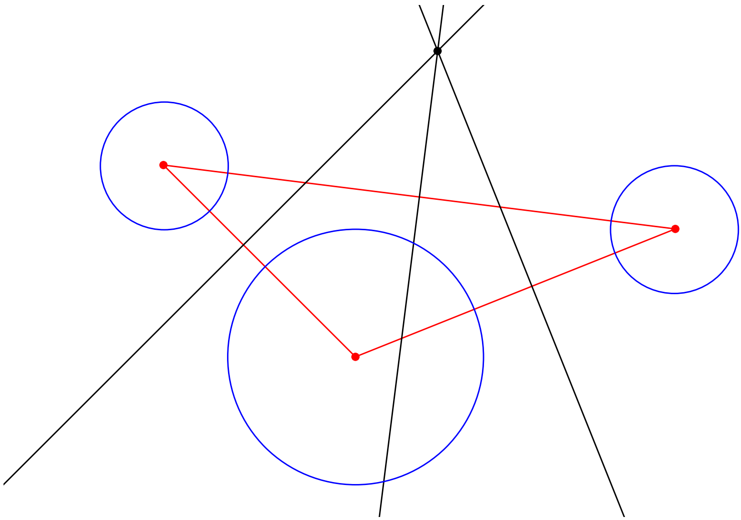
If you don't like computing determinants, there is alternative formula. Suppose circle C_i has centre (x_i, y_i) and radius r_i, $i = 1, 2, 3$. Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad C = \begin{bmatrix} x_1^2 + y_1^2 - r_1^2 \\ x_2^2 + y_2^2 - r_2^2 \\ x_3^2 + y_3^2 - r_3^2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Then the three lines meet at

$$\left(\frac{C^{\mathrm{T}} M Y}{2X^{\mathrm{T}} M Y}, \ \frac{C^{\mathrm{T}} M^{\mathrm{T}} X}{2X^{\mathrm{T}} M Y}\right). \tag{2}$$

To give you some idea of what goes on if the circles fail to intersect, suppose their centres are located at $(-3,1)$, $(0,-2)$ and $(5,0)$, and their radii are 1, 2 and 1 respectively. Then a simple calculation using (2) shows that the common point is at $(9/7, 39/14)$.



Example 8 Even more generally, suppose r_1, r_2, ..., r_n are non-negative numbers, and imagine a hypersphere S_i of radius r_i centred on C_i, $i = 1$, 2, ..., n. Suppose also that

$$s_i \;=\; \frac{C_i \cdot C_i - r_i^2}{2}, \qquad i = 1, 2, \ldots, n.$$

If S_i and S_j intersect in more than a single point, then $L_{i,j}$ is the $(n-2)$-hyperplane that passes through $S_i \cap S_j$. In the special case where there are such intersections between every pair of hyperspheres, the result generalizes Ollerenshaw's three-circles theorem to $n-1$ dimensions. The four spheres case is illustrated on the cover. As before, it doesn't matter if S_i and S_j do not intersect.

Solution 259.1 – Four primes

> Find a number n such that n is the product of four distinct primes and every group of order n is Abelian.

Tommy Moorhouse

The strategy We need to find an integer n which is the product of four consecutive primes and such that every group of order n is Abelian. The strategy will be to show that we can choose the prime factors p such that there is exactly one subgroup of each order p. By refining the choice of factors, guided by some group theory, we can show that any group of order n, where the factors of n are chosen in accordance with the criteria we uncover, must be Abelian.

Sylow's theorems We first invoke Sylow's theorems. We use the standard numbering of these theorems given in [Lederman]. Throughout we use the symbol e to denote the identity element of a group. Let $n = p_1p_2p_3p_4$ where the p_i are consecutive primes. Then Sylow's first theorem tells us that any group of order n has at least one subgroup of order p_i for each of p_1 to p_4.

Next we use Sylow's third theorem, which tells us that the number of distinct subgroups of order p_i is an integer of the form $r_i = 1 + kp_i$ for some $k \geq 0$ and that r_i divides n.

Suppose we can choose the set $\{p_i\}$ such that none of the integers $1+kp_i$ divides n for any $k > 0$. Then, by Sylow's third theorem, there is exactly one p_i-subgroup for each p_i. Each subgroup is Abelian, being cyclic. Note that there is a non-Abelian group of order 6, so some other criterion must be applied before we can deduce anything about the existence or otherwise of non-Abelain subgroups.

