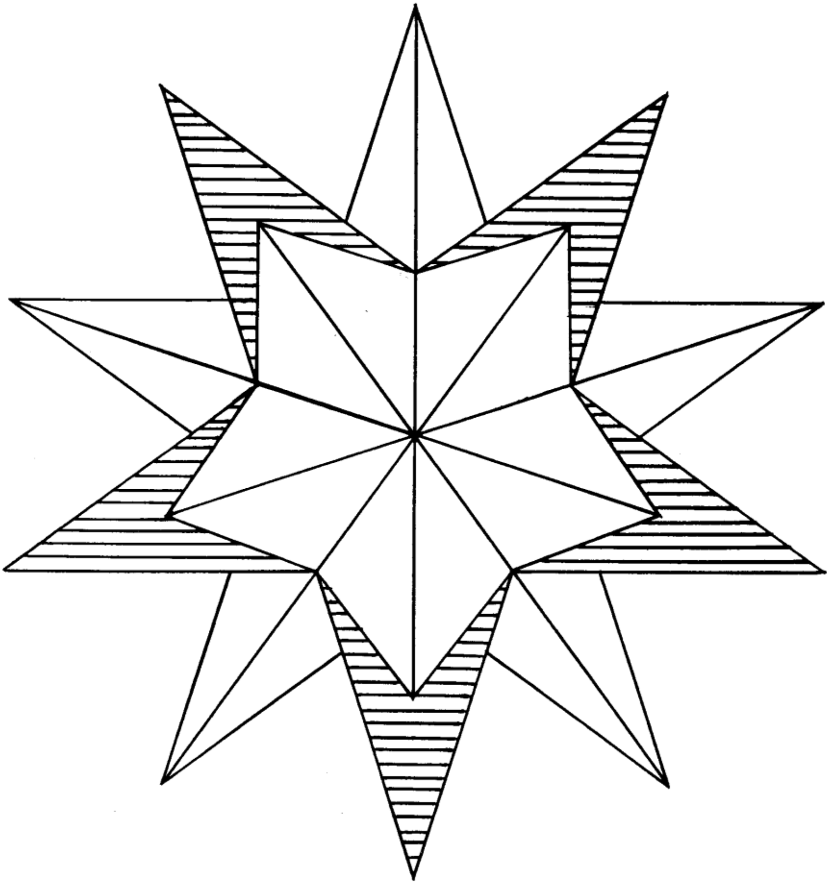


M500 48

M500 is a atudent-owned, student-operated magazine for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for public discussion of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names and addresses, with telephone numbers and details of past and present courses of voluntary members, by means of which private contacts can be made to share OU and general mathematical interests - or to form self-help groups by telephone or correspondence.

MATES is a special list of MOUTHS members who have explicitly volunteered for their MOUTHS details to be distributed to members in closed institutions such as prisons and special hospitals.

The views and mathematical abilities expressed in M500 are those of the authors and may not represent those of either the Editor or the Open University.

SAMPLE COPIES of M500 and application forms for membership are now issued by JOYCE MOORE. Please enclose 30p in loose, unused, small-value stamps, and also please state your current courses.

PUBLISHER Marion Stubbs cover designs, missing issues, missing pages

EDITOR Eddie Kent articles and letters for publication

PROBLEMS EDITOR Jeremy Humphries problems and solutions for publication

MEMBERSHIP SECRETARY Peter Weir Subscriptions (renewals and new) - changes of address - MOUTHS/MATES data and changes

TREASURER Austen F. Jones, ACA

WEEKENT ORGANISER 1978 Sidney Silverstone

PUBLICITY Nick Fraser

PRINTER Waterside Printing Company, 11 Rumbridge St., Totton, Southampton

M500 48 published January 1978. Subscription £4 for 10 issues. Cheques and postal orders should be payable to THE M500 SOCIETY, crossed "ACCOUNT PAYEE ONLY. NOT NEGOTIABLE" for safety in the post.

ANYTHING SENT TO ANY OFFICER OF THE SOCIETY WILL BE CONSIDERED FOR POSSIBLE PUBLICATION IN THE MAGAZINE UNLESS OTHERWISE SPECIFIED.

The cover design by Chris Pile is a diagram of the Great Stellated Dodecahedron.

VECTOR GEOMETRY BOB ESCOLME

With $a$ representing the point A, $b$ the point B, any point on AB can be represented by the vector $b+\lambda(a-b)$, $\lambda\in\mathbb{R}$, that is $\lambda a+(1-\lambda)b$ or $\lambda a+\mu b$: $\lambda+\mu=1$; $\lambda,\mu\in\mathbb{R}$. (Figure 1.)

This can be used to prove that the medians of a triangle are concurrent. Thus let the points A,B,C be represented by $a,b,c$ respectively. Then X, the midpoint of BC is represented by $\frac{1}{2}(b+c)$. (See figures 2 and 3.) And any point on AX is represented by $\lambda a+\mu(b+c)/2$, $\lambda+\mu=1$.

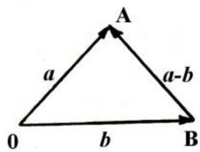



*Figure* 1

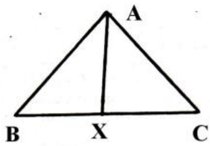



*Figure* 2

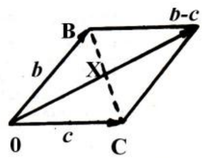



*Figure* 3

Choose $\lambda = \frac{1}{3}$, $\mu = \frac{2}{3}$ and we see that there is a point on AX represented by $(a+b+c)/3$. The symmetry of this result establishes the theorem.

