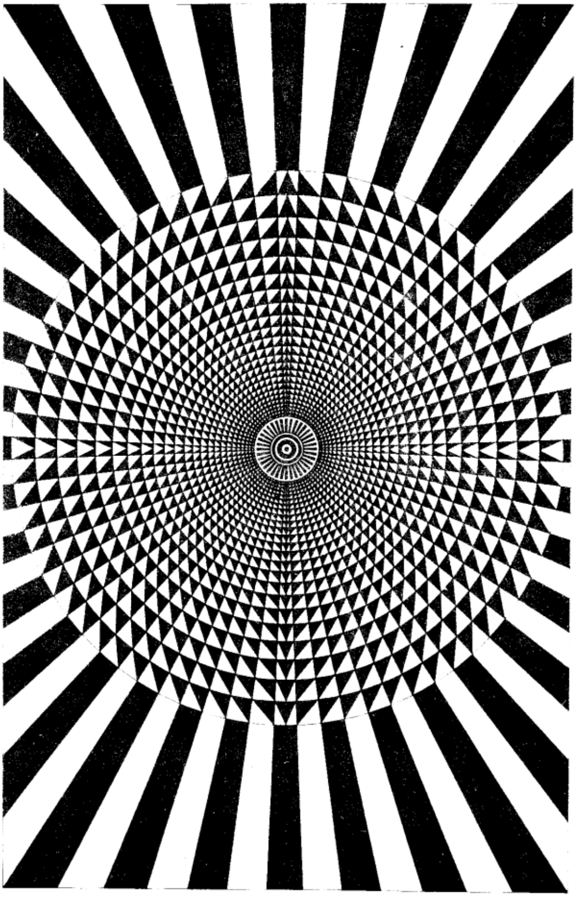

M500 34

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Articles and solutions are not necessarily correct but invite criticism and comment. Articles submitted for publication should normally be less than 600 words in length although subjects which require more space may be split by the authors into instalments.

MOUTHS is a list of names, addresses and telephone numbers together with previous and present courses of voluntary members, by means of which private contacts may be made by any who wish to share OU and general mathematical interests or who wish to form telephone or correspondence self-help groups.

The views and mathematical abilities expressed in M500 are those of the authors concerned and do not necessarily represent those of either the editor or the OU

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MATHEMATICS IS AN ART

Datta Gumaste

In M500 20 3 Sinbad asks "what is mathematics?" I am going to argue that mathematics is an art and will attempt a mathematical demonstration.

Definition I: we shall say that a pair (S,R) is a pattern whenever S is a nonempty set and R is a binary relation defined on it.

Example (i): Let S = {1,2,3}, R = $\le$. Then (S,R) is a pattern.

Example (ii) : Let S = P(T) = {t : t $\subset$ T; T $\neq \emptyset$} R = $\subset$.

Following the usual mathematical practice we shall call S a pattern when we mean (S,R) is a pattern.

Definition II: a pattern S has a form if $\forall$a,b$\in$S, aRb or bRa, a$\neq$b. Thus the pattern in (i) has a form.

Example (iii): S = {{i}, {1,2}, {1,2,3}}. R = $\subset$. Then S has a form

Definition III: A pattern S has movement if a,b,c – S aRb and bRc implies aRc (the familiar transitivity property). All the examples so far have movement.

Definition IV: Let S be some pattern. If $\exists$x$\in$S such that $\forall$a$\in$S, xRa, a$\neq$x, we shall say that S has a climax. We shall refer to the point x as climax point. In words x is a climax point of S if for any point a other than x, xRa. In (i) integer 1 is a climax point; in (ii) it is $\emptyset$.

Definition V: Let S be some pattern; if S has form, movement and a climax point we shall say that S is beautiful. Examples (i) and (iii) are beautiful.

Now let us have a look at some familiar mathematical objects which according to our definitions turn out to be beautiful patterns:

(1): S = Z , R = .

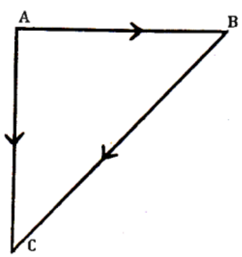
(2): The closed real interval [a,b] with the binary relation $\le$ or $\ge$.

(3): Let G be some group; S = {N_i: is normal in G and $N_i \supset N_{i-1}$ for each i}. R = $\supset$.

(4): Let {S_n} be some increasing or decreasing convergent sequence such that lim{S_n} = L. {S_n} $\cup$ {L} is beautiful.

(5) We can now restate the boundedness criterion for infinite series (See Spivak page 391): A nonnegative sequence {a_n} is summable iff the set of partial sums {S_n} $\cup$ {L} is beautiful. L = $\sum_{n-1}^{\infty} a_n$.

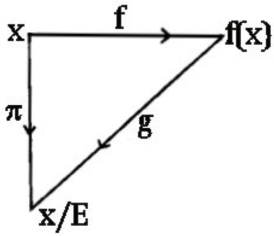
(6): consider the following commutative diagram



Here S = {A,B,C}
R is defined as follows:
xRy if there is an arrow from x to y, x,y $\in$ S.
It is easy to see that S is beautiful.

➔

(7). Perhaps the simplest beautiful diagram was introduced in M100 Unit 19. Here x is some nonempty set. f is a map on x to f(x). π is a natural map on x to x/E, the quotient set, E is the equivalence relation induced by f; and g is a map on f(x) to x/E.



(8): consider a proof of a mathematical theorem. It is simply a set of propositions which are true or false. Call such a set T = {t: t is a proposition}. Define on T the binary relation $\Leftarrow$ (is implied by), Then (T, $\Leftarrow$) is a beautiful pattern. The point of climax of T is the conclusion of the theorem. This shows that any mathematical proof is beautiful.