Finally, choose each $p|n$ such that p does not divide any of p_i-1, p_ip_j-1, $p_ip_jp_k-1$, where p_i, p_j and p_k are distinct factors of n and the indices range over the set {1,2,3,4}. We assert that in this case every group of order n is Abelian (and in fact cyclic).

The details Suppose on the contrary that G is non-Abelian. Then there exist elements a, b of G such that $ab \neq ba$. Not both of these elements can have order n because in that case each of a and b would be equal to a power of the other, and so the elements would commute. In fact we can choose the orders of a and b to be prime factors of n. For suppose that the element c of order p_1p_2 does not comute with b, say. The element c^{p_1} has order p_2. Suppose that it commutes with b. As p_1 and p_2 are relatively prime we can write

$$1 \;=\; sp_1 + tp_2$$

for relatively prime integers s and t. Now

$$cbc^{-1} \;=\; c^{sp_1}c^{tp_2}bc^{-tp_2}c^{-sp_1} \;=\; c^{tp_2}bc^{-tp_2}.$$

Since c does not commute with b we see that c^{tp_2} also does not commute with b, and this element is another element of order p_1. Thus we might as well take the non-commuting elements to be from different subgroups of prime order. This simplification is not essential, but it allows us to avoid questions of the existence of subgroups of G of order other than prime order. Suppose that the order of a, $|a| = p_a$ (that is, $a^{p_a} = 1$) so that a generates the cyclic group C_{p_a}. We denote this group A for brevity. Now consider the elements e, bab^{-1}, ba^2b^{-1} and so on. These elements are distinct and form a subgroup of G of order p_a as is easily seen by direct multiplication. We denote this subgroup bAb^{-1}. Let $|b| = p_b$. Both p_a and p_b are prime factors of n. This subgroup must be equal to A since we have noted that there is only one subgroup belonging to each p.

The idea now is that if G is non-Abelian then it is possible that $bAb^{-1} \neq A$. If the groups are equal, as they must be for our choice of n, then conjugation by b permutes the elements of A, preserving the subgroup structure. This means that $bab^{-1} = a^k$ for some $k > 1$. But since $ba^2b^{-1} = a^{2k}$ and so on we see, by repeatedly conjugating with b, that $b^{p_a-1}ab^{-(p_a-1)} = a$ so that b^{p_a-1} commutes with a. Two cases are possible: either $p_b | p_a - 1$ (so that $b^{p_a-1} = e$), alternatively $\gcd(p_b, p_a - 1) = 1$.

Suppose first that $\gcd(p_b, p_a - 1) = 1$ and $b^{p_a-1} \neq e$. It follows from the fact that b^{p_a-1} commutes with a that $b^{k(p_a-1)}$ also commutes with a for all $k > 0$. Since $\gcd(p_b, p_a - 1) = 1$ we can write $sp_b + t(p_a - 1) = 1$ for some integers s and t. Then $b^{t(p_a-1)} = b$ so that b commutes with a. In this case A and bAb^{-1} are equal. This does not introduce any additional restrictions on the factors of n.

This leaves us with the possibility that p_b divides $p_a - 1$, in which case it is possible that b and a do not commute. We avoid this possibility by choosing n such that none of $p_1, \ldots, p_4$ divides any of the expressions $p_i - 1$, $p_ip_j - 1$, $p_ip_jp_k - 1$ where $p_i | n$. This means that all possible non-commuting pairs of elements are absent and we see that any group of this order is Abelian.

Four primes A check on sets of small consecutive primes confirms that the set {19, 23, 29, 31} satisfies the restrictions above, so that all groups of order 392863 are Abelian.

References There is a wealth of literature on elementary group theory. The book referred to above is a 'classic' text: [Ledermann] W. Ledermann, *Introduction to Group Theory*, Longman, 1973 (reprinted 1977).

Problem 262.2 – Digit sum ratio

Vincent Lynch

I regularly visit the Missouri state university maths problem website and often send solutions. The challenge page is at http://people.missouristate.edu/lesreid/Challenge.html. Here is an example of a challenge problem.

Let $S(n)$ denote the sum of the (base 10) digits of n. Show that for any positive integer m there is an n such that $m = S(n^2)/S(n)$. For example, when $m = 4$, $n = 13$ works since $13^2 = 169$ and $S(169)/S(13) = 16/4 = 4$.

In addition, M500 readers might like to investigate bases other than 10. Do all number bases $b = 2, 3, \ldots$ have the stated property?

