M500 39

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Articles and solutions are not necessarily correct but invite criticism and comment. Nothing submitted for publication ought to be more than about six hundred words long; otherwise it may be split into instalments.

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CONGRUENCE CLASSES *Bob Escolme*

Problem M500 23 4 is: prove that if m, n are relatively prime then the groups $U_m \times U_n$ and U_{mn} are isomorphic. The following proof is lengthy because it is elementary and because it establishes the result by exhibiting the morphism function. It would be interesting to see an indirect proof. The one given here (checked by Bob Margolis) requires no more knowledge than that gained from the group Units of M100. Some sections are in outline and may be completed by the reader. The whole of the discussion is in $\mathbb{J}$, the integers.

DEFINITIONS

(1) $a \equiv b \bmod n \Leftrightarrow \exists s$: $a = sn+b$. It can be shown that $\equiv$ is an equivalence relation on $\mathbb{J}$ for any fixed n.

(2) $(a)_n = \{x: x \equiv a \bmod n\}$. Since for given n, $\equiv$ is an equivalence relation partitioning $\mathbb{J}$ into equivalence classes $(a)_n, (b)_n, \ldots$, we can say that $(a)_n = (b)_n \Leftrightarrow a \equiv b \bmod n \Leftrightarrow a \in (b)_n \Leftrightarrow b \in (a)_n$,

(3) $(a)_n + (b)_n = (a + b)_n$. This definition of 'addition' of equivalence classes is well defined; ie if $(a)_n = (s)_n$ and $(b)_n = (t)_n$ then $(a+b)_n = (s+t)_n$.

(4) $(a)_n(b)_n = (ab)_n$. This too is well defined.

(5) We have two sorts of scalar multiplication:
$p \times (a)_n = (pa)_n$ and $m \circ (a)_n = (ma)_{mn}$. Again well defined. When we come to use one of the 'scalar' multiplications in an equation in congruence classes we need to take care that we use the same one throughout. We will then drop the use of the symbols $\times$ and o, it being clear in each context which one we are using.

We now have an arithmetic for equivalence classes mod n, or congruence classes. Note that $(a)_n + (b)_m$ and $(a)_n(b)_m$ are defined only when $m = n$.

Write $(m,n) = 1 \Leftrightarrow m$ and n have no common factors other that 1, ie are relatively prime. ($(m, n) = 1 \Leftrightarrow \exists\, p,q$: $pm + qn = 1$. This is proved on page 18 of Herstein's *Topics in Algebra*.)

LEMMA A $(x,n) = 1 \Leftrightarrow \exists y$: $(x)_n(y)_n = (1)_n$. (Herstein p23 problem 13.)

PROOF $(x,n) = 1 \Leftrightarrow \exists y,z$: $xy + nz = 1 \Leftrightarrow \exists y,z$: $xy = (-z)_n + 1 \Leftrightarrow \exists y$: $xy \equiv 1 \bmod n \Leftrightarrow (xy)_n = (1)_n$ for some y.

THEOREM 1 $U_n = \{(x)_n: (x,n) = 1\}$ is an abelian group under the binary operation $(x)_n(y)_n = (xy)_n$, $\forall (x)_n, (y)_n \in U_n$.

PROOF See M202 Unit 11 page 12; but armed with our arithmetic of congruence classes it is almost trivial. Note that in the light of lemma A we can write U_n as $\{(x)_n: (x)_n(y)_n = (1)_n$ for some $y\}$. $(1)_n$ is clearly the unit element of U_n and our equivalent definition for U_n shows that a congruence class, mod n, can be a member of $U_n \Leftrightarrow$ it has an inverse in U_n. Use lemma A also to establish closure.

LEMMA B $(m,n) = 1 \Rightarrow \forall d$, $(d)_m \cap (d)_n = (d)_{mn}$.

PROOF $\forall x \in (d)_m \cap (d)_n$, $x \in (d)_m$ and $x \in (d)_n$. So $(x)_m = (d)_m$ and $(x)_n = (d)_n$ (✳). But $(m,n) = 1 \Rightarrow \exists p,q$: $pm + qn = 1$. Applying q and then n to the left-hand equation in (✳) and p and m to the right-hand one and using the definitions in Definitions 5 appropriately we get $(qnx)_{mn} = (qnd)_{mn}$ and $(pmx)_{mn} = (pmd)_{mn}$. Now add these two equations:
$(qnx + pmx)_{mn} = (qnd + pmd)_{mn}$. Recall that $pm + qn = 1$, and so that

last equation can be simplified to $(x)_{mn} = (d)_{mn}$, ie $x \in (d)_{mn}$. Thus $(d)_m \cap (d)_n \subset (d)_{mn}$. With $(d)_{mn} = \{x: x = smn + d, s \in \mathbb{J}\}$, it is clear that $(d)_{mn} \subset (d)_m \cap (d)_n$.

LEMMA C $(m,n) = 1 \Rightarrow$ for any $(a)_m$, $(b)_n$, $\exists d$: $(a)_m = (d)_m$ and $(b)_n = (d)_n$. This is Herstein page 23 problem 15.

PROOF If $a = b$ take $d = a$. Assume then that $a \neq b$. Since $(m,n) = 1$, $\exists p,q$: $pm + qn = 1$. Multiply both sides of this equation by a – b to produce, after some rearrangement of terms: $(b-a)pm + a = (a-b)qn + b = d$, say. The required result then follows.

COROLLARY Applying lemma B to lemma C we obtain $(m,n) = 1 \Rightarrow$ for any $a,b \in d$: $(a)_m \cap (b)_n = (d)_{mn}$. We will be using this result to produce our morphism function.

THEOREM ? $((m,n) = 1$, $(a)_m \in U_m$ and $(b)_n \in U_n) \Rightarrow (a)_m \cap (b)_n \in U_{mn}$.