There are many examples of beautiful patterns in maths courses, particularly in M100. (So cheer up M100 students!) Let (S,R) be a beautiful pattern. Let (S',R') be some pattern. Suppose f:S $\rightarrow$ S' such that $\forall$a,b$\in$S, aRb $\Rightarrow$ f(a)R'f(b). f is a kind of morphism which preserves beauty.

Definition VI: if f is afunction on (S,R) to (S',R') which satisfies the above condition then f is beautiful. There are anumber of beautiful functions around. Here are some.

(9) Let s: n $\rightarrow$ n+1 (n$\in$N); then s is beautiful. s is the root of construction of natural numbers.

(10) f aone-one continuous function on S to S' where S is a closed interval and S' a set of reals.

(11) it follows from (10) that log and exp defined on suitable domains are beautiful.

(12) various circular functions are beautiful when defined on suitable domains. This partly explains why the graphs of trig functions look so sexy. Is this why an analyst insists on analytical proofs and does not depend on graphs of functions? In fact it is remarkable that an analyst can prove anything at all when he is surrounded by so many passionate curves.

Consider the triple (C,R,U): C is a class of beautiful patterns; R is a binary relation on C; U is a set of objects such that for each y$\in$C, $\exists$a$\in$U with aRy.

Definition VII: we shall say that an object (C,R,U) is an art. In other words an object of art is a collection of beautiful patterns on which some binary relation is defined and there exists a set U such that if y is any beautiful pattern there is some a$\in$U such that aRy.

If B is some branch of mathematics it is essentially a collection of theorems and their proofs where each theorem is implied by a statement assumed or proved to be true. Let C = {t: t is a proof of a theorem}. U = {p: p is an axiom or definition}. R is implication ($\Rightarrow$). Then (C,R,U) satisfies the definition of art. But this shows that B which is any branch of mathematics is an art, and hence Mathematics Is An Art.

But when you are in love you do not need a proof, You simply love, and in loving you are love.

MAX AND MONGE

Rosemary Bailey

Dare I suggest that if articles were vetted for mathematical content before publication, the first two articles to be axed would be two of his own most recent contributions: that on Constructions in M500 31 and that on the Monge Shuffle in M500 33? Richard Ahrens has already commented on the first of these (in M500 32) so I shall confine myself to the second.

Why does Max always leap into a problem by putting it on a computer? (By which I mean a computing machine of course; I am a computer because I compute (ie I do sums) but that doesn't, I hope, make me a machine. This aside is just a personal plea for better use of language.) Computing machines are splendid for a) doing tedious numerical work and b) making exhaustive checks of possibilities when no more elegant method is available; but surely as mathematicians we should try to apply some mathematical reasoning <u>before</u> using the machine.

Max makes a great to-do about his statements i)(each card eventually returns to its starting point) and ii) (the longest cycle length for any card is N, $1 \leq H \leq 2p$) not being obviously true, and then uses the machine to verify that they are indeed true for $p \leq 50$. In fact they are both true for all values of p, and this can never be verified by a machine. I submit that even if they are not obvious they are at least extremely simple to prove. The function f represents a shuffle, and so it is <u>by definition</u> a one-to-one onto function on the set of cards; that is, a permutation of the set of cards. Hence no sequence such as 9,14,6,5,6,5,... could ever occur: each number x in the cycle defines uniquely what number succeeds it (f(x)) and what number precedes it ($f^{-1}(x)$). Thus if we write $f^n(x)$ for $f(f^{n-1}(x))$ and $f^1(x) = f(x)$ the sequence

$$x, f(x), f^2(x), \ldots, fn(x), \ldots$$

has no repitition until it returns to x, which it must therefore do as the set of cards is finite. Thus

i) $\exists N$ such that $f^N(x) = x$,

ii) since $x, f(x), \ldots, f^{N-1}(x)$ are all different, $N \leq 2p$.

So much is elementary, as any student of M202 will recognise, and in no way depends on the special shuffle f. I admit that the remaining problems, ie

i) what is the order, m, of f? in particular is it true that $(4p + l)k \pm p = 2^{m-2}$, as David Asche conjectures?

ii) do all cycle-lengths of f divide the longest one?

iii) is it true that all cycle-lengths of f are the same if and only if 4p +1 is prime?

are neither obvious or elementary and consequently I'll take the easy way out and leave them to others to solve. But please, <u>please</u> don't use machines to verify what can be proved with a little thought.

COINCIDENTAL BIRTHDAYS

Steve Murphy

Suppose that we have a group of m people. The number of ways in which each of them may have a different birthday is 365!/(365−m)!.

If we assume that birthdays are evenly distributed throughout the year the probability that two or more people have the same birthday is

$$p_m = 1 - \frac{365!}{(365-m)!365^m} \qquad (1)$$

Now there are tables giving values of $\log_{10} n!$ (eg the Biometric tables) and using these it may be shown that $p_{24} = 0.583$, $P_{23} = 0.5173$, $P_{40} = 0.891$. So that we only need 23 people to have a slightly better than even chance of having two or more with the same birthday. Equally we can deduce that if there are more than 40 people we have a better than 90% chance. (p_{41} = 0.903.)

In fact the following is a good approximation to p_m :-

$$p_m = 1 - \exp(-n(n-l)/730) \qquad (2)$$

and Sinclair programmable buffs could use the program below to calculate p_m :

Program Entry B/E /rcl/x/ '/365/'/−/x ↔m/enter/'/1/'/−/ x ↔m /enter/Δ / C/E (17 entries).

Operation 365/Δ/sto/ C/E/1/enter; then press exec m times.

Of course the chance of having two and only two people with the same birthday is a little different:

$$P_m = \frac{m(m-1).365!}{2(366-m)!365^m} = \frac{m(m-1)}{2(366-m)}(1-p_m) \qquad (3)$$

and $P_{23} = 0.36$, $P_{24} = 0.34$ which demonstrates that there is no number m such that we have a better than even chance of finding two and only two people with the same birthday.