Backward number words

Ken Greatrix

I spotted this one in the *Mail on Sunday*, Sept 28th 2014, on page 55 in the pub quiz section.

What number, when spelt out, is the only word in English to have all of its letters in reverse alphabetical order? (Of course, there are lots of words in which the letters are in reverse alphabetical order, e.g. 'toe', but I think they mean only words describing numbers.)

The answer given is 'one', but I have realized that there's at least one other. Would anyone like to guess what it is before looking at equation (1) on page 19? Are there any more?

Problem 262.3 – Binomial coefficient sum

Show that

$$\sum_{n=2}^{\infty} \frac{(2n-3)!}{4^n (n-1)!n!} = \frac{\log 4 - 1}{8}.$$

Problem 262.4 – Rational integral

Tony Forbes

Suppose a and b are positive integers and that $r > 1$ is a rational number. Show that

$$\int_0^1 (r^b(1 - x^{1/a}) + x^{1/a})^{1/b}\, dx$$

is rational. Hopefully it can be done without actually evaluating the integral. I say this because I computed some non-trivial cases and all I could do was observe the results with horror. For instance, when $a = 4$ and $b = 7$ you get this diabolical expression:

$$\frac{7\left(343r^{36} - 343r^{29} - 1595r^{28} + 4147r^{21} - 4292r^{14} + 2180r^7 - 440\right)}{3190\left(r^7 - 1\right)^5}.$$

Problem 262.5 – HH or TH

During one of his visits to M500, David Singmaster suggested this interesting opportunity for possible wealth enhancement.

'We all know that in tossing a fair coin you are just as likely to get a head followed by a head as a tail followed by a head. So what can the harm be in accepting this offer of a simple game? You will toss a coin repeatedly until you get a head followed by a head or a tail followed by a head. If it's a head followed by a head, I'll give you £2. If it's a tail followed by a head, you give me a £1.'

Was it wise to take up his kind offer?

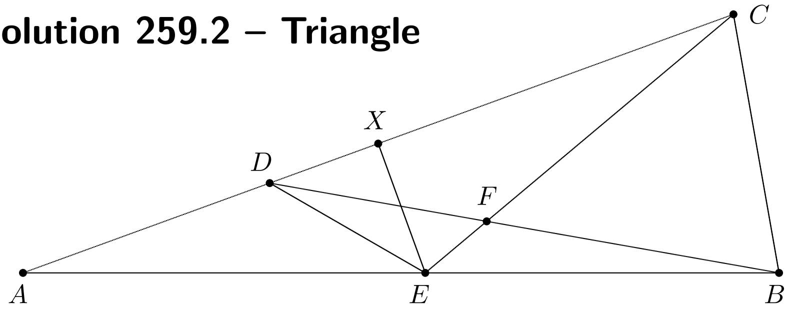
Cube

How many faces of a cube can you see at once, assuming:

(i) the faces are not opaque;
(ii) you have x-ray vision;
(iii) there is a convenient mirror nearby;
(iv) you are inside the cube;
(v) none of the above.

Before you put the cube away, here's another. Draw a diagonal line across one of the faces. Now choose one end of that line and from there draw a diagonal across another face. What's the angle between the two lines you drew? (By the way, we think the answer to (v) could be 5.)

Solution 259.2 – Triangle



(i) In the diagram $|AB| = |AC|$, $\angle BAC = \angle ACE = 20^\circ$ and $\angle ABD = 10^\circ$. What is $\angle AED$?

(ii) Let $\zeta = e^{\pi i/18}$. Show that

$$\frac{\zeta^3(\zeta^2 - \zeta^{10} + \zeta^{12} - 1)}{\zeta^4 + 2\zeta^8 - 2\zeta^{10} - \zeta^{14} - 3} = \frac{1}{\sqrt{3}}.$$

Steve Moon

Clearly $|AE| = |EC|$ and using basic angle sum properties we quickly get

$$\angle ACB = \angle CBA = 80^\circ, \quad \angle ECB = 60^\circ, \quad \angle CBD = 70^\circ, \quad \angle CFB = 50^\circ,$$

$$\angle CFD = 130^\circ, \quad \angle FEB = 40^\circ \quad \text{and} \quad \angle CDF = 30^\circ.$$

