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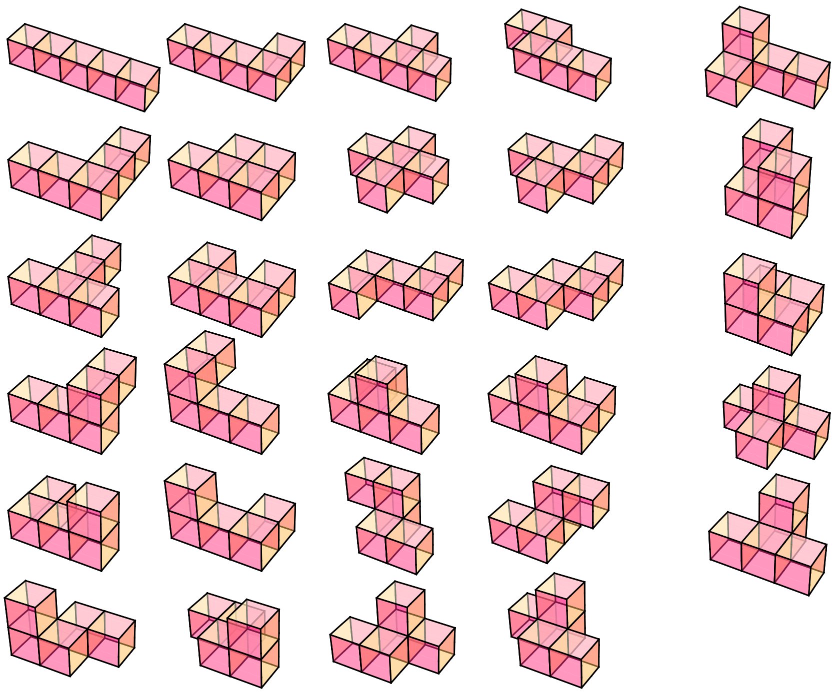
# M500 282



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# The Bedlam Cube

## Rob Evans

This article shall mainly be about how to solve the so-called 'Bedlam Cube' puzzle. This puzzle requires that one constructs a $4 \times 4 \times 4$ cube out of 13 different component pieces. See Figure BC on page 7 for illustrations of the said pieces.

Readers of this magazine whose subscription goes back far enough can confirm that this puzzle was referred to by Chris Pile in M500 **134** and written about at greater length by him in M500 **152**. The relevant articles appeared respectively under the titles 'Cube Dissection' and 'Pentacubes'.

In some ways the Bedlam Cube puzzle resembles the well-known $3 \times 3 \times 3$ 'Soma Cube' puzzle. This other puzzle has only 7 component pieces. Consequently, most intelligent people can solve it within 10 minutes and, moreover, do so without using a computer. However, the same thing most definitely cannot be said about the Bedlam Cube puzzle. Indeed, if one attempts to solve this puzzle without using a computer then one is more likely to be driven insane by it than to ever solve it!

In light of the last paragraph, let us agree to use a computer. That said, we still need to think seriously about what program (i.e. algorithm) shall ensure that our computer solves the puzzle efficiently. If we do this then we are likely to come to the conclusion that the program needs to have an appropriately designed recursive structure with as many levels as there are pieces of the puzzle.

After a proverbial 'month of Sundays', I eventually came up with just such a program. To cut a long story short, the said program was run on an old Amstrad and threw up a solution within an hour or so! However, I do not intend to give a complete listing of the program since it consists of hundreds of lines of code and would require me to provide a lot of explanation of all its details. Instead, I intend to present a very-much-simplified version of the program that highlights its recursive structure and its other fundamental features. I say 'a very-much-simplified version' since I do not want to give readers the impression that it can in some straightforward manner be translated into a working computer program.

Associated with the underlying concepts of the program we have the following useful notation:

$C$ denotes an arbitrary fixed $4 \times 4 \times 4$ cubical region of space;

$U(C)$ denotes the 64 unit cubes of which $C$ is composed;

$\mathbf{X}$ denotes a $1 \times 13$ row matrix, each of whose entries $x(1,1)$, $x(1,2)$, $\ldots$, $x(1,13)$ is a variable that ranges over $U(C)$;

$\mathbf{P}$ denotes the 13 pieces of the puzzle;

where $P_0 \in \mathbf{P}$, we have that $U(P_0)$ denotes the 4 or 5 unit cubes of which $P_0$ is composed.

Henceforth, all notation refers to how things are at a particular time in the running of our program. Next:

$L$ denotes the level of the recursive structure at which the computer finds itself;

APC denotes $\{P \in \mathbf{P} : P \text{ is configured so that } U(P) \subseteq U(C)\}$ ('APC' stands for 'are properly configured');

APO denotes $\bigcup_{P \in \mathrm{APC}} U(P)$ ('APO' stands for 'are properly occupied').