One possible approach to the generalised problem posed by Eddie Kent (if there are m people what is the probability that n of them share a birthday?) is to think of it in terms of a slightly different model.

Suppose that there are 365 pigeon holes and that m postcards are to be distributed randomly into these pigeon holes. Suppose that there are k. cards in the ith hole. The numfber of ways in which this particular arrangement can he acheived is

$$\frac{m!}{k1!k2!k3!...km!},\ k_i \in \{0, 1, 2, ..., m\};\ \sum_{i=1}^{365} ki = m.$$

If we suppose that no pigeon hole contains more than n-1 cards, then the number of ways in which suitable arrangements can be made is the coefficient of x_m in the expansion of

$$y: x \rightarrow m!(1 + x/1! + x^2/2! + ... + x^{n-1}/(n-1)!)^{365}.$$

➔

If a_m is the required coefficient of x^m we may deduce that the probability of finding n or more people with the same birthday in a group of m people is

$$p_{mn} = 1 - \frac{a_m}{(365)^m} . \qquad (4)$$

With n = 2 this method gives the result summarised in (4). I can't use it to give a neat closed formula for p_{mn} though for particular cases it seems to provide a method of calculation.

For example with n = 3 one would find the coefficient of x^m in the expansion of

$$g: x \rightarrow m!(1 + x + \tfrac{1}{2}x^2)^{365}.$$

writing for the jth derived function of g we have, after a bit of fiddling:

$$(1 + x + \tfrac{1}{2}x^2)\, g^{(j)} = (366 - j)(1 + x)(g^{(j-1)} + \tfrac{1}{2}(732 - j)(j - 1)g^{(j-2)}.$$

Put $g^{(j)}(0) = j!b_j$ and

$$jb_j = (366 - j)b_{j-1} + \tfrac{1}{2}(732 - j)b_{j-2} \qquad (5)$$

Now $b_0 = m!$ and $b_1 = 365m!$ so that (5) allows the calculation of any particular value of b_j, and in particular b_m. Then $a_m = b_m/m!$. (Equally put $b_0 = 1$ and $b_1 = 365$ and use (5) directly.) This would therefore give a method for calculating p_{m3} for particular values of m, and might be adaptable for other values of n.

M231 ANALYSIS

Graham Read

In my experience Analysis is quite a difficult subject for beginners, and I can certainly recall that I didn't find it easy when I was a student. I was never exactly sure what was wanted in a proof, and all those ε's and ∂'s drove me mad. The surprising thing was that a year later I could look back on the course and wonder why I found it so hard.

I taught the subject for many years to full time students, and their reactions were nearly always the same as mine, so perhaps this is a general aspect of Analysis or maybe its the way I teach it. Certainly it would be interesting to hear from OU students who took M231 in its first year, and particularly those who went on to third level courses.

Part of the Analysis examination paper is devoted to 'book-work', hence the restriction on taking material in with you. It is true that you can learn the statement and proof of theorems parrot fashion, but the examiners are not only trying to test your memory. Rather they are trying to test your analytic ability and details and methods of proof of standard theorems do indeed provide a good test.

REFERENCES
Richard Ahrens

I know we are in the Common Market but giving Frenchmen with reactionary views space in our learned journal is ridiculous (M500 32 10). (I couldn't even find his name on my MOUTHS list.) Giving references to sources when writing mathematics has nothing to do with humiliation. It is simply a matter of courtesy. The author who gives references avoids taking credit for someone else's good ideas and saves his readers a lot of frustration if they want to look further into the problem. M500 32 provides a splendid example where a reference would have been very useful: on page one Daniel Dubin should have said which book contains those gems of mathematical thought. We could all then cross that book off any list of titles that we might be tempted to consult. Another area where references might usefully be employed is in solutions to problems. I agree with the editor that we do not want detailed solutions to problems, but to just print a number with no clue to the method of solution is no better. The fact that the probability of getting a triangle by cutting a wire into 3 pieces, is 1/4, is of no interest to anyone. The method of arriving at this answer might easily be interesting, and a reference in lieu of a solution would do.

STAR TREK ADDICTS
Marion Stubbs

Following a rather nice review of Dice Star Trek in Don Featherstone's "Wargamers' Newsletter" of January 1976, another newsletter contacted me and has recently sent the following info:

"LOU ZOCCHI GAMES: We are currently selling 3" white plastic, snap together models of the United Federation Star-Fleet Cruiser for 2 dollars. This is the first ship into production and will be followed by the remainder of the Star Fleet and its enemies... Since the ship will be needed for the game, we have made them first because they are also beautifully detailed models that they have value even if you don't play games. The Star Fleet game rules will be done later this summer." Details &c from L Zocchi, 1513 Newton Drive, Biloxi, MS, USA, 39532.

Obviously this is not DST, but DST addicts might like to know. DST still continues to sell steadily but slowly, from unexpected sources - like six copies to the New Tork Star Trek Society, and another dollar-earner in Ohio, who sent details of a Space Games specialist dealer in Texas --- and so it goes, sweeping the vast American continent I

Anyone like MG (who shall be nameless) who objects to DST ad nauseum in M500 please note that I barely need to advertise the thing since it sells of its own accord, but about half of M500 subscribers have it now and might like odd snippets of info.

COMPUTERS AND CHESS
Max Bramer

I was most interested to see the recent correspondence on omputers and chess in M500, as this is a subject of which I have made a close study in recent years.

The history of the "chess-playing machine" easily predates both the discipline of Artificial Intelligence (in which computer hess has always played a leading part) and electronic computers themselves.

An electro-magnetic machine for the simple endgame King and Rook against King (with a mechanical arm to move the pieces) was demonstrated at the Paris World Fair of 1900, nearly fifty ears before the first modern computer was built.

"Von Kempelen's Turk", a life-size figure dressed as a Turk nd seated at a large box which served as a playing table, was demonstrated to Empress Maria Theresa of Austria over two centuries ago and later was demonstrated to large audiences throughout Europe and America for almost seventy years.