You may like to have a go at proving the following results using vector algebra. Some of them demand rather more of one's ingenuity with such algebra than that used above. They all appear early on either as examples or excercises in *Vector and tensor analysis* by Harry Lass (McGraw Hill International Student Edition).

i. Let ABCD be a quadrilateral in 3-space (ie the apices do not necessarily lie in a plane). Show that the lines joining the bisectors of the opposite sides bisect each other.

ii. Let the consecutive apices A,B,C,D of a quadrilateral lying in a plane be represented by the vectors $a,b,c,d$ respectively. Then ABCD is a parallelogram if and only if $a-b = d-c$.

iii. The lines joining a vector A of a parallelogram to the midpoints of the sides which do not Include A trisect the diagonal which does not include A.
(Hint: use ii.)

iv. (*Theorem of Ceva*.) The lines which join three points, one on each side of a triangle, to the opposite vertices of the triangle are concurrent if and only if the product of the algebraic ratios in which the points divide their respective sides is equal to $-1$.

v. (*Theorem of Menelaus*.) Three points, one on each side of a triangle, are collinear if and only if the product of the algebraic ratios in which the points divide their respective sides is equal to $+1$.

vi. The altitudes of a triangle are concurrent.

➔

vii. The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

viii. (*Desargue's Theorem*.) If two triangles in 3-spaoe are so situated that the three points of intersection of corresponding sides are collinear, then the lines joining the corresponding vertices are concurrent, and conversely.

(Ed - The above material was accompanied by a letter which managed to avoid the wpb, so you may as well read it:

The enclosed material is far too extensive for you to cosider its publication, though It could be of interest to those who have read M201. (In fact the knowledge required does not extend beyond that gained in M100.)

The interest I feel lies In the fact that one can prove some of the classical results of Euclidean and projective geometry by vector methods which (in spite of the first appearance of some of the symbol-full pages) are really quite neat.

The hardest looking proofs, for example, are the ones for Ceva's, Menelaus's and Desargue's theorems. However there is a symmetry about the way the symbols go; and although at the start it looks as though one has far too many to deal with, the results almost seem to fall out unasked. In particular, for Desargue's theorem, we get to a determinant which at first sight looks extremely messy. Yet it is easily expanded to show its zero value.

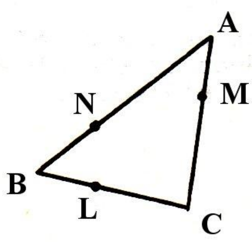
In fairness to myself and to the author mentioned in the first page of the enclosed material, I should mention that all the proofs are my own. If any of them were to appear in M500 therefore they should be checked first. Of course they might well appear in a text book and in that sense there is nothing original here.

However some M500 readers might well enjoy having a go at the proofs; though (as I found with my first attempts) if they don't keep their approach general, and don't spot the underlying symmetries, they could get some extremely unsatisfying algebra.

Incidentally, I feel there is little to be gained from producing proofs by classical methods: the essence of the thing is to use vectors.

and then examine a couple of the proofs. The rest can be set as Problem 48.1. (These proofs will not, of course, have been checked. After all who, as they say, am, as it were, I.).)

i. Let the four points A,B,C,D be represented by vectors *a,b,c,d*. The midpoint of AD, say X, has vector ½(*a*+*d*) while the midpoint of BC, Y, has vector ½(*b*+*c*). So the midpoint of XY has vector ¼(*a*+*b*+*c*+*d*) and the symmetry of this establishes the required result.




iv. *Theorem of Ceva*. Let the distinct vertices be *a,b,c,* none = 0. Let the points L,M,N have vectors $\alpha b+(1-\alpha)c$, $\beta c+(1-\beta)a$ and $\gamma a+(1-\gamma)b$ respectively, $\alpha, \beta, \gamma \neq 1,0$. Then the vectors of the general points on AL, BM, CN are:

$$x = \gamma(\alpha b+(1-\alpha)c) + (1-\gamma)a$$
$$y = \mu(\beta c+(1-\beta)a) + (1-\mu)b$$
$$z = \nu(\gamma a+(1-\gamma)\, b) + (1-\nu)c.$$

Rewrite them as

$x = (1-\lambda)a + \lambda\alpha b + \lambda(1-\alpha)c$

$y = \mu(1-\beta)a + (1-\mu)b + \mu\beta c$

$z = \nu\lambda a + (1-\lambda)b + (1-\gamma)c,$

$\lambda,\mu,\nu \neq 1,0.$

These are vectors of the same point if and only if the following set of equations have the same solution:

$$\frac{1-\lambda}{\lambda a} = \frac{\mu}{1-\mu}\cdot\frac{1-\beta}{\beta} = \frac{\gamma}{1-\gamma}$$

$$\frac{\alpha}{1-\alpha} = \frac{1-\mu}{\mu} = \frac{\nu}{1-\nu}(1-\gamma)$$

$$\frac{1}{1-\lambda}(1-\alpha) = \frac{\beta}{1-\beta} = \frac{1-\nu}{\nu\gamma}.$$

(*Divide the* ***a*** *and* ***b*** *coordinates of* ***x*** *by* $\gamma\alpha$ *the* ***a*** *and* ***b*** *coordinates of* ***y*** *by* $1-\mu$, *of* ***z*** *by* $\nu(1-\gamma)$ *and equate the results. This produces the first line of our equation. Similarly for the others.*)

Write $\gamma/(1-\gamma) = K_3$, $\alpha/(l-\alpha) = K_1$, $\beta/(1-\beta) = K_2$ and then write the equations as

A) $f_1(\lambda) = g_1(\mu) = K_3$

B) $g_2(\mu) = h_2(\nu) = K_1$

C) $f_3(\lambda) = h_3(\nu) = K_2$.