Each time $L$ increases / decreases in value by 1 an element of $\mathbf{P}$ is added to / taken from APC and the corresponding 4 or 5 elements of $U(C)$ are added to / taken from APO. Next:

if $P_0 \in \mathbf{P} \setminus \mathrm{APC}$, then $\mathbf{CONF}(P_0)$ denotes the configurations of $P_0$ for which $U(P_0) \subseteq U(C) \setminus \mathrm{APO}$;

if $P_0 \in \mathbf{P} \setminus \mathrm{APC}$ and $c_0 \in U(C) \setminus \mathrm{APO}$, then $\mathbf{CONF}(P_0, c_0)$ denotes the configurations of $P_0$ for which $c_0 \in U(P_0) \subseteq U(C) \setminus \mathrm{APO}$.

The respective contents of $\mathbf{CONF}(P_0)$ and $\mathbf{CONF}(P_0, c_0)$ vary in accord with the contents of APC and APO. However, at all times, by definition, $\mathbf{CONF}(P_0, c_0) \subseteq \mathbf{CONF}(P_0)$.

Using these concepts and this notation, a very-much-simplified version of our program is as follows.

Firstly, let APC = APO = $\varnothing$ and $L = 0$. Secondly, follow the instructions that make up the content of the next paragraph.

Firstly, increase the value of $L$ by 1.
Secondly, if APO $\neq U(C)$ then choose an element from $U(C) \setminus \mathrm{APO}$ to assign to $x(1, L)$ and follow the instruction that makes up the content of the next paragraph. If, on the contrary, APO $= U(C)$

then print off the corresponding solution.
Thirdly, decrease the value of $L$ by 1.

$(\forall P \in \mathbf{P} \setminus \mathrm{APC})(\forall \mathbf{conf} \in \mathbf{CONF}(P, x(1, L)))$ (Firstly, realize **conf** and make the corresponding changes in the respective contents of APC and APO. Secondly, follow the instructions that make up the content of the previous paragraph.)

Later in this article we shall state the criterion by which the computer chooses an element from $U(C) \setminus \mathrm{APO}$ to assign to $x(1, L)$. Until then, everything to be said about our program holds irrespective of that criterion (i.e. even if the said choices are made purely arbitrarily!).

Of course, the recursive structure of the program has 13 levels. But, in the actual running of the program, the computer spends most of its time moving up and down amongst the middle ones unable to ascend/descend to the highest/lowest ones. Nevertheless, occasionally it manages to pass through all 13 levels and print off a solution to the puzzle. Indeed, by the time our program has stopped running, it prints off each solution to the puzzle exactly 24 times! The unfortunate fact that the computer prints off each solution exactly 24 times (rather than exactly once) derives from the rotational symmetry of $C$. Later in this article, we shall modify our program so as to break this symmetry and, thus, avoid the repetition that corresponds to it.

Notwithstanding the repetition that was referred to in the last paragraph, our program runs fairly efficiently. Of course, this efficiency derives mainly from the fact that the program has an appropriate test at each level. However, the efficiency also derives from the specifics of this test, namely the requirement that $\mathbf{conf} \in \mathbf{CONF}(P, x(1, L))$.

If the said requirement were changed to $\mathbf{conf} \in \mathbf{CONF}(P)$ then the computer would discover every solution to the puzzle no fewer than $24 \times 13!$ times. Interestingly, the extra repetition implied by the factor of $13!$ would be avoided if the program as a whole required that the pieces of the puzzle become properly configured in a predetermined order. However, this alternative approach seems to leave no room for further significant improvement, and for sure leaves no room for improvement of the kind to be described next in this article!

Continuing from where we left off immediately prior to the last paragraph, we shall now consider how our program can be made still more efficient by ensuring the choice of an element from $U(C) \setminus \mathrm{APO}$ to assign to $x(1, L)$ is made wisely (rather than purely arbitrarily). Consequently, we now state the criterion for what constitutes the best choice here.

Where $b$ denotes the best choice from $U(C) \setminus \text{APO}$ to assign to $x(1, L)$, we have

$$\sum_{P \in \mathbf{P} \setminus \text{APC}} \#[\mathbf{CONF}(P, c)] \text{ is minimized for } c = b,$$

$$\text{where } c \text{ ranges over } U(C) \setminus \text{APO}.$$

If more than one $b$ satisfies this criterion then the $b$ can be chosen arbitrarily from all the said $b$s.

Unfortunately, if this criterion were adopted then, in terms of the efficiency of our program, the costs would quite likely outweigh the benefits! (In my case, given the limited working memory on my old Amstrad I could not have incorporated it into my working computer program even if I had wanted to.) However, if we bear in mind the theoretical correctness of the above criterion for $b$ then we should be able to come up with a more practical criterion for a good enough choice of an element from $U(C) \setminus \text{APO}$ to assign to $x(1, L)$. And, fortunately, intuition suggested to me an obvious example of such a criterion. In order to state this more practical criterion in a succinct manner, we introduce the following additional notation.