The Turk was actually a hoax operated by a strong human chessplayer inside the box, but genuine computers have been entered in ordinary tournaments against humans and a world computer chess championship was held at the IFIP Conference in Stockholm in 1974.

Computer Chess is not so much a game as a touchstone for he whole field of Artificial Intelligence. If a computer can be programmed to performed at master level in such a complex domain as chess, the implications for our view of man as an “intelligent" animal will be vast. If it cannot, the validity f Artificial Intelligence as a research area may be called nto question.

The strongest draughts-playing program is claimed to perform at just below world championship level but I cannot judge if his is true. (The main reference is Samuel, A L, Some Studies in Machine Learning Using the Game of Checkers. II - Recent rogress. IBM Journal of Research and Development volume 11 November 1967.)

Strong programs also exist for obscure games like Kalah, but Chess has remained elusive. The pinnacle of performance could be described as weak county-level play at best, but this has largely been achieved by storing thousands of standard opening moves from textbooks, to be played off by rote, combined with extremely deep analysis of possible variations. In the more difficult - but structurally simpler - part of the game, the endgame, the general level of play remains abysmally poor.

After a decade of little discernable progress, the emphasis has shifted away from whole games to endgames, particularly slementary ones such as King and Rook against King (KRK) and King and Pawn against King (KPK) which are usually dismissed in one or two pages in beginners textbooks.

These and other elementary endgames have been closely ;udied in Britain, America, Russia, Holland and doubtless elsewhere. Anyone who thinks that writing a program to play one ➔

of these is trivial is invited to try it! A few programs have been written but they are invariably highly complex and far from the slimple solution which it seems intuitively clear must exist for such an overtly simple problem.

My own interest is in devising a representation (model) of the overall framework which the reader of a textbook brings to his reading and in terms of which he interprets what he reads. Given the right framework, the algorithm itself should be easy to construct. Rather than describe my approach, may I invite I invite readers to play against programs I have put in the BASIC library for KPK and KRK.

$ROOK for KRK
$PAWN for KPK.

A large number of mathematics students played against earlier versions of the programs at last year's Summer Schools 3d and this helped me greatly to improve them. The latest versions print out informative messages to help use them and are (I hope) much improved. The 'move-finding' parts of the programs are of the order of 200 lines of BASIC each, which I hope may be the right size of solution to a simple problem. As a target, it might be useful to know that it has now been proved (by exhaustive counting!) that the stronger side can always win KRK in 16 moves or less and that when KPK can be won (which is not always possible) it never takes more than 19 moves.

I should be interested to hear of people's experiences, especially if they find any errors!

LETTERS

From David Wells - Allow me to introduce myself in my new guise: LONDON GAME CONSULTANTS. Myself and any of my colleagues who might be needed for a particular task.

Games invented, tested, assessed, written about — anything which the might suggest. All games dealt with, but especially educational games, because I used to be a teacher, and am still very interested in teaching; and abstract games because I play Chess and Go, and Guerilla, a game of mine which Philmar have just put on the market.

In case anyone wonders, I'm still puzzles editor of Games and Puzzles, but I'm not on the games testing panel any more. And as far as games are concerned I'm a free agent. So if you have any ideas, especially for anything educational, I shall be delighted to hear from you.

From Lynne Roberts - Many thanks for DST received safely. My husband asks me to point out that he is not a ludo playng nitwit, but that LUDOMANIAC is a generic term describing one who is addicted to board games!

(MS - Lynne ordered DST to (roughly speaking) keep her ludomaniac husband happy!)

THE PUBLIC SCHOOL SEQUENCE

David Asche

(In M500 33 Richard Ahrens defined a sequence $\{v_n\}$ where the number v_n is the remainder on division of u_n by n, $n \in \{l, 2, ...\}$.)

The elements of column 2 are the terms of the sequence given by $u_n = u_{n-2} + u_{n-1}$ $(n > 2)$, $u_1 = 1$, $u_2 = 3$-

The recurrence relation has solutions of the form $u_n = L^n$, where $L^2 - L - 1 = 0$. The solution which satisfies $u_1 = 1$, $u_2 = 3$ is

$$u_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{2n-1}\left(1+5\binom{n}{2} + 5^2\binom{n}{4} + \ \dots) \right).$$

Suppose n is prime. If n = 2 then $u_2 \equiv 1 \pmod 2$. If n is an odd prime then the terms $\binom{n}{2}, \binom{n}{4}, \dots, \binom{n}{n-1}$ are all divisible by n (Well-known result!)

Thus $2^{n-1}u_n \equiv 1 \pmod n$.
Also $2^{n-1} \equiv 1 \pmod n$(Another well-known result!!), so must have
$u_n \equiv 1 \pmod n$ (Clearly?) In other words the corresponding element in column 3 is $v_n = 1$.

2. The problem of finding out when v = 0 seems harder. The condition is <u>not</u> n = 6k where k is odd. One can check that $v_{30} \neq 0$. My conjecture is that $v_n = 0$ when $n = 2 \times 3^r$ (r = 1, 2, . . .).

3. If we consider a more general sequence in column 2, given by $u_n = ku_{n-2} + u_{n-1}$ $(n > 2)$, $u_1 = 1$, $u_2 = 1 + 2k + 2k$ we obtain the solution

$$u_n = \frac{1}{2n-1}\left(1 + \binom{1}{2} +(1 + 4k) + \binom{n}{4} (1 + 4k)^2 + \dots\right)$$

which again produces the corresponding $v_n = 1$ when n is prime, by the same reasoning. In the case of k =2 it seems that we probably get $v_n = 1$ also when n is a power of 2.

4. The investigation of such sequences (Fibonacci type, in the arithmetic of remainders modulo n - Ed) may have some point to it, but I am not really too convinced.