PROOF $\exists r,s$: $(a)_m(r)_m = (1)_m$ and $(b)_n(s)_n = (1)_n$ (†). This follows from the fact that $(a)_m \in U_m$, $(b)_n \in U_n$ and lemma A. Also, $\exists d$: $(a)_m = (d)_m$ and $(b)_n = (d)_n$ (by lemma C). Multiply by $(r)_m$ and $(s)_n$ respectively to obtain $(a)_m(r)_m = (dr)_m$ and $(b)_n(s)_n = (ds)_n$. Apply these equations to (†) above and we get $(dr)_m = (1)_m$ and $(ds)_n = (1)_n$. Now $(m,n) = 1 \Rightarrow \exists p,q$: $pm+qn = 1$, so multiply the immediately preceding equations appropriately to obtain $(qndr)_{mn} = (qn)_{mn}$ and $(pmds)_{mn} = (pm)_{mn}$. Add these two equations and use the fact that $pm+qn = 1$ to get $(qndr + pmds)_{mn} = (1)_{mn}$. And that is the same as $(qnr + pms)_{mn}(d)_{mn} = (l)_{mn}$. Thus $(d)_{mn} \in U_{mn}$ by our argument following the statement of theorem 1; and by lemma C and its corollary $(a)_m \cap (b)_n = (d)_{mn}$.

COROLLARY We have thus established a well defined function

$$\varphi : U_m \times U_n \rightarrow U_{mn}$$

for $(m,n) = 1$. $\varphi((a)_m, (b)_n) = (a)_m \cap (b)_n, \forall((a)_m, (b)_n) \in U_m \times U_n$.

THEOREM 3 The function φ as defined above for $(m,n) = 1$ is an isomorphism which is onto and so $U_m \times U_n \cong U_{mn}$ (when $(m,n) = 1$).

PROOF Let $X = ((a)_m, (b)_n)$ and let $V = ((\alpha)_m, (\beta)_n)$, with $X, Y \in U_m \times U_n$. In accordance with lemma C $\exists d, \delta$: $(a)_m = (d)_m$, $(b)_n = (d)_n$ and $(\alpha)_m = (\delta)_m$, $(\beta)_n = (\delta)_n$. And by appropriate multiplication we get $(a\alpha)_m = (d\delta)_m$ (<u>†</u>) and $(b\beta)_n = (d\delta)_n$(<u>±</u>). Also $\varphi(X) = (a)m \cap (b)n = (d)mn$ and $\varphi(Y) = (\alpha)_m \cap (\beta)_n = (\delta)_{mn}$. So $\varphi(X)\varphi(Y) = (d)_{mn}(\delta)_{mn} = (d\delta)_{mn} = (d\delta)_m \cap (d\delta)_n$ (lemma B) $= (a\alpha)_m \cap (b\beta)_n$ ((<u>±</u>)and (<u>†</u>)) $= \varphi((a\alpha)_m, (b\beta)_n) = \varphi((a)_m(\alpha)_m,(b)_n(\beta)_n) = \varphi((a)_m,(b)_n)\varphi((\alpha)_m,(\beta)_n) = \varphi(X,Y)$. Thus φ is a morphism.

It can be shown that if $\varphi(X) = (1)_{mn}$ then $X = ((1)_m, (1)_n)$ thus establishing that the kernel of the morphism is the unit element for $U_m \times U_n$ and so φ is one-one. Let $(d)_{mn} \in U_{mn}$ then $\varphi((d)_m,(d)_n) = (d)_{mn}$ with $((d)_m,(d)_n) \in U_m \times U_n$ so φ is onto.

COROLLARY $(m,n) = 1 \Rightarrow \phi(mn) = \phi(m)\phi(n)$ where ϕ is the Euler phi-function. This follows from the isomorphism $U_{mn} = U_m \times U_n$.

If m and n are relatively prime, the product of the number of numbers less than and relatively prime to m and the number of those less than and relatively prime to n equals the number of numbers less than and relatively prime to the product mn.

CLANK! Sue Davi es

Has anyone noticed that M500 is being taken over by machines? In recent months the number of items concerning computers and electronic calculators has increased alarmingly. I appreciate that the things are a necessary evil in a technological society and I have become reconciled to the sight of handymen producing these electronic monstrosities to calculate the number of square yards required to tile a twelve foot by twelve foot floor, and housewives in supermarkets frantically pressing buttons to discovern the price of six chocolate bars at eight and a half pence each. But why must the wretched things intrude into the pages of our precious M500? Yes I know computers and calculators are a necessary tool in mathematics, so is pencil and paper but no-one writes articles about them. Yet we have this endless stream of reports on the comparative technical merits of various types of calculator, and in 37 the suggestion that the ultimate in human stupidity is to use a comma in a computer program instead of a decimal point.

The extent to which dependence upon machines undermines human intelligence is clearly shown by the 196th root problem. This was an interesting problem which, as was pointed out in 37, is instantly solvable by inspection given that the answer is an integer. Those not wishing to use this information could indulge in a little elementary factorisation and the whole thing cancelled out nicely in a few minutes. It should be obvious to anyone that to feed a calculation of this size into an electronic calculator will produce rounding errors and only an approximate answer can be obtained. In which case, since an approximate answer is immediately obtainable by inspection, why in the sacred name of Babbage should anyone want to put it into a calculator. Nevertheless button-pushers by the dozen sent in 'solutions' to nine meaningless decimal places.

Then it was suggested that we should run the CoRA algorithm on a machine to find out in how many iterations 196 produces a palindromic. Who cares? The mathematical interest of the CoRA process is in whether it produces a palindromic for all numbers and if so, why?

The brainpower deterioration caused by excessive use of machines seems to extend to *Creative Computing* from which we have problem 38.1 billed as a hard problem requiring days of patient reasoning. It is not! (Don't believe everything you read in *CC*, Marion.) In fact it's a straightforward logical problem with only two basic alternatives to consider. I picked the wrong one first and still reached the solution in less than fifteen minutes.