Firstly, let $c_0 \in U(C)$. Then $N(c_0)$ denotes $\{c \in U(C) : c \text{ has a face in common with } c_0\}$. Here, '$N$' stands for 'neighbours'; $\#[N(c_0)] \in \{3,4,5,6\}$.

Using this notation, we now state the desired criterion. Where $g$ denotes the good enough choice from $U(C) \setminus \text{APO}$ to assign to $x(1, L)$, we have

$$\#[N(c) \setminus \text{APO}] \text{ is minimized for } c = g,$$

$$\text{where } c \text{ ranges over } U(C) \setminus \text{APO}.$$

If more than one $g$ satisfies this criterion then the $g$ can be chosen arbitrarily from all the said $g$s.

Having presented the above very-much-simplified version of our program and the above criterion for $g$ we return to the unfortunate fact that the computer prints off each solution exactly 24 times (rather than exactly once). As mentioned before, this repetition derives from the rotational symmetry of $C$ and can be avoided by modifying our program so as to break this symmetry. Fortunately the required modification can be made very easily. In order to give the relevant details in a succinct manner, we introduce the following additional notation.

Firstly, let $S^+(C)$ denote all rotational symmetries of $C$. Next, on the assumption that APC = APO = $\varnothing$ and $P_0 \in \mathbf{P}$ let $\mathbf{conf}_0 \in \mathbf{CONF}(P_0)$. Then

$$\mathbf{orb}(\mathbf{conf}_0) \text{ denotes } \{\mathbf{conf} \in \mathbf{CONF}(P_0) : \exists\sigma \in S^+(C)[\mathbf{conf} = \sigma(\mathbf{conf}_0)]\}.$$

In turn, $\mathbf{ORB}[\mathbf{CONF}(P_0)]$ denotes $\{\mathbf{orb}(\mathbf{conf}) : \mathbf{conf} \in \mathbf{CONF}(P_0)\}$.

Using this notation and our previous notation, we now give the details of the required modification. Between the existing first and second paragraphs of our program insert the following new paragraph.

> Firstly, increase the value of $L$ by 1.
>
> Secondly, choose an element from $\mathbf{P}$, say $P^*$.
>
> Thirdly, $(\forall \mathbf{orb}[\mathbf{conf}] \in \mathbf{ORB}[\mathbf{CONF}(P^*)])$ (Firstly, choose an element from $\mathbf{orb}[\mathbf{conf}]$, say $\mathbf{conf}^*$. Secondly, realize $\mathbf{conf}^*$ and make the corresponding changes in the respective contents of APC and APO. Thirdly, follow the instructions that make up the content of the next paragraph.)

This wording of the required modification has deliberately been made to conform as far as possible with the general style of the above wording of the original (i.e. unmodified) program. However, when it comes to writing a working computer program we save ourselves a lot of trouble if, instead, we make a predetermined choice of $P^*$ from $\mathbf{P}$ and, in turn, a predetermined choice of $\mathbf{conf}^*$ from $\mathbf{orb}[\mathbf{conf}]$ for each $\mathbf{orb}[\mathbf{conf}] \in \mathbf{ORB}[\mathbf{CONF}(P^*)]$. And, of course, this predetermined choice of $P^*$ from $\mathbf{P}$ should be made wisely (rather than purely arbitrarily). Consequently, we now state the criterion for what constitutes the best choice here.

Where $B$ denotes the best choice of $P^*$ from $\mathbf{P}$, we have

$$\#(\mathbf{ORB}[\mathbf{CONF}(P)]) \text{ is minimized for } P = B, \qquad \text{where } P \text{ ranges over } \mathbf{P}.$$

If more than one $B$ satisfies this criterion then the $B$ can be chosen arbitrarily from all the said $B$s.

A few straightforward calculations are (as readers can confirm) sufficient to demonstrate that $B$ is in fact the piece with the shape of a three-dimensional plus sign. Specifically, we have that $\#(\mathbf{ORB}[\mathbf{CONF}(B)]) = 2$. In other words for $L = 1$, there are just two configurations of $B$ that we need to consider.

A possible clue that the approach taken by this article is the one intended by the inventor of the Bedlam Cube puzzle comes from a remarkable number coincidence. The fact that we have adopted the above modification means that the recursive structure of our program now has only 12 levels—the same as the maximum number allowed for a general recursive structure in Basic 2 (the programming language in which I wrote my working computer program!). "Not much of a number coincidence," I hear you say? However, I suspect that this programming language (unlike many more modern ones) was in existence when the Bedlam Cube puzzle was conceived. Moreover, I suspect that with respect to the relevant capability most of the programming languages that were available then were identical.