Now A) and B) have a common solution if and only if $g_1(\mu)g_2(\mu) = K_3K_1$. Looking back to see what $g_1(\mu)$ and $g_2(\mu)$ are we get the result

$$\frac{\mu}{1-\mu}\cdot\frac{1-\beta}{\beta}\cdot\frac{1-\mu}{\mu} = K_3K_1;\ \frac{1}{K_2} = K_3K_1;\ K_1K_2K_3 = 1.$$

Similarly B) and C) have a common solution iff $K_1K_2K_3 = 1$, and C) and A) have a common solution iff $K_1K_2K_3 = 1$. And so AL, BM and CN are concurrent

$$\Leftrightarrow K_1K_2K^3 = \frac{\alpha}{1-\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{\gamma}{1-\gamma} = 1, \text{ ie} \quad \Leftrightarrow \frac{CL}{BL}\cdot\frac{AM}{CM}\cdot\frac{BN}{AN} = 1 \quad \Leftrightarrow \frac{LC}{BL}\cdot\frac{MA}{CM}\cdot\frac{NB}{AN} = -1.$$

wmwmwmwmwmwmwmwmwmwmwmwmwmw

A LETTER

From Norman W Lees: : Packed with goodies as it is, do you think it would be worthwhile having a "For the library" column where the more learned of our colleagues could describe their favourite maths books? Books are mentioned of course, but it is somewhat en passant in fashion.

PS - I do like the historical articles - particularly the Gauss and Littlewood ones.

*Ed - That PS was Positively Not Solicited*! *Also I agree with Norman*: *it would be nice to have books described. If everybody wrote a short article on the one mathematical book they are most pleased to have acquired, I'm sure it would be most welcome. I will try and do something myself.*

SITS VAC TRAINEE-ASST GRADE. SALARY NIL MARION STUBBS

Joyce Moore recently blithley volunteered as a sort of trainee-assistant Membership Secretary. This offer was gratefully accepted by Peter Weir, who now seems to entertain some vague hopes of eventual retirement from his enormous job. They are working out ways and means of sharing the labour, but it seems likely that Joyce will cope with the new enquiries for samples while Pete will concentrate on cash and subscriptions, and renewals. The pair now seem to constitute a "Membership Secretariat", not envisaged by our constitution. Pete will dole out labour to Joyce.

Meanwhile Nick Fraser is increasingly i/c publicity, also not envisaged. All SPls/1978 are by yours truly but it looks likely that Nick will get the total publicity job in his lap very soon, if not already. He has been doing great things during 1977 regarding Summer School and Day School displays, dealing with *Sesame* (rather abortively, unfortunately, but no worse than I ever achieved in that area!) and writing oddments for any other OU newsletter etc which requested some piece on M500 and its activities He is i/c the SP2-5 of M231/1978 offered recently by Robin Wilson.

I suppose it is possible that there are other people around who would vaguely like to help but are too shy to volunteer for a major, massive office. Eddie was getting desperate for help with the M500 Problems Section before he found Jeremy Humphries, who seems to be absolutely superb at the job, as we must all surely agree?

Austen Jones stays in the background, doing Treasurer's work which he reckons requires a very minimum of HND in Accountancy (preferably Chartered Accountant) to do. We don't hear much in M500 from Austen, but believe me he is labouring hard, with the most unenviable job of all. Our turnover is probably unpleasantly close to VAT level and he is perpetually shunting cash to and fro between Building Societies and bank accounts, of which we have so many that I cannot keep track. This summer, 1977, AJ added to my sundry traumas by casually commenting that he was not going to do any more maths and ought to retire! Given my assurance that he could be doing *arts* for all I cared he has kindly agreed to carry on as Treasurer. However, the writing is on the wall! Does anyone with min HND/HNC Accountancy feel like volunteering to 'help' AJ, with a view to eventually becoming Head of the Treasurer' Department?

Eddie does and did need a Problems Editor, and it is possible that he might like some sort of assistance elsewhere. Our Editor needs to be able to type, and to possess or have access to an electric typewriter. (Eddie has the M500 typewriter which cost about £500 from the equipment fund.) If there is a budding trainee-assistant Editor, or Problems Editor, around, no doubt something could be found for them to do. Even Eddie's and Jeremy's enthusiasm will eventually wane, and it would be useful to know who else is keen, if anyone.

Finally we come to my 'Publishers' job. Everyone now thinks I sit back and do nothing, or Make Decisions or something. In fact I slog away addressing 400+ envelopes per month, typing MOUTHS lists etc, and shoving everything into the 400+ envelopes per month as well as dealing with printers. This takes one complete week per month of my non-work-work time. From the moment the typescript arrives at 176 from Eddie I do nothing else at home➔

(not even housework) until a full week or more later. I also get a general average of five letters a day from members or potential members - although very recently this has died away to nil thanks be to examinations and Winter Holidays. I cannot see anyone else ever taking my job off me, and am happy to continue doing it even when 70-80 years old. But you never know - someone somewhere could be longing to do such a job but is too terrified to volunteer. What I need is a mechanical (non-computer) addressing machine for the massive job of *correctly* addressing 400+ envelopes per month. Such a machine costs around £500+ and does about 1000 envelopes per hour. Given that - and we already have c£300 now in the Equipment Fund - then the Publisher's job gets rather more trivial; especially if the Membership Secretaria*t* produced the stencils of the MOUTHS list which at present it does not do. So, given the machinery and so forth, maybe someone would actually LOVE my job? (I like it, but fair's fair, and if someone else would dearly love to have it then they can have a go.) Meanwhile Richard Shreeve has taken charge of the Special Issue, with three new student editors - and also with typing assistance from Joyce Moore.

So let us know if you are an embryo M500 officer or trainee-assistant same. M500 is now one of the great OU societies, despite lack of *Sesame* recognition, and there is plenty of work around for any volunteers. You don't have to be an elected official in order to help. Choose your scene and let's embark on a new era whereby various basic jobs are done by several people? Maybe Joyce Moore has triggered off a whole new system? It is all, sociologically speaking, very interesting as a development. I am so glad that our Constitution is flexible on an annual basis and can be changed in accordance with the needs of THE SOCIETY.

MMMMMMMMMMMMMMMMMMMMMMMMMMMMM

0YEZ! OYEZ! OYEZ!