Not that I object to easy problems. It's a great boost to the ego to actually be able to do an M500 problem. But I do draw the line at being asked in problem 38.3 (by Eddie of all people, who complained that the 196th root problem was trivial) to explain the remarkable fact that $n^n/n! = n^{n-1}/(n-1)!$. Hands up all those who managed to divide through by n without using a machine.

I do realise that M500 policy is to print most things sent in, and if large numbers of people want items about calculators they must be printed. But for the benefit of those of us who are searching desparately through the pages for some mathematics would it not be possible to group together all pieces written by, for or about machines in a pull-out section in the centre of the magazine.

Incidentally please include my name on any MOUTHS list to be sent to prisoners. Since I assume they have no access to computers or calculators their communications will be most welcome.

DOUBLE FACTORIALS   Colin Mills

Let the double factorial of a number n, written $n!!$, be defined as

$$n!! = n \times (n-2) \times (n-4) \times ... \times r!!$$

where $r = 0$ or 1, depending, on whether n is even or odd. It is of course not strictly necessary to define 0!! if one allows the defining product to terminate with $r = 2$ for even numbers, but it is useful to do so, since the (ordinary) factorial 0! can be and is defined. The gamma-function does not enter into my demonstration although anyone who can define $n!!$ in terms of the gamma-function is welcome to do so - I shall be very interested to read it. It is easy to see that $n!! = \frac{n!}{(n-1)!!} = n \times (n-2)!!$. Thus $2!! = \frac{2!}{0!!} = 2 \times 0!$; hence, to be consistent, 0!! must be defined as taking the value 1, and this table follows:

n	$n!$	$n!!$
0	1	1
1	1	1
2	2	2
3	6	3
4	24	8
5	120	15
6	720	48
7	5040	105
8	40320	384
9	362880	945
10	3628800	3840
11	39916800	10395
12	479001600	46080
13	6227020800	135135
14	87178291200	645120
15	1307674368000	2027025
16	20922789888000	10321920
17	355687428096000	34459425
18	6402373705728000	185794560
19	121645100408832000	654729070
20	2432902008176640000	3715891200
21	5109094217289440000	13749310470
22	1124000727803676800000	81749606400

I believe that I came across the Double Factorial at school (about 12-15 years ago), possibly in *Mathematical Pie*, but cannot remember any details. Perhaps it was a mathematical curiosity. The first few terms resemble the Fibonacci series but rapidly diverge from it as to be expected since the Fibonacci series soon includes primes. It might be interesting to express the double factorials as a function of sums of previous terms, or even try to adapt Stirling's Formula (for factorials of large numbers).

A PART OF LIFE *Brian Woodgate*

I feel that I must take issue with Alan Slomson (37 5); we do not study mathematics in ivory towers and we are not isolated from the real world.

Before the OU started we already had our own lives and personalities. The OU forms a part of this and takes its place alongside families, houses, jobs, interests, hobbies etc. Why should it be any more wrong for us to study pure mathematics than to paint pictures, build model boats, grow carnations or play golf? Does anyone suggest that people who do these things are deprived in some way? Surely it is up to the individual to study or enjoy what he or she likes; there have never been so many opportunities as now.

I am always wary of the "we know what is good for you" brigade and yet I believe in a broad education. This is the advantage of the OU where one has a free choice of study. But even so it is only a small part of life.

So let us go ahead and study Pure Maths or collect beer mats or what we like, as long as we enjoy it and it does no harm to anyone else.

SQUARE ROOT COMPETITION *Peter Wei r*

First, some background. I have been in commercial computing for years with hardly a chance to use any maths. However, recently I found a use for a bit of numerical work.

The situation was this. You have the annual sales of a product over five years, say:

1972	1973	1974	1975	1976
100	110	121	140	147.

From 1972 to 1973 the sales change by +10%. From 1975 to 1976 they change by +5%. The requirement is for an 'average' change from 1972 to 1976: ie what change, if it occured every year, would take the sales from 100 to 147? The answer is not 47/4 = +11.75%. So what is it? Derive a general formula.

If your answer agrees with mine you need to work out 4th roots. I write programs in a stone-age dialect of the computer language Cobol, which can only multiply and divide and not extract roots.

What I in fact did was to write out a square root routine and use it twice. The routine is not impressive, and probably takes a bit of time to execute.

Hence the competition: write a routine (in English, a flowchart, OU BASIC or anything) to work out the square root of a real number from 0 to 20 000 to an accuracy of 0.002, using only add, subtract, multiply and divide. Send your attempts to me. Prizes will be awarded though I haven't decided on them yet. Don't take it too seriously - I might give first prize to the daftest non-solution. Bribery is allowed, the preferred currency being Swiss francs. Anything worth publishing probably will be. Closing date will be about a month after publication date.

CONTINUOUS ASSESSMENT   Max Bramer

The start of a new academic year seems a good time to raise a question in which I have been interested for some time: how do students - and tutors - view the amount of work involved in continuous assessment?

To be specific, what is a reasonable amount of time to expect to have to spend in writing/marking a TMA (or answering a CMA) for the reward (financial or spiritual) offered?

It would be interesting to know how both 'sides' see it from each other's viewpoint as well as their own, and for continuing students (or as the year goes on, for new ones) how the practice matches up to the theory.

I should prefer mostly serious answers, but I have no objections to pseudonyms!

LETTERS

From Lytton Jarman - I always seem to choose an OU course which no one else locally appears to fancy. In the past Marion Stubbs has invariably put me in touch with a fellow student with whom I have been able to correspond. For 1977 I have gone even more obscure and selected D301 (on the recommendation of M500 reader Mr Williams). Despite the lack of an M in the code the course is half statistics and half history. Is any other reader down for D301 in 1977?