Next, we consider a somewhat subtle point that has up to now been conveniently overlooked. Throughout this article we have assumed that if the pieces of the Bedlam Cube puzzle can be put together à la *Star Trek* then they can be put together (with the same spatial relationships amongst them) without them leaving ordinary three-dimensional space. But, I am not 100 per cent sure that this claim is true! After all one can easily imagine a similar puzzle where this claim is false. (See Figure 2P for a puzzle with just two pieces, where this claim is obviously false.) In the fictional world of *Star Trek* each of the pieces could first be dematerialized and then be rematerialized in the right place! However, in the real world we come up against the obvious constraints implied by material solidity of the pieces.

With regards to the point discussed in the last paragraph, readers shall no doubt be relieved to learn that for each of the solutions that my working computer program came up with during the time that it was run for I was able to put together (with the corresponding spatial relationships amongst them) the materially solid pieces of the Bedlam Cube puzzle without them leaving ordinary three-dimensional space!

Lastly, should some readers think that the whole of this article should be dismissed as recreational they might like to consider the fact that problems of three-dimensional tessellation (that are similar in some ways to the Bedlam Cube puzzle) have been faced by those with the challenge of assembling detector equipment for use in physics experiments at the Large Hadron Collider facility in Switzerland.

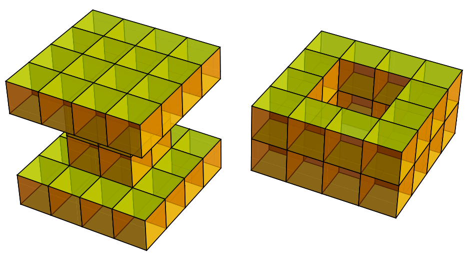


Figure 2P

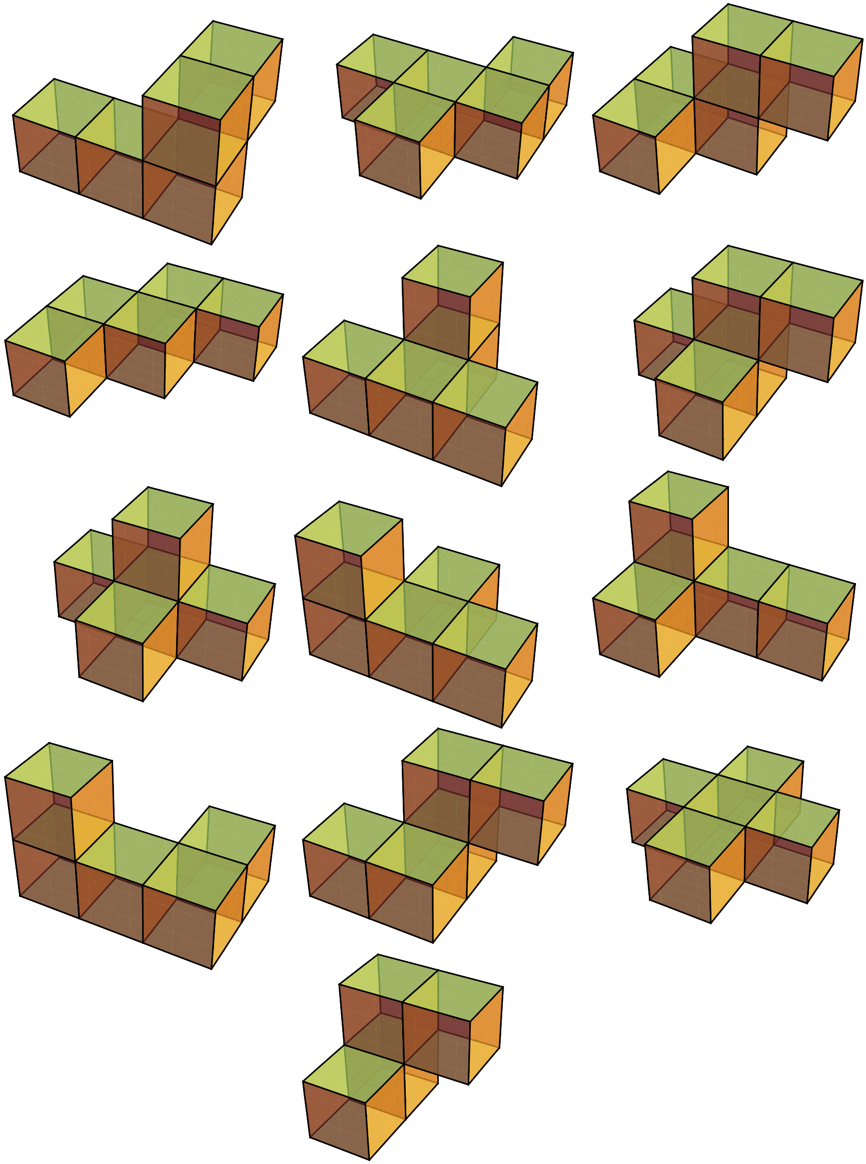
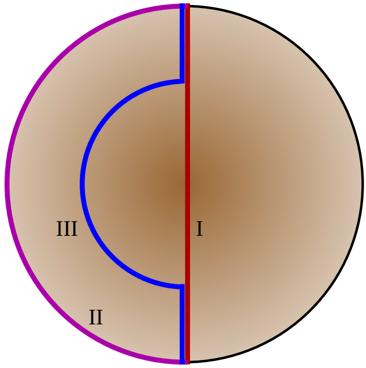