NICK FRASER

*Oyez*!1: *OUSA Open Day*. It has taken me a long time to write this report because of the rude interruption of exams and mine was not until 3rd November. Still I would like to tell you what a success the Open Day was for the Maths Faculty in attendance terms. Marion led me to believe that the Faculty was considered to be the province of the untouchables and that hardly anyone came. I am pleased to say it was packed (literally) all day and I met many present members of M500 and even convinced a couple that it might be worthwhile joining. I spent a lot of time talking to new students who are to do Ml01 in 78 explaining what OU maths (and M500) is all about. The main attraction to the Faculty were the puzzles which you may remember from Summer School. I was amazed at the intelligence of some of the children doing the puzzles. Someone asked me to explain how to do a puzzle and confidently ended up in a mess which this brat came along and sorted out in ten seconds. It is the nearest I have come to murder. Nice little boy!!

The reason for the Faculty being so full had nothing to do with the fact that the other faculties put on very little and that the faculty was in direct line of the car park to the entrance hall.

➔

An enjoyable day apart from the fact that I did not realise how awful Milton Keynes was.

*Oyez*! 2: *That Weekend*. Well there we are MWWI Mk III has been and gone. Successful without a doubt, better than the previous year, certainly. Coventry was a bit further to travel for me but the facilities and surrounding area were much better than at Aston. However we must not be complacent for if we wish to have a better Weekend next year we must have constructive comments NOW. For example, I found the Friday evening session of less use, being tired from the days work and the drive down. Personal feeling it may be, but reflected by a few people I spoke to. How about a General lecture on the Friday evening as well. As to subject matter of the lecture rather than on the OU admin (interesting though they were) on a mathematical topic. Invite a speaker just for that purpose rather than exhaust our already overworked tutors. Are we not a bit OU Maths oriented and wouldn't this help alleviate it.

*Oyez*! 3: *Membership Drive*. Richard Shreeve approached me at the Weekend about sending a circular to students in his area. I agreed to write for him a piece and get it duplicated if required. I offer the service to anyone else who has access to names and addresses of students in their area. For example in Region 8 (my own region) regional office issues a list of Self Help group organisers with addresses. Does your region do it? If so use it. More members mean a better financial base for the SOCIETY and the magazine. At present I gather there is a minimum print run of 500 each month and with 420 members that means about 80 magazines left over. If we had 1000 members then the consequent economies would be significant, for it is in the setting up that the costs of magazines mount. More members mean a better MOUTHS list and more likelihood of isolation being totally alleviated (unless it is desired by the student). More members mean a better magazine in that more contributions come in and it reaches a wider audience. If you feel you can help let me know.

MMMMMMMMMMMMMMMMMMMMMMMM

M321 DANIEL DUBIN

On 14th January 1977 I sent a letter to all those students who had sat the final examination of M321 the previous autumn. This letter was written on behalf of the examination board; we were particularly concerned because the students in question did not choose what were meant to be the easy options and their marks suffered accordingly.

The great majority of students chose to start at the second part of the exam, the "long" questions, leaving the "short answer" questions until later; unfortunately, later often never came.

The survey result here was fairly uniform. This was a chosen strategy to overcome the lack of time. The students felt the examination to be fair but overlong: anywhere from six to nine hours

(*continued on page* 11 ➔

# GAUSS III

JEREMY GRAY

I mentioned last time that Gauss's work on the 17-gon drew on his mastery of the theory of numbers. One nears the heart of Gauss's work with the mention of number theory, and I shall fail in my task If I cannot communicate some of the excitement that Gauss's number theory generates for mathematicians. He was not, of course, the first number theorist; even of the modern period. But he was the man who turned it from an isolated bag of results into a theory. Let me at once give you a great result of his: *Theorem aureum* or golden theorem; the law of quadratic reciprocity, proved by him on the 8th of April 1796, only nine days after his construction of the 17-gon.

First some terms. We say $a \equiv b$ (mod $n$) ($a$ is *congruent* to $b$ mod $n$) if $n$ divides $a$–$b$, and we ask the question for what values $q$ does the congruence $x^2 \equiv q$ (mod $p$) have solutions? Such values of $x$ are called *quadratic residues mod p*. For example the quadratic residues mod 10 are 0, 1, 4, 5, 6, 9, which are the last digits of the squared numbers. We write $\left(\frac{q}{p}\right) = +1$ if $q$ is a residue mod $p$ and $\left(\frac{q}{p}\right) = -1$ otherwise ($q$ is a 'non-residue'). We can equally well ask if $x^2 = p$ (mod $q$) has any solutions (is $p$ a quadratic residue mod $q$?) It turns out that the theory of quadratic residues for composite numbers reduces to that for primes: for instance, 0, 1, 4, 5, 9 are also the quadratic residues mod 2 and mod 5 (where they reduce to 0, 1 and 0,1,4 respectively). Gauss discovered - as Legendre had conjectured in 1785 (Memoire: Hist, Acad. Sciences) - and proved (which Legendre could not) that for primes $p$ and $q$ the questions are very intimately related. Indeed

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left[\frac{p-1}{2}\right]\left[\frac{q-1}{2}\right]}.$$

So, if $p$ and $q$ are not both congruent to 3 mod 4 then $p$ is a residue mod $q$ if and only if $q$ is a residue mod $p$, whereas if $p$ and $q$ are both congruent to 3 mod 4 the reverse is the case.