The Public School Sequence continued:
John Reade

Use of Chebyshev Polynomials

If we write $A = (1 + \sqrt{5})/2$, $B = (l - \sqrt{5})/2$ then we have $u_n = A^n + B^n$. Also $B = -1/A$ so if n is even = $u_n = A^n + 1/A^n = 2T_n(½(A + 1/A)) = 2T_n(\sqrt{5}/2)$ and if n is odd $u_n = A^n - 1/A^n (A - 1/A)U_{n-1}(½(A + 1/A)) = U_{n-1}(\sqrt{5}/2)$. T_n, U_n are the nth Chebyshev polynomials of the first and second kind and are defined by

$$T_n(\cos\theta) = \cos n\theta$$
$$U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta.$$

These polynomials satisfy various identities which yeild properties on $\{u_n\}$. For example

$$T_m(x)T_n(x) = ½(T_{m+n}(x) + T_{m-n}(x))$$

gives $u_m u_n = u_{m+n} + u_{m-n}$; m,n both even.

Note that the relation $u_n = u_{n-1} + u_{n-2}$ defines u_n uniquely for $n \leq 0$. In fact $u_0 = 2$ and $u_{-n} = (-1)^n$, $n \geq 1$ so the abve equation holds for all m,n (both even).

The corresponding equations for other parities of m,n are

$$u_m u_n = u_{m+n} - u_{m-n}; \text{ m even, n odd}$$
$$u_m u_n = u_{m+n} + u_{m-n}; \text{ m odd, n even}$$
$$u_m u_n = u_{m+n} - u_{m-n}; \text{ m,n both odd.}$$

Hypothesis (2) (Is $v_{2k} = 2^k - 1$ a theorem?)

From above we have $(u_n)^2 = u_{2n} + u_0 = U_{2n} + 2$, n even. Therefore $U_{2n} = (u_n)^2 - 2$. Assume as an induction hypothesis that $u_{2^n} \equiv -1 \bmod 2^n$. Then $u_{2^n} = k2^n - 1$ for some k. Therefore $u_{2^{n+1}} = (u_{2^n})^2 - 2 = (k2^n - 1)^2 - 2 = k^2 2^{2n} - k2^{n+1}1 \equiv -1 \bmod 2^{n+1}$. This result obviously holds for n = l so is true for all n.

Question (3) (For which n is $v_n = 0$?)

A partial answer: $u_n \equiv 0 \bmod n$ if $n = 2.3^k$.

For n even we have $u_n u_{2n} = u_n + u3_n$. Therefore $u_{3n} = u_{2n} - u_n = u_n(u_n^2 - 2) - u_n = u_n^3 - 3u_n$. Assume as an iduction hypothesis that $u_{2.3^n} \equiv 0 \bmod 2.3^n$ then $u_{2.3^n} = k.2.3^n$ for some k so $u_{2.3^{n+1}} = (u_{2.3^n})^3 - 3_{2.3^n} = 8k^3 3^{3n} - 2k3^{n+1} \equiv 0 \bmod 2.3^{n+1}$, etc.

SMOKING AT SUMMER SCHOOLS
Marion Stubbs

My world, during the past thirty years, has been progressively more poisoned, deafened and congested by motor cars. My mother, who retired to a peaceful village, now has a junior motorway running through her back garden. To parody John Wills (M500 33) no-one needs to demonstrate how grand they are by travelling around in fancy wheelchairs; travel by car was rare in years 1-12 of my schooldays. We walked, or cycled, or in true Socialist spirit, used Public Transport.

All except the physically handicapped are blessed with two strong legs, yet my friends and neighbours take their pet cars ;o the local shops and regard a one-mile walk as an athletic feat.

As a smoker, is it entirely obvious that I study less well nan those who do not? With no mathematical background I have so far) managed to meet every due date for every TMA and CMA ince 1971 and never (yet) used the cut-off date. This is on op of a full time job and running M500 and doing other things such that readers and other friends ask plaintively how I "manage to do it".

Sometimes I receive photographs of my friends' favourite or newest motor car, with or without the owner standing beside it. They do not know that cars are my bete noir, unless a car happens to be essential for some reason - I am not so biased as to object to legitimate and necessary use of private vehicles where significant distances are involved, or carriage of bulky goods, or where the Public Transport system does not exist. They would surely be astounded if I sent them a photograph of my favourite cigarette packets?

Now cigarettes may be killing me, but I am not convinced, since I have solid reasons for saying that people die from diseases said to be caused by smoking when they do NOT smoke; on the other, they do not kill <u>you</u>. However, your motor cars are liable to kill not only you but also me and even children who happen to get in your way. When you are driving you become aggressive beyond belief and positively hate other drivers; the air around is rather bluer than the air around my placid cigarettes.

But, unlike non-smokers, I am tolerant. You roll up to your Summer Schools, and even to Aston Weekend, in your cars - some of you travelling through my mother's ex-garden - with my consent; just let me have my smokes with equal tolerance? Point less to the splinter in my eye and behold more the plank in yours.

Moriarty: How are you at Mathematics? Seagoon: I speak it like a native.

Spike Milligan: "Marie Celeste".

It is now quite lawful for a Catholic woman to avoid pregnancy by a resort to mathematics, though she is still forbidden to resort to physics and chemistry.

H L Mencken: "Notebooks".

(both from Ian Turner.)

M331: INTEGRATION AND NORMED SPACES
David C Dowson

While I recognise that every student has the right to describe a course as he or she sees it, there is one case of such blatant misrepresentation of M331 in the 1976 Special Issue of M50O that I feel bound to give my view as a course tutor.

That M331 "turns out to be a set collection of methods for reducing problems in many-dimensional spaces to unitary spaces" will, I suspect, come as a great surprise to the course team. The course is certainly not a collection of methods; it is about the concept of integration. Moreover, only two of the fourteen units deal with integration in more than one dimension and the notion of unitary space does not appear at all. The correspondent's other remarks on M331 are equally unreliable.