Figure BC

# Solution 280.4 – Mud




There is a circular field of radius 1. The field is quite muddy on its circumference and the muddiness increases as you go towards the centre. More precisely, if $x$ is the distance from the centre of the field, the muddiness (in some suitable units) is given by

$$m(x) = \begin{cases} \dfrac{12(\pi-2)(1-x)}{5(1+x)^4} + 1, & x < 1, \\ 1, & x \geq 1. \end{cases}$$

In particular, at the centre of the field $m(0) = (12\pi - 19)/5 \approx 3.73982$. You want to get from a point on the circumference to the diametrically opposite point. What path will minimize your exposure to mud?

The picture shows some typical routes. Your exposure to mud whilst walking in a straight line across the diameter (I) is $2\int_0^1 m(x)dx = \pi$, the same as by going around the circumference (II); hence the bizarre nature of the formula for $m(x)$. But if you combine two straight paths and a semicircle, you can do better (III). However, the optimum route is likely to be of a completely different nature.

## Dick Boardman

The problem is symmetrical about both axes. An optimum path in one quadrant can be reflected in the axis to produce the optimum solution to the whole problem. There will be two solutions, one on either side of the centre of the field. The answer to this problem is a minimum path, not just a number. This makes it a problem in the *Calculus of Variations*.

The Calculus of Variations has a long history with work done by many mathematicians including Euler and Lagrange. There exists a general solution by Euler and Lagrange which converts the unknown path into the solution of a differential equation. Unfortunately in this and many other cases the differential equation is very difficult to solve.

Euler suggests that an approximate solution can be found by splitting the optimum path into a number of straight line sections. This will produce a path which is very close to the optimum and adequate for all practical purposes. Brute force computing can compare many thousands of paths and find the best approximation.

In the statement of the problem you [TF] consider three paths, circular around the edge, straight across the middle and a combination a straight line along the axis and a semi-circular path. These paths collect varying amounts of mud.

This particular mud function has a very imprecise minimum with a lot of paths giving a result very close to the minimum.

I consider first a single straight line from $(0, y_1)$ to $(1, 0)$ where $y_1$ is the point where the line meets the $y$-axis. See the left-hand diagram, below, and note that the original picture has been rotated by $-90$ degrees. We need to integrate the function $m(x)$ over the path, and for this purpose we write down the integral over a general line segment from $(X_1, Y_1)$ to $(X_2, Y_2)$:

$$S((X_1, Y_1), (X_2, Y_2)) \;=\; \int_{X_1}^{X_2} m\left(\sqrt{x^2 + (\lambda(x - X_1))^2}\right) \sqrt{1 + \lambda^2}\, dx,$$

where

$$\lambda \;=\; (Y_2 - Y_1)/(X_2 - X_1)$$

is the slope of the line through $(X_1, Y_1)$ and $(X_2, Y_2)$. For the path in question, we need to minimize the function

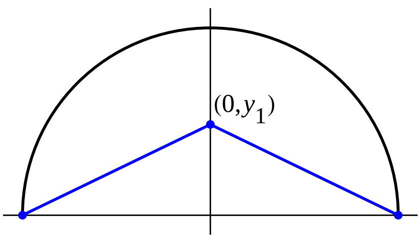
$$M(y_1) \;=\; 2S((0, y_1), (1, 0))$$

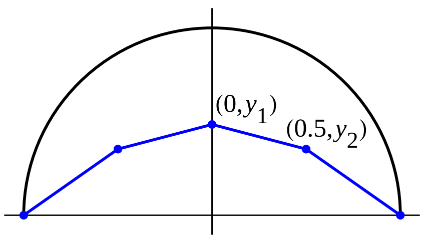
over the variable $y_1$ subject to the constraint $0 \le y_1 \le 1$. It seems that this can only be solved numerically. So I use MATHEMATICA's `NMinimize` function. The minimum occurs when $y_1 = 0.483691$ and gives $S((0, y_1), (1, 0)) = 1.30852$ and hence $M(y_1) = 2.61705$.

I next consider, two straight lines from $(0, y_1)$ to $(0.5, y_2)$ and from $(0.5, y_2)$ to $(1, 0)$ (right-hand diagram). The function to minimize is

$$M(y_1, y_2) \;=\; 2\big(S((0, y_1), (0.5, y_2)) + S((0.5, y_2), (1, 0))\big).$$

Now, `NMinimize` gives $y_1 = 0.475093$, $y_2 = 0.442376$, $M(y_1, y_2) = 2.60566$.