I illustrate this with 5 and 11. The quadratic residues mod 5 are 0, 1, 4 so $x^2 \equiv 11$ (mod 5) (or equivalently, $x^2 \equiv 1$ (mod 5)) can ➔

be solved, and $\left(\frac{11}{5}\right)$ = +1. The quadratic residues mod 11 include 5: $4^2 \equiv 5$ (mod 11). So $\left(\frac{5}{11}\right)$ = +1 also. And $\left(\frac{11}{5}\right) . \left(\frac{5}{11}\right) = +1 = (-1)^{5 \cdot 2}$. And to illustrate the technique's immense range, what about 65537 and 257? Is 257 a square mod 65537? Well

a. 65537 ≡ 2 (mod 257) - just divide.

b. 257 ≡ 1 (mod 2)

so $\left(\frac{65537}{257}\right) = \left(\frac{2}{257}\right)$ from equation (a)

and $\left(\frac{257}{2}\right)$ = +1 from eouation (b)

257 ≡ 1 (mod 4)

so, by the golden theorem,

$$\left(\frac{257}{2}\right)\left(\frac{2}{257}\right) = +1;$$

or $\left(\frac{257}{2}\right) = \left(\frac{2}{257}\right)$

so $\left(\frac{2}{257}\right)$ = +1.

Hence

$$\left(\frac{65537}{257}\right) = 1$$

and 257 likewise is a quadratic residue mod 65537.

Perhaps I should give some idea of Gauss's proof. But which one? In the course of his long life Gauss gave seven; two in the *Disquisitiones Atithmeticæ* alone. Besides, the book is hard to read - at his talk at the Gauss bicentennial in Toronto this year (1977) the distinguished number theorist Atle Selberg said he was very glad he had not begun his reading in the subject with this book - it is written in an austere formal Latin (a somewhat inaccurate English translation exists). Full understanding of Gauss's work only began with Dirichlet's *Vorlesungen über Zahlentheorie* of 1863. We must remember that Gauss's was the first systematic treatise on number theory.

We know a little about the way in which Gauss was led to discover the result. In the *Disquisitiones Atithmeticæ* (§151) Gauss noted that equivalent results had been stated by Euler in the 1740s, although without proof. Euler had considered congruences of the form $x^2 + ny^2 \equiv 0 \pmod{p}$, where $x$ and $y$ are relatively prime; and discovered that the existence or non-existence of solutions depends ➔

on whether or not $p \equiv 1 \pmod 4$ or $p \equiv -1 \pmod 4$. Gauss also acknowledged Legendre's work, and by 1808 hailed him as "the discoverer of this most elegant theorem". But Gauss's methods of obtaining his proofs remain firmly his own.

Perhaps I may say some things about the proofs. They are generally difficult, I believe. Gauss sought to generalize them, as did other mathematicians, to deal with cubic and biquadratic reciprocity laws. Above all, they illustrate Gauss's belief in the essential unity of mathematics, a belief he maintained even while extending mathematics into wholly new fields. But I should like to expand a little on a broader question: why number theory? and attempt to defend the theory that number theory is one of, even the chief glory of mathematics.

There is surely no one who would deny that mathematics finds remarkable application in the natural sciences. The interaction between mathematics and physics, engineering, and astronomy is a huge fund of energy for mathematics, which in turn gives the arguments in those theories their consistency and explanatory power. But no man can rest content with being the servant of another, least of all a pure mathematician. Mathematics is both the Queen and the Servant of the Sciences - in the famous remark - and as Queen she cannot for ever say: I am important because Science uses me. Now, of all the candidates for the paradigm objects of mathematical thought, four stand out: numbers; geometry; analysis; and foundations (say, sets or categories). Analysis seems to rest on numbers and geometry - in a rather murky way - and foundations is too basic for most mathematicians to care for it. So they are driven to numbers and geometry, where they feel both that mathematical intuition has something to work on and yet they are dealing with truly basic ideas. And of these two number theory is the older, the more unifying, and the more mysterious. So to number theory then goes the crown. And, as I have said, number theory is, as a theory, chiefly due to Gauss. We shall see that in his hands it is indeed both mysterious and unifying.

Mysterious? Well it is strange indeed that there should be any connection at all between *p* being a quadratic residue mod *q* and *q* being a quadratic residue mod *p*. How remarkable that these two ➔

questions about different primes should have interrelated answers. And why is it that the classification of primes into those of the form $4n + 1$ and those of the form $4n + 3$ should matter? In fact number theory is full of these surprising results, easy to suspect in some cases, very difficult to spot in others. The common feature is that the proofs are almost always hard! But a theory, if it is not merely a collection of facts, should explain and make luminous the objects with which it deals by exposing their intimate relations; so what did Gauss's theory achieve?

His first proof of the law in the *Disquisitiones Arithmaticæ* merely added to the mystery. H J S Smith said of it that it was "presented by Gauss in a form very repulsive to all but the most laborious students" (*Report on the Theory of Numbers*, Part I, §18, 1859). As improved by Dirichlet it rests on induction: the law is assumed true for all primes less than $p$ and then demonstrated for the prime $p$. The law is trivially true for the primes less than 7, which starts you off. Others amongst Gauss's proofs opened the way for very deep connections with complex function theory, and one (the second, also given in *Disquisitiones* ) was of immense significance for algebra.

I shall be attempting to describe some of this work in later episodes, but not in the next one which will be devoted to Gaussian problems. There should be some reward for anyone who tries to penetrate the icy reaches of the man's thought.

Bibliographical note

Some of Euler's and Legendre's work can be found in translation in the magnificent *Source Book in Mathematics* 1200 - 1800 edited by D J Struick (Harvard University Press, 1969).

Illllllllllllllllllllllllllllllllllllllllllllllllllllllllllllllllllllllll

Jeremy Gray is a course assistant with the Open University at Milton Keynes, at present working on M203. He is also secretary of the British Society for the History of Mathematics.

*continued from page* 6)

would have been about right.

We feel that a two-fold attack on this problem is necessary - and this is a lesson for all courses. First of all we have tried to make the short questions truly short and the long questions of reasonable length. Together with the fact that new and tricky material does not appear on the exam, this is a fair contribution by us. Secondly the students must work faster. It is my feeling that the amount of time a student allows himself to do a problem at home is unrealisticly long, particularly the second time it is encountered. During an examination, when speed is a (valid) variable which is tested, such slowness is a great luxury. Unfortunately, speed is increased only by practice.