Milada (Book Swaps) Mitchell - I have kept all my books from day one - 1969 (yes '69), even have two copies of the first Sesame and beastly Polya rests on a shelf. I have always been a hoarder but now something must be done: a large bookcase delegated to the OU is full, the top is weighed down with as many units as it can take and something must be done to make space for 1977 and on. Does anyone else collect the original 1970 CMA forms and TMA envelopes that don't have glue that tastes of bus drivers' socks? Perhaps Milada would like to start a museum for early OU maths literature. Who has got an autographed copy of Polya?

Joyce (Brotherhood of Man) Moore. With sixty-eight thousand million ancestors possible in 1066 and a two million or so population I have a greater chance of being better than 1011 in line for the throne than getting a Littlewoods first divi.

King Lytton the First.

From Norman Lees - I have two sets of Units M231 and D282 for any takers. I would willingly offer accomodation for M201, M202 and MST282 Units.

As an introduction to M231 I would suggest *Elementary Calculus* by T M Cronin (Crosly, Lockwood and Staples); dead easy. *An Introduction to Real Analysis* by Derek G Ball (Pergamon Press) is nearer the standard of the course.

From Max Bramer - I have a spare set of that excellent course PM951 (which should be an undergraduate course in 1978!) - sixteen units plus eight 'course books' printed by the OU. Also I have units for that nearly equally good course M251 and perhaps others (TM221?) if I search hard enough. What I am missing is M202, M332 or even M331. Are there any offers?

From Eric Lamb - Concerning the M202 exam, I think everyone had the same experience: it was too long. As I left the exam room I heard such comments as 'Do you believe in fairies, because I'm gonna need some' and 'I spent an hour on the first question'. A much more serious comment was 'I got into such a panic; you know I couldn't add up,' and this came from an individual who in my opinion is the best mathematician in our tutorial group. It seems a great pity that this student could fail the exam not because of any mathematical inability but because he was not at ease in the exam room.

For myself I also made childish mistakes because of exam nerves and insufficient time. I can see why Roy Nelson gets so many exam papers which 'haven't been checked by the students before handing in' to quote one of his statements at the Ml00 Summer School last year.

From J A Chappell - In some ways I am sorry that I did pass M201 last year since I did not get as much out of the course as I could have. I wouldn't have minded doing it again. As it was I spent the winter months re-reading Laplace Transforms and Wave Equations.

I am looking forward to M231. The set book seems to be very readable (the loo, in the bath, in bed, &c) unlike the M201 books which I found a bit of a slog.

I'm not sure that I fancy learning all those proofs, but really I've got to learn how to integrate functions of functions one day.

From Peggy Chapman - I really enjoyed the concentrated work at the Weekend; I know it relieved the doubts of a lot of points in M231 so am looking forward to being straight-ened out in September.

From Beryl Brayshaw - I have been a member of M500 for over a year now and my mathematical background consists of O-level maths followed over twenty years later by M100 and M201. The mathematical problems in M500, except for a few exceptions, I find are way over my head. I therefore conclude that I am no mathematician so the question arises should I continue to subscribe to M500?

You are appealing for more members but do you mean mathematicians or students more of my level? I would have thought that most of the mathematicians in the OU already subscribe to M500 or have tried it and decided against it. You have about 2000 potential new subscribers when Ml00 starts but even if you gave everyone of them a sample copy very few would join M500.

Surely it would solve your problem and mine if you included some problems of a simpler nature. Not necessarily mathematical. How about a logic problem or a number crossword? At least it would stimulate interest until the knowledge is obtained to solve the harder problems.

Do other OU students with a similar mathematical background agree with me? Perhaps the lady who joined M500 and asked for advice on M100 last year would care to comment.

I think I'd better carry on for another year. You never know, when I get to the end of M231 and M251 I may be able to do some of the problems. I enclose a deposit for the Aston Weekend (if you'll have me after this). It will be nice to meet Marion again. After all I probably would not have joined M500 but for meeting Marion at Stirling last year. You would probably get all the new M100 students to join if you could arrange for Marion to meet them all.

From Peter Ost - I have now returned to the UK and hope to resume my OU studies having had to withdraw twice from M332 due to lack of TV, tutors or anyone else to talk to in Germany.

From Paul Luft - Thanks a lot for M500 38 which I appreciated even more than usual. I wonder if the journal could possibly give more potted versions of the different maths and maths-related courses, say every two issues - with comparisons on the basis of (i) work load, (ii) intellectual demands, (iii) legibility of the course; how it is written - whether it is just thrown together, (iv) enjoyment, satisfaction or otherwise obtained from the course.

I think that the value of bringing previews and overviews of various courses would be great for the newer (F and G) year students who don't really know of all the paths ahead (or what just the titles mean) and would be tremendous.

From Pete Chapman - I personally found the Summer School the high point of my M100 year and regret that M231 does not have one. Thus I enclose a deposit for your Weekend at Aston University.

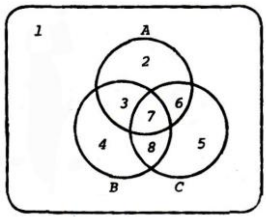
From Graham Read - I would very much like to hear the views of students who have completed M231 on the question of examinations in Analysis, certainly the Course Team spend many hours discussing this particular point. Is it the case that the M231 examination is a higher hurdle than other exams? The pass rate would seem to indicate that it isn't.

From Sidney Silverstone - Looking at the number of courses completed by MOUTHS subscribers it strikes me that a fair number of them will be obtaining Hons this year or next and wondering what they should do afterwards. Would it be possible to persuade one of the staff MOUTHS or perhaps somebody from Walton Hall to write an article for M500 (or the Special Issue) about higher degrees. Full time university students are in contact with post-graduate students and so can get a feel for what is involved. We have no such opportunity.

Cahn's Rule: When all else fails, read the instructions.

SOME LAWS OF THOUGHT *Roger Claxton*

Starting from a Venn Diagram (taking a three-variable case *A,B,C* as an example) we can number the separate sectors as shown in the diagram.