On repeating the exercise with more line segments, I chose to tighten the constraints in order to reduce the computational effort. Denote by $x_i$

the $x$-coordinate corresponding to $y_i$, so that for a path with $n$ segments, we have $x_i = (i-1)/n$. Then, assuming the parabola $y = -0.48(x^2-1)$ is a crude approximation for the path, we restrict the $y_i$ to

$$\max\left[0,\ -0.48(x_i^2-1)-0.2\right] \ \le\ y_i\ \le\ \min\left[\sqrt{1-x_i^2},\ -0.48(x_i^2-1)+0.2\right].$$

With four segments we have

$$\begin{aligned} M(y_1,y_2,y_3,y_4) \;=\; & 2\big(S((0,y_1),(0.25,y_2))+S((0.25,y_2),(0.5,y_3)) \\ & + S((0.5,y_3),(0.75,y_4))+S((0.75,y_4),(1,0))\big), \end{aligned}$$

and the minimum, $M(y_1,y_2,y_3,y_4) = 2.60566$, occurs at

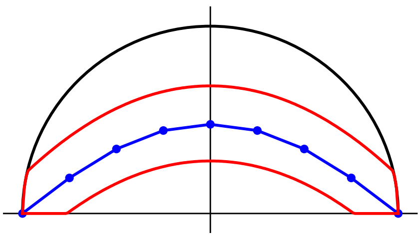
$$y_1 = 0.475093,\quad y_2 = 0.442376,\quad y_3 = 0.343862,\quad y_4 = 0.189437.$$

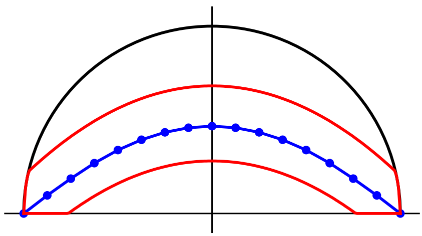
Finally I considered eight straight lines:

$$M(y_1,y_2,\ldots,y_8) \;=\; 2\left(\sum_{i=1}^{7} S\left((x_i,y_i),(x_{i+1},y_{i+1})\right)+S\left((x_8,y_8),(1,0)\right)\right),$$

giving

$$\begin{aligned} &y_1 = 0.465759,\quad y_2 = 0.457059,\quad y_3 = 0.432778,\quad y_4 = 0.393063,\\ &y_5 = 0.338012,\quad y_6 = 0.268181,\quad y_7 = 0.185955,\quad y_8 = 0.0966756,\\ &M(y_1,y_2,\ldots,y_8) \;=\; 2.60301. \end{aligned}$$





If the path is converging to something, then it seems to be doing so rather slowly. Obviously I am substituting a lot of brute force for a lot of elegant mathematics, and I have no real idea of how close to the minimum I am. Whether this matters depends on your personal taste. For a real problem, I would do a lot more computing; and computer time is a lot cheaper than engineer time these days.

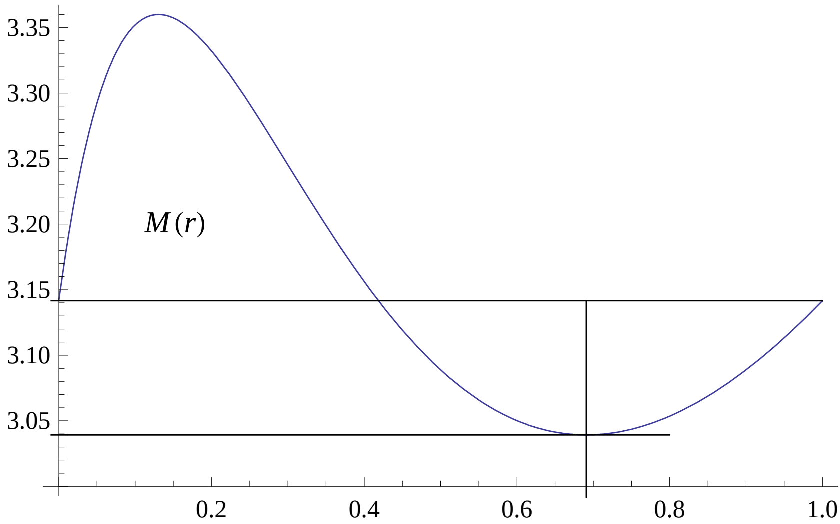
In working on this problem I have been greatly helped by a book, *Applied Calculus of Variations for Engineers* by Louis Komzsik.

## Tony Forbes

It is interesting I think to observe that even Dick Boardman's first approximation (two straight lines, left-hand diagram on page 9, where the integrated muddiness amounts to 2.61705) is better than any path of type III. Just to prove this, we compute the best route consisting of a semicircle preceded and followed by equal straight 'vertical' segments.