The other main concern of the board was the fact that very few students chose to do the numerical methods questions, yet such questions earned these students good points on continuous assessments. The reply to this question was clear: too many chances for error in doing so many numerical computations. Our feeling is that this is a chimera: there is not all that much numerical computation involved in these questions. Rather there is emphasis on *structure, technique, and applicability*. We shall, in any event, try to emphasize these points in the future.

mnmnmnmnmnmnmnmnmnmnmnmnmnmnmnmnmnmnmnm

SPECIAL ISSUE 1978   RICHARD SHREEVE

The contributions are coming in but please, please send more. I'm sure that there are more members of M500 with opinions and contributions than the faithful regulars that we have had to rely on in the past.

If you had any particularly tough experiences, then put them in print. Likewise, any courses or combinations that really clicked.

I think that the Weekends must have helped some people who otherwise were really struggling. If this is true then do please say so.

Also really what we want is lots of short articles on combinations of courses - those which went well together and those which were in no way complementary. And if you think  that a half credit course is nearer a full credit - just write it down and let us have it.

*rtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrtrt*

A NOTE FROM OUR FOUNDER:

WANTED - COVER DESIGNS.   Situation currently desperate! Please do them in black ink on plain white paper (to fit the top ¾ of A4 paper leaving space for "M500 4-" underneath) ready for camera reduction to A5. Send to Marion Stubbs. Many thanks.

## COURSE REPLACEMENTS AND COURSE REVIEWS COLIN MILLS

Having mentioned the double factorial in M500 39 Wallis's formula turned up inside a month - Spivak page 328 (with a very much untidier notation). However, I am still carrying out scattered research and would be interested to hear what others know or can find out. My main purpose in writing though has much more to do with Professor Pengelly's guest lecture at the Lanchester Mathematics Week-end Work In at the beginning of September. Like many students with the OU I tend to find the Course Handbook an *embarras de choix* , and I've tended to restrict myself to courses I consider to be applicable. Where I've read of courses of more general interest to me I've been deterred from investing time in full credits - for instance I haven't studied M201 or M202, not just because of a feeling that they would be too difficult (probably exaggerated) but also, as full credits, they would restrict my choice of degree profile, and I am sure there are many others (mostly scientists, engineers and teachers, but not entirely) with an interest in Maths who would feel as I do. For these reasons I was rather disturbed to hear that M231 and MST282 were being absorbed into M203 and M204.

Professor Pengelly said in his lecture that the challenge of rewriting courses was essential in order to maintain staff morale in the faculty - which seems to be change for change's sake - and after producing some suggestive but contentious figures, stated that for reasons of internal politics it was essential that the number of students passing through post-foundation courses be increased, especially the numbers in individual courses. While these arguments are important I do not believe that more students will be attracted to these courses when the number of second level mainstream Maths courses has been decreased from four to two. It is not only mainstream mathematicians who must be catered for, but also users of mathematics, whether pnfessionals or amateurs, who should be encouraged to take half-credit maths courses which would also be taken by mathematicians. I believe that M203 should continue to be available as M211 and M212 - it is possible to alternate assignments and broadcasts for two related half-credit courses so that they can be taken together as a full credit course - and M204 should be available in the same way. It was also a disappointment by the way to hear that the standard of M203/4 should be lower than that of M201/2.

I also think that more students could be attracted to third level courses, which suffer particularly from small numbers, adopting the idea I raised in *Sesame* 6/5, August 1977 of an MA by examination, using 3rd and 4th level courses. The latter would be especially valuable if they contained a substantial proportion of project work, in which students used the subject. The MA could thus act as a bridge between a broadly based BA (hons) and postgraduate research.

This does not necessarily represent the Open Science Society policy as we are rather wary of giving the OU advice on what courses to run (unlike the OU Medical Society), but we do believe that the OU can benefit from feedback on courses which can be channelled via the student societies and used for the preparation of a handbook for ST courses - unlike OUSA we believe M500 deserve the credit for this idea: *Sesame* 6/6, September/October 1977.

POSGRADUATE DEGREES

Professor Oliver Penrose has asked me to bring the following information to the attention of M500 members.

*Conference of Professsors of Applied Mathematics in: Postgraduate Degrees by Course and Research in Applied Mathematics in United Kingdom Universities.*

*The following Universities offer a wide variety of postgraduate opportunities in Applied Mathematics leading to the degrees of MSc and PhD by research.*

Aberdeen University
Aberystwyth University
Aston University
Bangor University College
Bath University
Belfast, The Queen's University
Birmingham University
Bradford University
Bristol University
Brunel University
Cambridge University
Canterbury, University of Kent
Cardiff, UWIST
City University, London
Cranfleld Institute of Technology
Dundee University
Durham University
East Anglia University, Norwich
Edinburgh University
Exeter University
Heriot-Watt University, Edinburgh
Hull University
Keele University
Lancaster University
Leeds University
Liverpool University
London University
London, Imperial College
London, King's College
London, Queen Mary College
London, Royal HolIoway College
Loughborough University
Manchester University
Manchester, UMIST
Newcastle upon Tyne University
Nottingham University
Open University
Oxford University
Reading University
St Andrews University
Salford University
Sheffield University
Southampton University
Strathclyde University
Sussex University
Swansea University College
Warwick University
York University

*A booklet* University Postgraduate Degrees by Course and Research in Applied Mathematics 1978-79 *describing these courses and showing how applications should be made can be obtained from the Mathematics/Applied Mathematics Departments of any of the Universities named above*. Alternatively a free copy may be obtained by ... .

---

**Apples may fall**

Headline in *The Times*, August 5, 1977.
(Newton would be pleased!)