As one might expect each of these vectors represents a different boolean relationship between *A*, *B* and *C*. Since each of *A*, *B* and *C* can have two states (true or false, off or on or whatever) there are 2 × 2 × 2 = 8 different relationships. These correspond to the eight sectors as shown in the table below.

A	*B*	*C*	*sector*	*relationship*
0	0	0	1	$\overline{A} \cap \overline{B} \cap \overline{C}$
0	0	1	5	$\overline{A} \cap \overline{B} \cap C$
0	1	0	4	$\overline{A} \cap B \cap \overline{C}$
0	1	1	8	$\overline{A} \cap B \cap C$
1	0	0	2	$A \cap \overline{B} \cap \overline{C}$
1	0	1	6	$A \cap \overline{B} \cap C$
1	1	0	3	$A \cap B \cap \overline{C}$
1	1	1	7	$A \cap B \cap C$

If we are given the Venn diagram we can construct its truth table simply by setting the appropriate row to 0 if the sector is unshaded or 1 if the sector is shaded. Thus if sectors 2, 5 and 6 only are shaded the truth table is on the right (where *X* is a function of *A,B,C*).

A	*B*	*C*	*X*
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

This enables a comparison to be made between a Venn diagram and a boolean expression. The expression is evaluated by truth tables and like rows compared. If the resultant values correspond to the appropriate entry under the *X* column then the diagram and the expression are equivalent.

Once the truth table has been constructed the corresponding boolean expression can easily be extracted. In fact the rows of the table where *X* = 1 give that expression. In our previous example the expression is $(A \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap \overline{C}) \cup (A \cap \overline{B} \cap C) = X$ and this expression; the truth table and the Venn diagram with sectors 2, 5 and 6 shaded all convey equivalent information.

Note that for any given expression, if it is evaluated through a truth table and the rows extracted where x = 1 then a usually different but equivalent formulation of the expression will result. For example see page 10. An equivalent formulation is thus $A \cap B \cap \overline{C}$ = X, slightly more efficient especially if it is to be set up as a switching circuit.

A	B	C	A	$\cap$	$(\bar{B}$	$\cup$	$(\bar{C}$	$\cup$	$(B$	$\cap$	$\bar{A}))$	$)\cap$	B
0	0	0	0	0	1	1	1	1	0	0	1	0	0
0	0	1	0	0	1	1	0	0	0	0	1	0	0
0	1	0	0	0	0	0	1	1	1	1	1	0	1
0	1	1	0	0	0	0	0	1	1	1	1	0	1
1	0	0	1	1	1	1	1	1	0	0	0	0	0
1	0	1	1	1	1	1	0	0	0	0	0	0	0
1	1	0	1	1	0	0	1	1	1	0	0	1	1
1	1	1	1	0	0	0	0	0	1	0	0	0	1
												$=X$	

This process is linked to more general minimization techniques and the form that results in this case is known as the standard sum form. 'Sum' is confusing and is used because of the similarity between addition and 'or'. There is also a Standard Product Form and there are techniques for minimization that do not need the truth table, but these must wait for a future article.

ACKNOWLEDGEMENTS

1. John Rowland for the stimulus.
2. *Introduction to Boolean Algebra and Logic Design* by Gerhard E Hoernes and Melvin F Heilweil (Staff engineers, IBM). (No publisher or date; the above article is based on notes I took from this book some years ago. The book gives an excellent exposition using programmed instruction techniques.)
3. Ml 00 Unit 11.

HOLIDAY REMINISCENCE *Miek Warden*

Subscribing to M500 as a new student remarkably resembles taking AM289, *History of Mathematics* (especially the calculus option) after Ml00 as a first foundation course.

The encouraging cries of 'Stick to M- courses and you will understand it all in the end', does very much sound like Graham Flegg saying: 'But there are four units on differentiation and integration in M100! Besides, the mathematics in the course is only there as an illustration.'

Both advocates are right in saying: 'You should find enough in M500/AM289 of interest and enjoyment.'

I can honestly say that I enjoyed the course, although I would have preferred to have given it more time than I had, with S100 to cope with as well. In fact the experience was partly a reason to change my 1977 option from S- to M- courses.

As for the success of the course, Graham Flegg seemed well satisfied when I met him at the Hatfield conference of the 'History of Applied Mathematics' (yes, I became that keen) and the following information was imparted to me:

Approximately 600 students took AM289, 100 dropped out at the beginning, another 100 did not make it, one way or another, and the positive results that did materialize caused him (GP) to be jubilant (he did not have a little list on him unfortunately). The course suffered less snags than any other (or did he say most other?) OU course(s) to date and he struck me as a happy man.

Dare I look forward to M201 with confidence? Is Ml00 the most severe of mathematical shocks? Any contact with M201 colleagues will be most appreciated.

PRISONERS AND MOUTHS

Nick Fraser

Are we an open or a closed society? Should we not reflect the principles of the body that brought us together, that of an Open University? The OU accepted them at their face value, that they wished to be given the chance to learn. I am perfectly happy for my name to be put forward but am against the Special List idea. I hope that the motto of the Society will not be 'If thine eye offends thee pluck it out.' I therefore ask all members that they agree to have their names put on this list.

FORTHCOMING EVENTS

The following were brought to our attention.

From Region 13 Newsletter:
"An open lecture entitled 'The 4-colour problem solved at last' will be given at Christ Church College, Canterbury by Fred Holroyd and Robin Wilson on or about May 4th. Please look out for final details in Sesame and the April mailing".

From LOUSA:
"Feb 25th - Informal evening with Les Holloway. Mount Pleasant Hotel, 53 Calthorpe St WC1. 8pm. 50p.

"May 20th - 'How to grow trees': talk by Robin Wilson. Mount Pleasant Hotel, 53 Calthorpe St WC1. 7.30pm. 50p."

Les Holloway is of course the editor of *Sesame* and Robin Wilson is the OU's complex analyst who wrote a book on graphs. Is nothing happening in the outskirts of Britain that would be of interest to mathematical readers?