Let $r$ denote the radius of the semicircle in III. Then the exposure to mud is given by

$$\begin{aligned} M(r) &= \pi r m(r) + 2\int_r^1 m(x)dx \\ &= \pi r - 2r + 2 + \frac{(\pi-2)(r-1)((r^2-1)(r+5) - 12\pi r)}{5(r+1)^4}, \end{aligned}$$




and

$$\frac{dM}{dr} = (\pi - 2)m(r) + \pi r\,\frac{dm}{dr}.$$

We could try to solve $dM/dr = 0$ but that doesn't seem to work very well. Instead, by looking at the plot of $M(r)$ through a powerful microscope we see a clear minimum in the vicinity of $r = 0.69072$ and that $M(0.69072)$ is approximately 3.03927. The path for this value of $r$ is the one marked III in the diagram on page 8.

# Solution 279.7 – Two or three dice

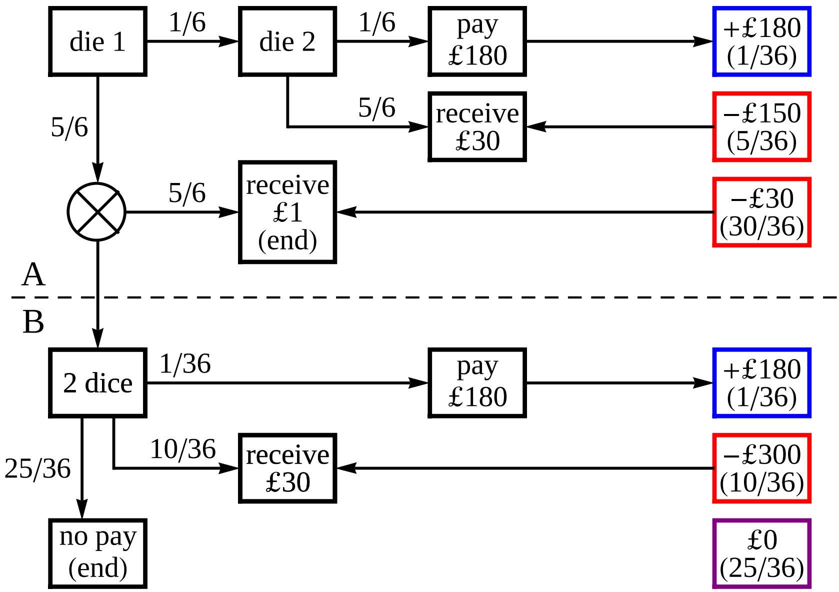
This is like Problem 274.1 – Two dice, except that there might be three dice. I offer you the following game, which is repeated many times. The rules are simple. How should you play?

I throw a die. If it is a 6, I throw a second die.
If the second die is a 6, I pay you £180;
otherwise you pay me £30.

If the first die is not a 6, I invite you to give me £1.
If you accept, the game ends;
Otherwise I throw two more dice.
If double 6 appears, I pay you £180.
If precisely one 6 appears, you pay me £30.
If no 6 appears, nobody pays anybody anything.

## Chris Pile

I don't think that I would be tempted to play this game! The rules can be summarized in the following diagram, which splits the game into two games, A and B, with game B optional. The right-hand boxes show the outcome after 36 throws.



In game B the probability of getting £180 is 1/36, but the probability of having to pay £30 is 10/36, or £300 on average over 36 throws of the two dice. Thus B loses £120 on average over 36 throws and hence it is best to avoid this game.

In game A the probability of getting £180 is again 1/36, but the probability of having to pay £30 is 5/36, or £150 on average over 36 outcomes of die 1 and die 2. However, to keep playing game A costs £1 for 30 of the 36 outcomes, or £30 over 36 throws. Thus game A in the long run breaks even. If your pockets are deep enough to survive a losing run of 35 (or more) payments of £1 or £30, there should be a time when a winning double-six comes along after a shorter run and puts you ahead. Then quit!

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# Problem 282.1 – 25 pentacubes

## Tony Forbes

Prove that any 25 distinct pentacubes can be assembled to make a $5 \times 5 \times 5$ cube. Or find a counter-example.

A *pentacube* is a solid object constructed from five unit cubes stuck together in a sensible manner. After a certain amount of experimentation one will convince oneself that are precisely 29 distinct varieties. As you can see from the illustration on the front cover (or the front cover of M500 **152**), twelve are flat (in the sense that the centres of the five cubes lie in a plane) and the remaining seventeen occur as six chiral pairs plus five which have reflection-in-a-mirror symmetry. You can make a complete set of pentacubes from 145 dice and a quantity of glue, or you can send the appropriate instructions to your 3D printer.

Can you solve the problem without actually trying each combination in turn? If not, a computer program built along the lines suggested by Rob Evans in his article, 'The Bedlam Cube', should have little trouble working through all 23751 cases. What if the 25 pieces are not necessarily distinct? The number of cases to consider is now $\binom{29+25-1}{29-1} = 903936161908052.$

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# Problem 282.2 – Isosceles triangle

There is an isosceles triangle $ABC$ with $BC$ on the $x$-axis and $A$ above it. The side lengths are $|AB| = |AC| = 191$ and $|BC| = 60$. Find all points $X_k = A + (k, 0)$ such that $k$ is a positive integer and $|BX_k|$ is an integer. Note that $AX_k$ is parallel to $BC$.