Let $\angle AED = \theta$. Then $\angle CDE = \theta + 20^\circ$. Construct EX perpendicular to AC and note that X bisects AC. [Here, you might want to think seriously about annotating the diagram with known angles and line segment equalities. Or not, if you wish to test your powers of concentration whilst following the rest of the article. — TF]

Part (i) To simplify the next steps, assume $|AE| = |EC| = 1$. Then

$$|AX| = |XC| = \cos 20^\circ, \quad \text{and} \quad |XE| = \sin 20^\circ. \tag{1}$$

Using the sine rule,

$$\frac{|AD|}{\sin 20^\circ} = \frac{|AB|}{150^\circ} = 2|AB| \;\Rightarrow\; |AD| = 4\cos 10^\circ \cos 20^\circ.$$

Therefore

$$|DX| = |AX| - |AD| = \cos 20^\circ - 4\cos 10^\circ \cos 20^\circ$$

and

$$|XE| \;=\; |DX|\tan(\angle CDE) \;=\; (\cos 20^\circ - 4\cos 10^\circ \cos 20^\circ)\tan(\theta + 20^\circ). \quad (2)$$

So, putting (1) and (2) together, we have

$$\tan(\theta + 20^\circ) \;=\; \frac{\tan 20^\circ}{1 - 4\sin 10^\circ}.$$

Using the formula $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$ this becomes

$$\tan\theta \;=\; \frac{4\sin 10^\circ \tan 20^\circ}{1 - 4\sin 10^\circ + \tan^2 20^\circ}. \qquad (3)$$

At this point it is tempting to resort to a calculator, but pressing on:

$$\begin{aligned}
\tan\theta \;&=\; \frac{4\sin 10^\circ \sin 20^\circ \cos 20^\circ}{\cos^2 20^\circ - 4\sin 10^\circ \cos^2 20^\circ + \sin^2 20^\circ} \\
&=\; \frac{2\sin 10^\circ \sin 40^\circ}{1 - 2\cos 20^\circ(\sin 30^\circ - \sin 10^\circ)} \;=\; \frac{\cos 30^\circ - \cos 50^\circ}{1 - \cos 20^\circ + 2\cos 20^\circ \sin 10^\circ} \\
&=\; \frac{\sqrt{3}/2 - \cos 50^\circ}{3/2 - \cos 20^\circ - \sin 10^\circ} \;=\; \frac{\sqrt{3}/2 - \sin 40^\circ}{3/2 - (\sin 10^\circ + \cos 20^\circ)} \\
&=\; \frac{\sqrt{3}/2 - \sin 40^\circ}{3/2 - 2\sin 40^\circ \cos 30^\circ} \;=\; \frac{\sqrt{3}/2 - \sin 40^\circ}{3/2 - \sqrt{3}\sin 40^\circ} \;=\; \frac{1}{\sqrt{3}}
\end{aligned}$$

using a lot of standard trigonometric identities. Hence $\theta = 30^\circ$. I do wonder if there is somewhere a handy geometry theorem I have overlooked.

Part (ii) Given $\zeta = e^{\pi i/18}$, we have

$$\zeta^n \;=\; (\cos 10^\circ + i\sin 10^\circ)^n \;=\; \cos 10n^\circ + i\sin 10n^\circ,$$

and we will need these readily calculable expressions at some stage:

$$\zeta^3 = \cos 30^\circ + i\sin 30^\circ = \frac{\sqrt{3}+i}{2}, \quad \zeta^6 = \cos 60^\circ + i\sin 60^\circ = \frac{1+\sqrt{3}i}{2},$$

$$\zeta^9 = i, \quad \zeta^{12} = \frac{-1}{\zeta^6} = \frac{-1+\sqrt{3}i}{2}, \quad \zeta^{15} = \frac{-1}{\zeta^3} = \frac{-\sqrt{3}+i}{2}.$$

Consider the numerator and denominator separately. Thus

$$\begin{aligned}
N \;&=\; \zeta^3(\zeta^2 - \zeta^{10} + \zeta^{12} - 1) \;=\; \zeta^2\zeta^3 - \zeta\zeta^{12} + \zeta^{15} - \zeta^3 \\
&=\; \frac{1}{2}\left(\zeta^2(\sqrt{3}+i) - \zeta(-1+\sqrt{3}i) - \sqrt{3} + i - \sqrt{3} - i\right) \\
&=\; \frac{1}{2}\left(\zeta^2(\sqrt{3}+i) + \zeta(1-\sqrt{3}i) - 2\sqrt{3}\right)
\end{aligned}$$