Men began to multiply on the face of the earth. Genesis VI 1.

ARRAY

In M500 21 4 John Earl raised the problem of placing points in an $n \times n$ array so that no three are collinear. It was known that the maximum number cannot exceed 2*n*, and for some small numbers *n* the figure 2*n* was reached. But it was felt that for large *n* the obtainable maximum would fall and, from probabilistic arguments, to $\leq((2\pi^2/3)^{1/3})n$ (but $\geq n$). Dorothy Craggs immediately set to to fill in some of the gaps for small *n*, then in October 1976 Martin Gardner of *Scientific American* took up the problem. Thus it was reported in the editorial of M500 38 that for $n \leq 13$ and for $n = 15$, 2*n* points are possible.

On January 26 this was received, from Dorothy Craggs:

Here are some (of many) solutions. I think *n* =16 is about the limit for solution by hand, unless someone can produce a bit of useful theory.

In coordinate form:

n=14 (0,5), (0,9), (1,3), (1,5), (2,9), (2,10), (3,2), (3,12), (1,0), (4,2), (5,12), (5,13), (6,6), (6,7), (7,6), (7,7), (8,0), (8,1), (9,11), (9,13), (10,1), (10,11), (11,3), (11,4) (12,8), (12,10), (13,4), (13,8).

n = 16 (0,4), (0,5), (1,2), (1,8), (2,3), (2,14), (3,10), (3,13), (4,7), (4,15), (5,3), (5,15), (6,6), (6,9), (7,1), (7,11), (8,4), (8,14), (9,6), (9,9), (10,0), (10,12), (11,0), (11,8), (12,2), (12,5), (13,1), (13,12), (14,7), (14,13), (15,10), (15,11).

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"Hair by hair
You may pluck a tiger bald."

Ted Hughes, *Moon-bells*

CONFERENCE

The Open University, believing that the word *mathematical* should not be restricted to its purely numerical connotations, is planning to run a course in Mathematical Education, of relevance to the in-service training of teachers.

They would like people with an interest - teachers in the middle school area, inspectors, lecturers - to attend a one day working conference at Walton Hall.

There will be two speakers: H O Pollak, Director, Mathematics and Statistics Research Centre, Bell Telephone Laboratories (formerly President of the Mathematical Association of America); and T J Fletcher, Staff Inspector, Mathematics. And a morning and afternoon session.

In the morning people will be in groups sharing common subject interests; probably

ART, MUSIC, DRAMA, CRAFTS, HOME ECONOMICS, ENGLISH, LANGUAGES, GEOGRAPHY, HISTORY, SCIENCES.

The later session will have grouping according to members' organisational or administrative commitments.

There will be a charge of £2.50 for lunch and expenses. Anyone wishing to go get in touch with Mrs Joan Sweet Faculty of Mathematics The Open University Walton Hall Milton Keyenes stating their area of interest from the list above. And be fast: it's quite soon.

UNIT BINDERS

Bob Escolme

At the risk of confusing means and ends many OU people may agree that the course units are attractive productions, well worth looking after even if not for subsequent reference. I keep mine in binders bought from Easibind Ltd (4 Uxbridge Street London W8; 01 727 0686) who supply gold lettering so that one can title them. As well as preserving the units in easy referable form the growing row of binders on one's bookshelf have the added bonus of impressing, oneself if no one else.

At £2.50 each the binders are not cheap. You need 3½ for the M100 units, but since the binders can only be bought in integral numbers the otherwise unwanted half can be used to accomodate such things as the BA Degree Handbook, supplementary material and assignment booklets. Two binders will house the M202 units and supplementary material. I guess but do not know yet that one binder would accomodate the material for a half unit.

Perhaps THE M500 SOCIETY or the Open University itself might take up the matter. If sufficient binders were assured of being sold I imagine Easibind Ltd could offer a reduced price. To allay the suspicions of the uncharitable I should add that I am in no way connected with Easibind Ltd. However, if the idea turned out to be fruitful I might suggest to them that they bought me a pint or two of bitter.

(When ordering binders you should specify that they are to fit A4 size magazines, with a three and a quarter inch binding and state the title you want put on.)

HYPERNOTE *Lewis Johnson*

If we draw circumscribing circles round the four black squares surrounding a white square on the chess-board, then the white square is not fully enclosed by the circles.

Likewise if we circumscribe spheres about the (six) black cubes surrounding a white cube on a three dimensional chess-board we do not wholly enclose the white cube.

However, if we circumscribe a hypersphere about the (eight) black hypercubes surrounding a white hypercube on a four dimensional 'chess-board' then we more than (so to speak) enclose the white hypercube.

PROOF Let the side of the four dimensional hypercube be 2 units. Then its semi-diagonal is $(1^2+1^2+1^2+1^2)^{1/2} = 2$) and this is the radius of the circumscribing hypersphere. This hypersphere therefore reaches the centre of the adjacent hypercubes.

MAGIC SQUARES *Mervyn Savage*

Referring to problem 34.1 (knight's move) this may be of interest: formation of odd number magic squares. There is a connection.

The next number goes one square above and one square to the right of the present one; where number 1 in a row or column follows number *n* in the same row or column. If this is not possible go down one square and continue as before. Begin with 1 in the middle of the top row.

This technique always produces a magic square for every odd number *n*. And every pair of numbers *a* and *a*+*n* are a knight's move apart. eg

816
357
492

SOLUTIONS

<u>34.4(11) *Find the next terms*</u>*:* 1221121221221121122121121221121121221221 ... .

The sequence consists of 1 one, 2 twos, 2 ones, 1 two, 1 one, 2 twos and so on, the arabic numerals in this sentence give the sequence itself, which is thus self-generating. - Anne Williams.

Ed - So the next terms are 21221211211221221121...; *what I want to know is, when does it start to repeat itself?*

<u>35.1 *Shuffle*</u>: *Prove that* $2^{m+1}x_m \equiv 2^m \pm (2x_0\text{-}1)$ mod 4p+1.