PROBLEMS CORNER JEREMY HUMPHRIES

This section is meant to be short because Eddie has so much for you in this issue.

There is one thought I've had. Do any of you contact each other directly? That is one of the ideas of M500. Quite often X writes to me to ask me what Y means by this, or what does Z think of that. Why not discuss among yourselves sometimes - perhaps coming to the magazine with the results of your collaborations?

SOLUTION 46.1 OLYMPIAD VI . Sequence $u_n$ is defined by $u_0 = 2$; $u_1 = 5/2$ and $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, $n$ = l,2,... . Prove that for positive integral $n$,

$[u_n] = 2^{2^n-(-1)^n/3}$ . $[x]$ is the greatest integer $\leq x$.

MICHAEL MCAREE and MIKE PURTON sent solutions to this. Mike's was shorter, so here it is.

We can express

$u_0 = 2^0 + 2^{-0} = 2$

$u_1 = 2^1 + 2^{-1} = {}^5/_2$

and the formula for $u_{n-1}$ gives

$u_2 = 2^1 + 2^{-1}$; $u_3 = 2^3 + 2^{-3}$; $u_4 = 2^5 + 2^{-5}$;

$u_5 = 2^{11} + 2^{-11}$.

Thus we find by induction $[u_n] = 2^{v_n}$ where $v_{n+1} = v_n + 2v_{n-1}$. Therefore $v_1, v_2, v_3, \ldots$ = 1,1,3,5,11, etc. The auxilliary equation is $m^2 - m - 2 = 0$ so that $m=2$ or $m=-1$. Hence

$$v_n = a(2)^n + b(-1)^n.$$

Substitute at $n$ = 1, 2: $2a-b=1$ and $4a+b=1$ which implies that $a = ⅓$ and $b = -⅓$. Hence $v_n = ⅓(2)^n - ⅓(-1)^n$, and

$$[u_n] = 2^{(2^n - (-1)^n)/3}.$$

SOLUTION 46,2 MINIMUM POINT: Given three points, A,B,C, determine the point O such that OA+OB+OC is a minimum.

If A,B,C define a triangle in which no angle is greater than 120° then the point O is inside the triangle. It is at the point where the sides of the triangle all subtend angles of 120°.

If at any point A, B or C the angle is ≥ 120°, then O is at that point.

BOB ESCOLME sent two methods of showing this. One used differential geometry and the other one he summarised thus:

To prove the first case consider the ellipse with foci B and C, passing through the O so constructed. For all X on the ellipse XB+XC = constant = a, and for Y outside the ellipse YB + YC > a. Use the 120° fact to show that the circle with centre A and radius AO has its tangent at O in common with the tangent of the ellipse at O. For X ≠ O on that tangent, X is outside the circle and the ellipse. You can establish six directions in which AY+BY+CY

➔

increases above OA+OB +OC as you move away from O. A calculus result shows that these six directions are sufficient to establish that YA+YB+YC increases in every direction.

For $\hat{A}$ = 120° construct A' outside triangle ABC so that A'$\hat{A}$ B = A'$\hat{A}$ C = 120°, thus making A the O-polnt for A'BC. You can then deduce that A is the required point for ABC. This result can be used to deduce that A is still the point if $\hat{A}$ > 120°

MIKE PURTON sent a solution also, which used calculus to find the point O. He says he remembers it as a problem from his M100 Summer School. STEVE AINLEY tells me that it is dealt with in Courant and Robbins: *What is mathematics*? pp 354-8.

When STEVE MURPHY and I were writing to each other about this a few months ago I found a very neat demonstartion for $\hat{A}$ , $\hat{B}$ , $\hat{C}$ < 120° in *Introduction to geometry* by Coxeter (John Wiley) pp 21-23. He says O is called the Fermat point.

If you want a mechanical demonstration of this problem, drill three holes, A,B,C, through your dining table. Hang equal weights on strings, one through each hole, and tie the upper ends of the strings together. Release the system and the knot will come to rest at the point O.

SOLUTION A6.3 MEDWAY LEAGUE You will have noticed the clue to 12 across wasn't printed. It was "eleven times the Hawks' score". Without that you cannot decide whether the Hawks scored 15 or 16. People who sent either one solution or the other or both are STEVE AINLEY, MARJORIE BREW, CHRIS LYONS, MICHAEL GREGORY and SIDNEY SILVERSTONE.

The final order of the teams was: Eagles, Hawks, Redwings, Tigers, Lions, Etceteras; there were 8071 spectators at the last match. The solution is:

ACROSS 1,18 3,15 5,30 6,77 7,72 8, 60 9,81 10,324 12,165.

DOWN 1,13 2,8071 3,17 4, 576 10, 36 11,25.

Chris wants to know if anyone can tell him where to find more Crossnumbers like these from *Fun with figures*. I don't know of any more - perhaps someone would like to make one up?

SOLUTION 46.4 MEASURES You have a 6- a 10- and a 15-pint measure and an unlimited water supply. An operation is filling or emptying a measure or transferring water from one to another. Obtain one pint in each of two measures in the smallest number of operations.

I should have said that these were the *only* operations. Marking of measures, using other containers, etc, are not allowed. Several people managed this using ten permitted operations: STEVE AINLEY, MARJORIE BREW, CHRIS LYONS, SIDNEY SILVERSTONE, MARION STUBBS and BRIAN WOODGATE.

It is possible in nine!

6 pint 0 0 0 0 6 0 6 1 1 10 pint 0 10 0 5 5 5 5 10 1 15 pint 15 5 5 0 0 6 6 6 15

SOLUTION 46.5 CLOCK PATIENCE The 52 cards are placed face down In 13 piles of 4 which are labelled A, 1, 2, ..., Q, K. The top card of the K-pile is turned up and placed next to the pile whose label maps to its denomination in the natural way. Then the top card of that pile is taken, and so on. Success comes when all cards are turned over. What is its probability?