and

$$\begin{aligned} D &= \zeta^4 + 2\zeta^8 - 2\zeta^{10} - \zeta^{14} - 3 \;=\; \zeta\zeta^3 + 2\zeta^2\zeta^6 - 2\zeta\zeta^9 - \zeta^2\zeta^{12} - 3 \\ &= \frac{1}{2}\Big(\zeta(\sqrt{3}+i) + 2\zeta^2(1+\sqrt{3}i) - 4i\zeta - \zeta^2(-1+\sqrt{3}i) - 6\Big) \\ &= \frac{1}{2}\Big(\zeta^2(3+\sqrt{3}i) + \zeta(\sqrt{3}-3i) - 6\Big). \end{aligned}$$

Dividing N by D yields $1/\sqrt{3}$, as required.

Tony Forbes

The two parts are closely related. From Steve Moon's analysis of the triangle we have

$$\tan\theta \;=\; \frac{4\sin 10^\circ \tan 20^\circ}{1 - 4\sin 10^\circ + \tan^2 20^\circ} \;=\; \frac{4\sin(\pi/18)\tan(\pi/9)}{1 - 4\sin(\pi/18) + \tan^2(\pi/9)}, \qquad (4)$$

which is (3) together with its translation from degrees to radians. However, when Dick Boardman, who drew my attention to this problem, asked MATHEMATICA to simplify the expression on the right of (4) it produced

$$\tan\theta \;=\; \frac{(-1)^{1/6}\left(-1 + (-1)^{1/9} - (-1)^{5/9} + (-1)^{2/3}\right)}{-3 + (-1)^{2/9} + 2(-1)^{4/9} - 2(-1)^{5/9} - (-1)^{7/9}}. \qquad (5)$$

On substituting ζ^{18} for -1 we get the expression in part (ii) of the problem. And, as we have seen, both $(\tan\theta)$s are equal to $1/\sqrt{3}$. Nevertheless a question is raised. How did MATHEMATICA get from (4) to (5)?

Problem 262.6 – Tans

Bryan Orman

Establish the following result:

$$\frac{\tan\dfrac{\pi}{20}}{\tan^3\dfrac{3\pi}{20}} \;=\; \frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}.$$

Solution 241.6 – Flagpole

> Denote the radius of the (perfectly spherical) Earth by R. A flagpole of height 1 is observed at a time chosen at random on a sunny day. What is the expected length of its shadow? Assume that this takes place near the equator on a day when the sun is directly overhead at midday.

Tony Forbes

Readers may recall that this was solved by Vincent Lynch in M500 **244** at various levels of approximation. The length of the shadow is

$$L \;=\; \frac{2}{\pi}\left(\int_0^{\alpha} R\theta\, d\theta + \int_{\alpha}^{\pi/2} Rt\, d\theta\right) \;=\; \frac{R\alpha^2}{\pi} + \frac{2R}{\pi}\int_{\alpha}^{\pi/2} t\, d\theta, \qquad (1)$$

where

$$\alpha \;=\; \arccos\frac{R}{R+1}, \qquad t \;=\; \theta - \arccos\left(\frac{R+1}{R}\cos\theta\right). \qquad (2)$$

With $R = 6378137$ numerical integration in (1) gives

$$L \;\approx\; 0.63662 + 4.88973 \;\approx\; 5.52635.$$

Vincent also showed that since R is large we can replace (2) by

$$\alpha \;\approx\; \sqrt{\frac{2}{R}}, \qquad t \;\approx\; \tan\theta - \sqrt{\tan^2\theta - \alpha^2}, \qquad (3)$$

and again numerical integration gives $L \approx 5.52635$. But now the integral on the right of (1) is doable. So we set it as Problem 244.6 and in M500 **258** Steve Moon obtained the exact solution

$$\int_{\alpha}^{\pi/2}\left(\tan\theta - \sqrt{\tan^2\theta - \alpha^2}\right) d\theta \;=\; \log\frac{\sin\alpha + \sqrt{\sin^2\alpha - \alpha^2\cos^2\alpha}}{2}$$

$$+\;\frac{\sqrt{1+\alpha^2}}{2}\log\frac{(1+\sqrt{1+\alpha^2})\left(\sqrt{\sin^2\alpha - \alpha^2\cos^2\alpha} - \sqrt{1+\alpha^2}\sin\alpha\right)}{(1-\sqrt{1+\alpha^2})\left(\sqrt{\sin^2\alpha - \alpha^2\cos^2\alpha} + \sqrt{1+\alpha^2}\sin\alpha\right)}$$