This can be proved easily by induction: (a) trivially true for m=0, (b) assume $2^{m+1}x_m = 2^m \pm (2x_0-1)+(4p+l)k$, k an integer, then for x_m odd, $2x_{m+1} = 2p + x_m + l$ by the definition of Monge's Shuffle. So $2^{m+2}x_{m+1} = 2^{m+1}x_m + 2^{m+1} + 2^{m+2}p = 2^{m+1} + 2^m \pm (2x_0-1) + (4p+l)k + 2^{m+2}p = 2^{m+1} \pm (2x_0-1) + (4p+1)(k+2^m)$, which is of the same form for m+1. Similarly for x_m even. - Max Bramer.

(There was another and longer proof from Steve Murphy, with pictures; it has gone into the pot with other Monge esoterica and might distil into something one day. Among the other results is a proof, from John Reade, that with a pack of 2n *cards the minimum number of Monge shuffles needed to return every card to its original positionis N, the first N such that* $2^N \equiv \pm 1$ (mod $4n$+1).*)*

<u>36.5 *The black ace*</u>: *A pack of cards is shuffled and dealt until the first black ace appears. Where is it most likely to be*?

If P_n is the probability the first black ace appears in position n, then $P_n = 2(52-n)/52\cdot51$, n = 1, ..., 51. The most likely *individual* position is first (as Jeremy Humphries points out), but the most likely average position over a large number of trials is the *expectation* $\sum_{n=1}^{51} nP_n = 17\tfrac{2}{3}$ - Max Bramer.

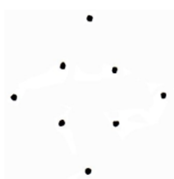
(*This second result is the same as one received last September from Chris Pile and rejected because* a) *it was different from Jeremy's and* b) $3 \times 17\tfrac{2}{3} \neq 52$.)

<u>35.5 *Rational termination* II</u>: *Find a number system in which all rational fractions have finite decimal expansion.*

The solution to this was actually given in M500 36 2. Any positive rational <1 can be expanded uniquely as a finite sum of the form $\sum_{i=2}^{m} a_i/i!$ where $a_i \in \{0, 1, ..., i-1\}$. (eg $\tfrac{2}{3}$ = 1/3! + 1/2!) Many other variable bases would probably also work, eg reciprocals of Fibonacci numbers, but would not necessarily give a unique representation.
Max Bramer.

<u>3.7.1(a) *Pairs*</u>: *Place eight points in the plane so that the perpendicular bisector of the line joining any two points passes through two other points.*

Place four points in a square. Construct an equilateral triangle on each side. The apices of these triangles are the other four points. Brian Woodgate.



<u>37.3 Powers</u>: *Find all positive integers m,n* ($m \neq n$) *with* $m^n = n^m$.

By looking at the graph of $(1/x)\log x$, it is clear that the only integral solution with $m>n$ is $m = 4$, $n=2$. David Asche.

Analytic proofs arrived ftom Tony Crilly, Krysia Broda and Brian Woodgate, who also produced the graph. Basically we have $m^n = n^m \Rightarrow n = m\log_m n = mk$; also $n = m^{n/m} = m^k$ hence $m^k = mk$. $\therefore$ $k = 1 + \log_m k$, $m = k^{(1/k-1)}$ whose only integral solution for k is 2.

<u>37.4 Planes</u>: *Given* 4 *distinct parallel planes show that it is always possible to construct at least one regular tethahedron with a vertex on each plane and show that the length of the side of any such tetrahedron is unique*.

Let the required tetrahedron have coördinates (0,0,0), k(l,l,0), k(1,0,1) and k(0,1,1) and let the four planes have unit normal (l,m,n) and be distant 0, a, b and c units from the origin. Then the condition for the vertices to lie on the planes is $k(l+m) = a$, $k(l+n) = b$ and $k(m+n) = c$. We see that we can solve to get $l = 1/2k(a+b-c)$, $m = 1/2k(a-b+c)$ and $n = 1/2k(-a+b+c)$, where k is determined by $l^2+m^2+n^2 = 1$. The length of side is $k\sqrt{2} = ({}^3/_2(a^2+b^2+c^2) - (ab+bc+ca))^{1/2}$. David Asche.

<u>37.5 Pshaw</u>! *How many commutative groups are there*? Answer, 10^{12} since only Abelian groups are commutative. Of course the American answer is 10^9. JH.

<u>38.3 Natural expansion</u>: *Why are the nth and* $(n+1)$*th terms of the Taylor expansion for* e^n *equal*?

This was no contest really! Solutions from Michael Masters, Mervyn Savage, Sue Davies, Brian Woodgate, R J Adams, Dave Diprose (who noticed that I had 25/2 = 13.5), Max Bramer, Ron Aitken and Marion, but she decided not to send it.

<u>38.2 Berwick's seven sevens</u>: 7375428413 × l25473 = 58781. Sue Davies; Ron Aitken.

<u>38.1 Therefore fireengines are red</u>: The Norwegian drinks Scotch, the Irishman shoots teal. Nick Fraser sent a matrix and commented "it took me about an hour and a half on Boxing Day. Ouite a feat to overcame the effect of Xmas cheer;" Mervyn Savage who just stated it; Michael Masters: another matrix - he asks "how do you get hold of *Creative Computing*?" (According to M500 36 15 you send £5 to *Creative Computing*, 60 Porchester Road, Southampton, Hants, S02 7JD); Sue Davies says there are only two possible orders for the colours of the cabin; one gives a logical impossibility and the other gives the solution; Nigel Graves sent five 5 × 5 truth tables showing the fate of various hypotheses. L S Dunmore sent an attractive diagram which might have made a nice cover if he hadn't written all over it. R J Adams: another matrix; Anne Williams: "this took about forty minutes but I was probably lucky;" Willem van der Eyken: "please don't ask me how I arrived at these canbinations; it seemed a highly unmathematical process." Russell Brass sent a picture of five coloured huts and some cards and said "it's quite easy if you use only common sense... . Time about an hour including paper clipping." Dave Diprose: "a fairly relaxed fifty five minutes and there will be faster times presented." Max Bramer: "Alas, even this will not persuade me to buy *Creative Computing*." And, just before we close, Ron Aitken. (*You will be glad to know that there are more of the same on file*!) And a late solution from Michael McAdee who also did 38.3.