The same correspondent, in his remarks on M202, makes two remarks which I think are pertinent to M331 and which need correcting. Firstly, "that all other courses are merely applications of the techniques and theorems which are now in your grasp" is so breathtakingly fatuous that no one can take it seriously. Secondly, your correspondent advises students not to use terms and definitions arising in M202 in other courses as "there will be no allowance for the elegant M202 approach". I hope that other students will recognise the silliness of this remark. As a course tutor, I will give full credit to any correct solution. Credit will not be given, however, to work showing little or no understanding of the points at issue and which is heavily camouflaged by an almost overwhelming, indiscriminate and inaccurate use of irrelevant concepts from another course. Fortunately for course tutors, the incidence of this disease is very small. (I personally know of only one such case.)

I notice that your correspondent is much given to giving advice to other students and my advice is - ignore his advice.

I personally think that M33I is an interesting, but not easy, course in pure mathematics. It will test the student’s capability in analysis to the full. However, the conscientious student can do well, witness the results from my own group of whom over half obtained grades I and II.

ADVICE TO ASPIRANTS - If you go in for a spelling bee, and are brought down at the first shot, mind not to be waspish.

Punch 25 March 1876 (MS)

The Hospital for Lady-Students to Enter At - The Middle-sex.

Punch 10 June 1876 (MS)

MORE MACHIN PI - Peter Hartley

To show that $\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$ is an identity. Let A = arctan(l/5), B = arctan(l/239) then

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2/5}{1-(1/25)} = \frac{1}{12}.$$

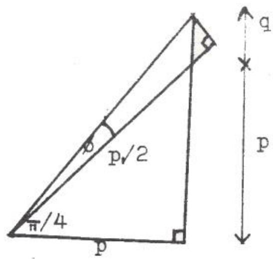
Similarly $\tan 4A = \frac{10/12}{1-(25/144)} = \frac{120}{119}$. Therefore

$$\tan(4A-B) = \frac{\tan 4A - \tan B}{1+\tan 4A \tan B} = \frac{(120/119)\ -(1/239)}{1+\frac{120}{119\times 239}} = 1.$$

So $4a - B = \frac{\pi}{4} + n\pi$ and n = 0 by inspection.

How did Machin find it. Perhaps some kind of right angle triangle construction. Notice tan2A = 5/12 so 2A is in a 5-12-13 triangle; also $239^2 - 1 = (4)\ (119)\ (120)$, 239=119 + 120, tan4A = 120/119 (!)

Consider trying to write $\frac{\pi}{4}$ in the form arctan x ± arctan y where x and y are small so that the series $\arctan x = x - x^3/3 + \ldots$ converges quickly. In fact lets try to make x = 1/n and y = 1/m where n and m are "large" natural numbers (ie 2,3,4,...). To this end try to associate $\pi/4$ with a small angle, as shown.



$$\tan\phi = \frac{q\sin 45^{o}}{p\sqrt{2} + q\cos 45^{o}} = \frac{q}{2p+q} = \frac{1}{2L+1},$$

where L = p/q and $\tan(\pi/4 + \phi) = \frac{p+q}{p} = \frac{1+L}{L}$.

So $\frac{\pi}{4} = \left(\frac{\pi}{4} + \phi\right) - \phi = \arctan\frac{1+L}{L} - \arctan 1/(2L+1)$.

Well L/(L + 1) isn't the reciprocal of a "large" natural number. But suppose we could rewrite it as an integer multiple of arctan1/n $n \in N$, n > 1. Try twice:-

$$\frac{L+1}{L} = \tan(2\arctan\frac{1}{n}) = \frac{2/n}{1-(1/n^2)}$$

$$n^2 - 2\left(\frac{L}{L+1}\right)n - 1 = 0. \quad n = \frac{L + \sqrt{(L^2+(L+1)^2}}{L+1}.$$

For this to be a natural number L and L+1 must be the first two numbers in a Pythagorian triad; eg L = 3 n = 2; so with 2L + 1 = 7

$$\frac{\pi}{4} = 2\arctan\frac{1}{2} - \arctan\frac{1}{7}.$$

(No other choice of L will work.)

Now try four times (you can try three if you like):-

$$\frac{L+1}{L} = \tan(4\arctan\frac{1}{n}) = \frac{2((2/n)/(1-(1/n^2)))}{1-((2/n)/(1-(1/n^2)))^2} = \frac{4n(n^2-1)}{(n^2-1)^2-4n^2} = f(n).$$

Solve for n in terms of L? No thank you - I'll just try some values for n: n=2, f(n) = –24/7; f(3) = 24/5; f(4) = 240/161; f(5) = 120/119 - Eureka! ie L = 119, 1/2L+1) = 1/239 and Machin's result follows. Any more values that work? No; f is decreasing: $f'(n) = -(n^2+1)((1/n^2) + (1/(n^2-1)^2))/f^2(n)$.

Try five times? Etc.

MRS HEAD'S KNITTING MACHINE (problem 31.2) Richard Ahrens

Krysia Broda has supplied a necessary and sufficient condition which must be satisfied if it is possible to knit all the wool without rewinding a spool (M500 32 15), viz: The largest spool must not contain more than one kth of all the wool (k is the number of threads used simultaneously). She has also given an algorithm which guarantees getting all the wool knitted in at most n−1 stops for spool changes (n is the number of spools, n≥k), viz: Use the k spools with the largest amount of wool on them and run the machine until one or more of the spools is empty or the (k+1)th spool contains one kth of the remaining wool. Repeat this process. This algorithm certainly works but I would like to point out that it is not always the most efficient.-

Take k=3, n = 8; spools contain quantities of wool equal to 13, 11, 9, 6, 5, 4, 3, 3. Krysia's algorithm requires five runs of the machine to knit all the wool but it can be done in four runs:

Krysia									better							
13	11	9	6	5	4	3	3		13	11	9	6	5	4	3	3
4	2	0	6	5	4	3	3		7	5	9	0	5	4	3	3
4	2	0	2	1	0	3	3		7	0	4	0	0	4	3	3
1	1	0	1	0	0	0	0		0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0									

Can anyone produce an algorithm which would have managed these spools in four runs of the machine?