(after correcting my obvious misprint in equation (7) on page 8 of issue **258**). I then wondered if the integral can be easily computed approximately on the assumption that α is small. To test this idea I got Mathematica to compute the approximation

$$\int_{\alpha}^{\pi/2}\left(\tan\theta-\sqrt{\tan^2\theta-\alpha^2}\right)d\theta \;=\; \frac{-2\log\alpha+\log 4-1}{4}\,\alpha^2+O(\alpha^4), \qquad (4)$$

which is sufficiently accurate to solve the original flagpole problem to 5 decimal places. Being somewhat surprised to see the weird form of the coefficient of α^2 I offered yet another problem to M500 readers: Prove (4) by hand.

Well, here goes. Calling the integral T, expanding the square root and observing that the two $\tan\theta$ terms cancel gives

$$\begin{aligned} T &= \int_{\alpha}^{\pi/2}\left(\tan\theta-(\tan\theta)\sqrt{1-\alpha^2\cot^2\theta}\right)d\theta \\ &= \int_{\alpha}^{\pi/2}\left(\frac{1}{2}\alpha^2\cot\theta+\sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-2}(n-2)!n!}\alpha^{2n}\cot^{2n-1}\theta\right)d\theta. \qquad (5)\end{aligned}$$

Integrating a general power of $\cot\theta$ is possible but messy. However, for small θ there is the approximation $\cot\theta = 1/\theta + O(\theta)$ and since α is small the important part of the integral is near the lower limit, where $\cot\theta$ is large. So perhaps we can get away with replacing $\cot^{2n-1}\theta$ in (5) by $1/\theta^{2n-1}$. Then

$$\begin{aligned} T &\approx \int_{\alpha}^{\pi/2}\left(\frac{1}{2}\alpha^2\cot\theta+\sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-2}(n-2)!n!}\cdot\frac{\alpha^{2n}}{\theta^{2n-1}}\right)d\theta \\ &= \alpha^2\left(\frac{-\log\sin\alpha}{2}+\sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-2}(n-2)!n!}\int_{\alpha}^{\pi/2}\frac{\alpha^{2n-2}}{\theta^{2n-1}}\,d\,\theta\right) \\ &= \alpha^2\left(\frac{-\log\alpha}{2}+\sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-2}(n-2)!n!}\cdot\frac{1}{2n-2}\right)+O(\alpha^4)\end{aligned}$$

since the upper limit of the final integration involves a power of α^2 and can therefore be ignored. But

$$\sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-2}(n-2)!n!}\cdot\frac{1}{2n-2} \;=\; \sum_{n=2}^{\infty}\frac{(2n-3)!}{2^{2n-1}(n-1)!n!} \;=\; \frac{\log 4-1}{4}$$

(see page 14). Hence

$$T \;=\; \alpha^2\left(\frac{-\log\alpha}{2}+\frac{\log 4-1}{4}\right)+O(\alpha^4),$$

as required.

M500 Mathematics Revision Weekend 2015

The **M500 Revision Weekend 2015** will be held at

Yarnfield Park Training and Conference Centre,
Yarnfield, Staffordshire ST15 0NL
between Friday 15th and Sunday 17th May 2015.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £285. The standard cost for non-residents, including Saturday and Sunday lunch, is £170. There will be an early booking period up to the 16th April with a discount of £20 for both members and non-members.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after the 28th February must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications.

A shuttle bus service will be provided between Stone station and Yarnfield Park on Friday and Sunday. This will be free of charge, but seats will be allocated for each service and must be requested before 1st May. There is free on-site parking for those travelling by private transport.

For full details and an application form see the Society's web site at www.m500.org.uk.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

Triskaidekaphobia

Eddie Kent

Arnold Schoenberg was fascinated by numerology. He was born on 13 September (in 1874) and always remained wary of the 13th of any month. In 1951 a friend pointed out that this year he would be 76, and $7 + 6 = 13$, which he hadn't noticed. When he also discovered that in July the 13th fell on a Friday he decided to spend the day in bed, to be safe. At shortly before midnight his wife looked in on him to point out how foolish he'd been; he just said 'Harmony' and died. It was 13 minutes to midnight on Friday 13th in his 76th year. There's glory for you!

04.02

Front cover: Four spheres and six planes; see page 6, especially Example 8 on page 11.