PROBLEMS

39.1 NEW YEAR RESOLUTIONS 1977: Tony Forbes

I managed to keep only five of my twelve new year resolutions in 1976 (M500 28) and so I am going to try again with another twelve resolutions for this year. Which ones will I keep?

1. To keep no more than four of the following resolutions.
2. Not to keep resolution (4).
3. To keep exactly one of the previous resolutions.
4. Same as (13).
5. To keep exactly two of the previous resolutions.
6. Same as (5)
7. To keep exactly two of the following resolutions.
8. Same as (7).
9. To keep exactly one of the following resolutions.
10. Same as (9).
11. Not to keep resolution (9).
12. To keep no more than four of the previous resolutions.

39.2 WHAT'S INTERESTING ABOUT 1977? Alan Slomson

Most people know the story of Ramanujan and the Taxi-cab. Once when the Indian mathematician was in hospital he was visited by C H Hardy who remarked that the number of his taxi-cab, which was 1729, was very uninteresting. But Ramanujan replied "No Hardy; It is a very interesting number. It is the smallest number which can be expressed as the sum of two cubes in two different ways." ($1729 = 1^3 + 12^3 = 9^3 + 10^3$)

Suppose that the number of the taxi-cab had been 1977. What might Ramanujan's reply have been in that case?

39.3 THE BLACK ACE II: Max Braner

i. The pack of cards is shuffled until the second black ace appears. Where is it most likely to be? (See solution 36.5.)

ii. What is the expectation for the rth of m designated cards in a pack of n cards?

39.4 TRUTH: C S Evans

A cynic suggested to me that VERSATILITY is 1001 times better than VERACITY. On this reckoning what is the value of RELATIVITY?

(Based on a problem in *The Sphinx Problem Book*, Herbert Phillips, 1934.)

39.5 ETHIOPIAN MULTIPLICATION: D Inman

The Ethiopians, it is said, were unable to multiply by any number greater than 2. If they had two numbers to multiply together they set them down side by side and successively divided by two on the right while at the same time doubling up on the left. If the right-number is a power of 2 this process ends with a 1 on the right and the product on the left. Otherwise, fractions appear, which mess up the calculation. So they ignored fractions, and to counteract the error they struck out all lines which had an even right-number and added up what remained in the left column, for this sum gave the required product. Why?

EDITORIAL

Sorry we didn't run to a January edition but the fact is I had such a good time over Christmas that there was no opportunity for work of any kind - I shall suffer for that no doubt during 1977.

While at one of the numerous parties that made up my lifestyle within this enviable period it chanced that a copy of the December M500 was lying around. A guest picked it up and leafed through it, eventually pausing at my own *Department of Victorian Parlour Tricks*. So she tried it and failed. I pointed out what she was doing wrong but it was no use; she swore she couldn't possibly attempt it with glasses on and without them her eyes just slithered around. Then I tried it on others present with the same success rate. So my problem is this: am I alone in the worid in being able to do it or was the experience merely a combination of whisky and low IQ? (They are mostly teachers.)

Everyone knows of course that if you draw three circles on a piece of paper and draw tangents from one to another in every possible way extending the tangent lines until they cross if they are going to you end up with three collinear points, I am told this can be proved but I wouldn't even like to see the proof. However when Professor Sweet at the department of engineering at some University (I'll be glad of some hard information here as I seem to have lost ny reference) was shown the effect in 1916 he said It was obvious. How? Think three-dimensionally.

I suppose you are aware that according to the Bible there are an uncountably infinite number of planets in the Universe? Have a look at Revelation VII 9: ... and, lo, a great multitude, which no man could number, ... stood before the throne, ... Since the number of people born on earth up to any time, no matter how far in the future you go, is finite and therefore necessarily countable, and since the same would presumably apply to any other planet which supports, or will support life, the only possible explanation is that the actual number of planets is uncountable.

In the editorial for 37 I passed on the information that Kenneth Appel and Wolfgang Haken had found the only possible kind of proof for the four-colour-map-problem, one using computers (and noise no doubt). Now Joyce Moore has sent me a cutting from the *Sunday Times* of December 19 1976 about a "maverick Cambridge mathematician, called G Spencer-Brown" who was due to read a paper on the 20th on what he claims is a mathematical proof, using his own Brownian arithmetic, developed in his *Laws of Form* of 1969. Since then it seems he has produced another proof and I understand the subject is going into the primary school curriculum after Easter. Is there no reader of M500 who would like to write a short article on this intriguing subject. Maybe after Robin and Fred's talk (see page 11).

Another thing sent to me recently (by Marion as it happens) is *HP65 Notes* volume 3 number 10, December 1976 which is a monthly publication of the HP65 Users Club (edited by Richard Nelson, S18 including overseas airmail cost). It seems by its designation to have been running about as long as M500 has, and there is a note to say that they have just bought an IBM typewriter - but I see from the 'disbursment from funds' column that it is a mark II correcting machine (cost $898.88). I still use Tlppex, I hope Mr Nelson and his friends don't mind if I copy some of his lists one day; they have an extraordinary amount of them. All we have is π to something like 79 places and e somewhere.

Going back to our interminable series on calculators and other mind-softeners it is interesting to note that the process can go far further than we would dream of taking it. *65 Notes* 3/10 page 22 has a comparison between HP65 and SR56 recently completed by a subscriber, Hal Kinne, "in his doctoral studies at the University of Texas at Dallas Management School."

Eddie Kent.