Question (ii) said, with the numbers $t_1, ..., t_n$ (t_i is the amount of thread on the ith spool) such that it is possible to knit all the wool without rewinding a spool; show that it is possible to knit all the wool with no more than n−1 stops to change spools, but that in some cases n−1 stops will be necessary. My answer is: If the quantities of wool on the spools are n real numbers which are linearly independent over the rational numbers, then n runs of the machine are necessary, eg we could use $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, ..., \sqrt{p_n}$ where p is the nth prime number.

I will leave it to you to convince yourself of the truth of this claim (or else give me a ring).

EULER'S POLYGON DIVISION (problem 31.1) Richard Ahrens

Krysia's solution to this problem (In how many ways can a plane convex polygon be divided into triangles by diagonals?) is of course correct.

$E_n = \frac{2.6.10.....(4n-10)}{(n-1)!} = \frac{2.4.6.8.10.....(4n-8)}{(n-1)!\,.4.8.....(4n-8)} = \frac{1.2.3.....(2n-4)}{(n-1)!1\ 2\ 3...(n-2)}$ (after cancellimg a lot of 2s and 4s)

$= \frac{(2n-4)!}{(n-1)!(n-2)!}$, Krysia’s result with n − 2 replaced by n.

SUPERFIELDS (a solution to problem 32.1)

Krysia Broda

Consider $(R^{+}. \bullet, \oplus)$; $\bullet$ is multiplication and $a \oplus b = 2^{\log_2 a \log_2 b} = a^{\log_2 b}$. $(R^{+}, \bullet)$ is an Abelian group

$(R^{+} - \{1\}, \oplus)$ is an abelian group since

Closure: if $a,b \in (R^{+} - \{1\})$ we can write $a = 2^{p}$, $b = 2^{q}$ where $p,q \in R^{+}$; then $a \oplus b = 2^{pq} \in (R^{+} - \{1\})$ since $pq \neq \emptyset$.

Commutative: $b \oplus a = 2^{qp} = 2^{pq} = a \oplus b$.

Associative: $(a \oplus b) \oplus c = (2^{pq}) \oplus 2^{r}$ (for $c = 2^{r}$, $r \in R$) $= (2^{pq})^{r} = 2^{pqr} = a \oplus 2^{qr} = a \oplus (b \oplus c)$.

Identity: $a \oplus 2 = a^{\log_2 2} = a$ so 2 is the identity element.

Inverse: If $a \oplus b = a^{\log_2 b} = 2$ then $\log_2 a \,.\, \log_2 b = 1$ ie $pq = 1$, so $q = 1/p$ and $b = 2^{q} = 2^{1/p} \in R^{+} - \{1\}$, $p \neq \emptyset$.

$\oplus$ is distributive over $\bullet$: $c \oplus (a \bullet b) = c \oplus ab = c \oplus 2^{p+q} = c^{p+q} = c^{p}c^{q} = (c \oplus a)(c \oplus b)$.

Now define $a^{o}b$ as $2^{(\log_2 a)\log_2{}^{(\log_2)b}}$, $a,b, \in (R^{+} - \{1,2\})$. Put $a = 2^{2^{p}}$, $b = 2^{2^{q}}$, $c = 2^{2^{r}}$, $p,q,r \in R^{+} - \{1\}$; and $2^{2^{p}} \in (R^{+} - \{1,2\})$; hence closed, since $a^{o}b = 2^{(2^{p})^{q}} = 2^{2^{pq}}$ $p,q \in R^{+} - \{1,2\}$.

Commutative: $a^{o}b = 2^{2^{pq}} = 2^{2^{qp}} = b^{o}a$.

Associative: $(a^{o}b)^{o}c = (2^{2^{pq}})^{o}c = 2^{2^{pqr}} = a^{o}(2^{2^{qr}}) = a^{o}(b^{o}c)$.

Identity: 4, since $a^{o}4 = 2^{2^{p}} = a$.

Inverse: for each $a \in R^{+} - \{1,2\}$ if $a^{o}b = 2^{2^{1}}$ we have $pq = 1$, $q = 1/p$; hence $b = 2^{2^{(1/q)}}$, ie inverses exist.

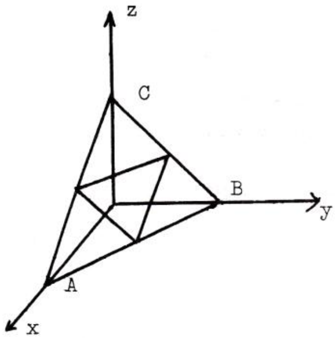
^{o} distributes over $\oplus$: $a \,^{o}\, (b \oplus c) = a \,^{o}\, (2^{(2^{q} \cdot 2^{r})}) = 2^{(2^{p})\, q+r)} = 2^{2^{pq}} \bullet 2^{2^{pr}} = (a \,^{o}\, b) \oplus (a \,^{o}\, c)$.

Therefore $(R^{+}, \bullet, \oplus, ^{o})$ is a superfield.

^{o} is not distrubutative over $\bullet$.

CUT WIRE (problem 31.5)
Richard Ahrens

There is a nice geometric argument that produces the answer 1/4 (see "References", page 6) - I don't know how convincing you find it.



If the piece of wire has length 1, each point on the triangle ABC corresponds to one way of cutting the wire into three pieces because in this triangle $x + y + z = 1, x > 0, y > 0, z > 0$.

The inner triangle gives these triples (x, y, z) which will form a triangle, because in this triangle we have $x < ½, y < ½$, and $z < ½$.