General opinion is that for a successful outcome the last card turned up must be a king. This was an easy one. Probability is therefore $^1/_{13}$ sent by STEVE AINLEY, MIKE PURTON and SIDNEY SILVERSTONE.

PROBLEM 48.1 FATHER CHRISTMAS

Father Christmas likes to start loading his sledge at noon on Christmas Eve. This gives him no trouble - he uses a point set method he got from Banach and Tarski - but first he must get the presents out of his warehouse.

The door has an electrical lock with twenty lights, numbered 1 to 20, each with its own switch. The system is so devised that a light's switch is inoperative unless the next lower numbered light is on and all the lower numbered lights are off. The first light can be turned on and off at all times. When all lights are on the door opens. All lights are initially off to save electricity.

An operation is turning on or off a light, and a team of gnomes can make one operation a second.

When must they start opening the lock?

*Hint: Try with fewer lights - find a general formula for even and odd numbers.*

PROBLEM 48.2 MODULUS ERIC LAMB

If $|x| < 0.1$ and $|y| < 0.1$ what is the probability that $|x - y| < |xy|$ ?

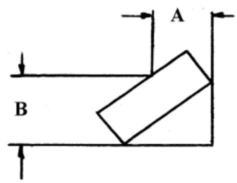
PROBLEM 48.5 SCRAMBLE

Write down a four digit number (=$p$). Scramble the figures to make another four digit number ($q$).

Tell me any three of the digits in $|p - q| = r$ (leading zero if necessary).

I can nearly always tell you what the fourth digit is. What is the probability that I get it right?

PROBLEM 48.4 RECTANGULAR PLATE STEVE AINLEY




Steve wants to know what is the largest rectangular plate which will go round a 90° corner in a corridor with arms of width A and B.

He says he can solve it for the longest straight line but the rectangle eludes him.

PROBLEM 48,5 OLYMPIAD LXVIII/l (IMO 1968)

Consider the sequence ($c_n$) defined by

$c_1 = a_1 + a_2 + a_3 + ... + a_8$,

$c_2 = a_1^2 + a_2^2 + ... + a_8^2$ ,

... ... ...

$c_n = a_1^n + a_2^n + ... + a_8^n$,

... ... ...

where $a_1$, $a_2$, ... , $a_8$ are real numbers, not all equal to zero. Being given that among the numbers of the sequence ($c_n$) there is an infinity equal to zero, determine all the values of $n$ for which $c_n = 0$.

(*This problem was taken from* Mathematical Spectrum 2/1, *page* 4. *where it is noted that of the* 108 *competitors at the International Competition, held in Yugoslavia, only one was a girl* (*from Bulgaria*!)

MMMMMMMMMMMMMMMMMMMMMMMM

HOW MANY INFINITIES? BRIAN WOODGATE

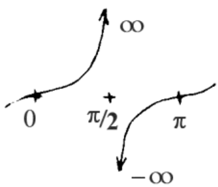
Reading Sid Finch's article (46 page 5) produced the following thoughts on infinity.

I can see no contradiction in the circle example given, as in each case we progress to a larger positive number.

Now consider two more examples:

i. the real number line $-\infty \leftarrow$------------------0------------------$\rightarrow \infty$

ii. tan θ




One way of viewing the above is to consider the 'roller' type of blackboard. The graph can then be drawn to 'join up' around the back of the board.

ie, to paraphrase Einstein, "Number space is curved".

So I would like to add one more alternative:

There is only one infinity. We just see it from different points of view.

EDITORIAL

I think we are now about up-to-date on issues; ie that this is the tenth one this year. (If I am wrong tell me somebody. I never was much good with figures.)

Harking back to Norman Lees's suggestion on page 3, I would like to ask anyone who does respond to give us their own reasons for being pleased with the book they have chosen. We do not really want to hear the author's stated intention or the publisher's sincere statement of belief. Did you hear about the little boy in a picture gallery who on being told that a certain painting was supposed to represent a man on a horse asked, well why doesn't it.

This is not a very Christmassy-looking issue; there were too many items jostling to get in and several nice things had to be left out; but this, from John Jaworsky, is essential:

Eddie Kent

In M500 46 Tony Brooks played around a little with the operation * defined by

$$m*n = m^{m^{m^{\cdots}}} \qquad \text{(taken to } n \text{ exponents).}$$

Leaving aside the more mathematical questions posed in the article I was interested to note that Tony observed how compact a notation * provided. For example 4*3 is larger than the number of elementary particles in the universe, 10*3 is larger than the largest number printable in 'ordinary' notation in a 400-page book.

Compact notations such as this bothered Turing (of machine fame) as well. He had his own contribution.

Consider the function $F_n(x)$ defined as follows:

$F_1(x) = 2x + 1$

$F_{n+1}(x) = F_n(F_n(F_n \ldots (F_n(x)) \ldots))$

where the composition is taken $F_n(x)$ times. For example, with $x = 2$ we have

$F_1\,(2) = 5$

$F_2\,(2) = 95$

and $F_3(2)$ is beyond pencil and paper calculation.

Turing was actually bothered by the comparability problem. He posed it thus. Suppose we offer a substantial prize for the person who manages to write the largest finite number on a postcard. A poor bet would be to write a sequence of very small 9s. A few factorials dotted around, mixed in with some exponents would obviously improve your chances. But how would you judge the competition? Which is larger, $F_8(9)$ or 10!!!!!!!!!!! or 2*5 or whatever?

I am prepared to offer an insubstantial prize. To the person who can write the largest finite number in ten or fewer symbols and, on being informed of all other entries, can prove that his entry is the largest. Normally acceptable notations, with * and $F$ as defined above. To avoid a trivial competition two entries have already been received:

| | |
|---|---|
| Entry $F_1(0)$: | $F_{99}(F_9(9))$; |
| entry $F_2\,(0) - 1$: | $F_{9*9}(9*9!)$. |