The area of the inner triangle is one quarter the area of the triangle ABC.

SOLUTIONS

29.1 SEVEN SPIROS MAKE AN AGNEW

From Dave Diprose: I have not seen a solution to this (SPIRO × 7 = AGNEW in how many ways). I could only find one: 14076×7 = 98532, unless S is allowed to take the value 0 when there are about eight more solutions. The only patterns were that 0 always occurs in SPIRO (well there are not many jokes in mathematics) and the digit totals of SPIRO (and hence AGNEW) were always 9.

31.1 EULER'S POLYGON DIVISION and 33.2 THE KING'S MOVE

From Richard Ahrens: A good treatment of triangulations of polygons is, I believe, to be found on page 102 of G Polya, "Mathematics and Plausible Reasoning", Princeton University Press, 1954.

Jeremy Humphries's problem of the King's Move prompts me to ask the following question: If the king can only move North or East and cannot move so that the number of Northerly moves exceeds the number of Easterly moves we get the following pattern of paths:-

Each number represents the number of paths from "start" to that square.

			14			
			5	14	28	48
		2	5	9	14	20
	1	2	3	4	5	6
start	1	1	1	1	1	1

Why are the numbers 1,2,5,14,... which, appear on the North-East diagonal line the same as the numbers in Euler's triangulation problem?

33.5 MAXNIM; Max Bramer: Rather than give a solution to this problem I shall attempt to put a computer program $3NIM to play the game on the computer system library, in time for Summer School. If anyone manages to beat the program I shall be very interested to hear of it.

34.1 MAGIC SQUARES E Kent

I found this as a manuscript addition to a Victorian book of that turned up in a jumble sale and wondered if anyone can improve on it:

47	10	23	64	49	2	59	6
22	63	48	9	60	5	50	3
11	46	61	24	1	52	7	58
62	21	12	45	8	57	4	51
19	36	25	40	13	44	53	30
26	39	20	33	56	29	14	43
35	18	37	28	41	16	31	54
38	27	34	17	32	55	42	15

Knight’s move from 1 to 64. Discovered by a pensioned Moravian officer named Wenzclldes. Columns and rows add up to 260 but not the two diagonals. Knight's move + verticals and horizontals + diagonals is unsolved. Of course this may no longer be true and to most people it isn't even interesting, but I drop it as a pebble.

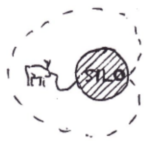
MORE BALLS Richard Ahrens

Six balls in three pairs: one red, 1 white, 1 blue. One ball of each colour is heavy; one ball of each colour is light. All heavy balls are of equal weight, so are all light ones. Find the three heavy balls in two weighings on a balance.

GOAT AND FIELD II John Parker

The goat is tethered to a point on the perimeter of a circular silo. The length of the rope just permits him to graze as far as the diametrically opposite side of the silo. What is his grazing area?

(no solution known)



(NOT TO SCALE)

34.4 FIND THE NEXT TERMS These, together with last months' (which no one has solved yet) are taken from the Journal of Recreational Mathematics who have very kindly given us permission to use them. It's a nice magazine. You have to find the next two terms and a rule.

11. 1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 1 2 1 1 2 1 2 2 1 1 2 1 1 2 1 2 2 1 2 2 1 1 ...
17. 1 2 4 10 26 76 232 764 2620 9496 ...

COPRIMES Max Bramer

Given n+1 positive integers all less than or equal to 2n prove that least two must be coprime.

EDITORIAL

The M500 Society will soon have its own typewriter. On order (six to twelve weeks delivery): One IBM Selectric 875 typewriter Open University (educational) discount £416 and four golfballs (2 free) £16; total £432. There is £323 in the equipment fund. The difference of £109 has been lent to the Society by Marion Stubbs who would like it back. So keep it in mind that the Equipment Fund is grateful for all contributions however large, however small. (The next objective is a mechanical addressing machine.)

I have been asked by several people not to put in solutions until at least two issues after the appearance of the problems. It quite often happens that many solutions arrive after the manuscript has gone to the printers so the authors never get cited for their work. Fair enough. I'll give it a try but if it makes my work harder something somewhere will have to give. At the moment my policy is to print the first respectable looking solution that arrives and just mention the composers of others that arrive later. This obviously keeps the number of decisions I have to make at a managable (or even manageable) level. And please don't make the mistake of assuming that any solution in M500 (or anywhere else for that matter) is correct. None is (except trivially) ever any more than a falsifiable thesis.

For next month we have a nice expert piece on Coincidental Birthdays from Alan G Mumford of Southampton University; a Public School ring from John Reade; applied pure mathematics from Rosemary Bailey; and the final word on Point Construction from Bob Margolis. (That was supposed to have been in this issue but we ran out of room - anyway that's my rationalisation; actually it's so full of Greek that it will almost have to be done by hand unless Mr Ibm hurries himself.)

Also there are many things on hand which just don't get in because I am not up to tackling the illustrations. Can I suggest that if you have something that needs a picture you draw it on a separate piece of WHITE, PLAIN paper in BLACK ballpoint - no bigger than necessary but remembering that it will be reduced by half in the printing. And while on the theme of illustrations, we need cover designs, please. Anything mathematical, or pseudomathematical, pretermathematical or even preterpseudo-mathematical.

And a last paragraph to mention Chez Angelique, the Bumper Late Night Problem Book from John Jaworski, John Mason, Alan Slomson & Al. You can get it from John Mason at the Mathematics Faculty, WH for £1 and then waste many happy hours in the company of wives, ladies and oranges to mention only these. There is also a section on games: my favourite is called Finchley Central where two players take it in turns to name stations on the London Underground. The first to say “Finchley Central" wins. There is a full analysis of this and other games as a non-Polya style discussion of problem solving. If you haven't got a copy send your pound along right away.

Eddie Kent.