---

# Solution 279.5 – Half

> Using the digits from 1 to 9 once only, how many equivalent fractions of 1/2 can you make. Can you be sure you have got them all? For example,
>
> $$\frac{7329}{14658}.$$

## Dave Wild

The single PYTHON statement

```
', '.join(['/'.format(i,2*i) for i in range(6234,9877)
if set(str(i)+str(2*i)) == set('123456789')])
```

produces the 12 solutions

6729/13458, 6792/13584, 6927/13854, 7269/14538,
7293/14586, 7329/14658, 7692/15384, 7923/15846,
7932/15864, 9267/18534, 9273/18546, 9327/18654.

---

## Tony Forbes

And here is a suggestion if you want to do it in MATHEMATICA.

```
Do[If[Sort[IntegerDigits[20001i]]==Range[9],
      Print[i,"/",2i]],{i,6234,9876}];
```

In case it's not obvious, the lower limit is 6234 because: (i) doubling a number less than 5000 produces not enough digits, (ii) 2 times 5xxx is either 10xxx or 11xxx, (iii) 2 times 61xx is 1xxxx.

Perhaps we can make this into some kind of competition. Try to find the shortest possible code in your favourite programming language.

---

# Problem 282.3 – Array

Which positive integers cannot be represented in the form $ab + c$, where $a$, $b$ and $c$ are non-negative integers, $b = a$ or $b = a - 1$, and $c = 0$ or $c = a - 1$.

This might have some application. Suppose you wish to arrange $n$ objects in a square array. If $n$ is not a square, we must compromise. Perhaps a rectangle with $a$ rows and $(a - 1)$ columns will work. And if that can't be done, maybe we can add a column of height $a - 1$. Thus $19 = 4 \times 4 + 3$, for example.

# Problem 282.4 – Binomial identities

## Tommy Moorhouse

The following identities came to my attention when I was working on an expression for

$$\exp \circ N(n) \;=\; \delta(n) + \sum_{k=1}^{\infty} \frac{1}{k!} N^{\circ k}(n),$$

where $N(n) = n$ with $n$ a positive integer, $\delta(0) = 1, \delta(n) = 0$ for $n > 0$ and

$$a \circ b(n) \;=\; \sum_{k+m=n} a(k)b(m).$$

Here $N^{\circ k}$ is the $k$-fold product $N \circ N \circ \cdots \circ N$. Note that the operation $\circ$ is associative (so that $(a \circ (b \circ c)) = ((a \circ b) \circ c))$.

**An identity for squares** A little trial suggested to me that

$$\sum_{k+m=n} km \;=\; N \circ N(n) \;=\; \binom{n+1}{n-2}.$$

Here $k$ takes the values 1 to $n-1$. The first part of this problem is to prove this identity.

**Identities for other powers** The second part of the problem is to prove the more general identity

$$N^{\circ k}(n) \;=\; \binom{n+k-1}{n-k}$$

for $n \geq k$.

**Associative property in action** Show from the associative property of $\circ$ that

$$N^{\circ k} \circ N^{\circ m} \;=\; N^{\circ (k+m)}$$

and deduce a set of identities for the binomial expressions.

I suspect that these binomial identities would be difficult to prove more directly!

---

**Q** Why did you turn your back on the six honest serving-men?

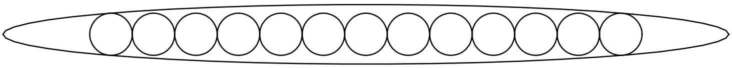
**A** Because ************** invite **************.

What are the missing 14-letter words, which are anagrams of each other?

— Sent by Jeremy Humphries

# Solution 279.9 – Circles and an ellipse

There are $n$ unit circles in a straight line. An ellipse encloses them. Show that the area of the ellipse is at least $2\pi(n-1)$.



## Tony Forbes

I put this in M500 because I thought there might be an easy answer. Well, if there is, then I am unaware of it. So instead we attempt to solve the more difficult problem of actually finding the minimum area of the bounding ellipse.

Case $n = 1$ is left to the reader and $n = 2$ was done by Dick Boardman and Ted Gore in issue **281**. We assume $n \geq 3$. If we place the circles along the $x$-axis symmetrically about the origin, the centre of the rightmost circle will be at $(n-1, 0)$. Clearly the centre of the ellipse will be at $(0,0)$ with its long diameter on the $x$-axis. Denote its radii by $a$ and $b$. To find the points of contact with the rightmost circle we solve

$$(x-n+1)^2 + y^2 \;=\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1$$

for $x$ to get

$$x \;=\; \frac{-a^2 + a^2 n \pm \sqrt{a^4 - a^4 b^2 + a^2 b^4 - 2a^2 b^2 n + a^2 b^2 n^2}}{a^2 - b^2}.$$

But we are not interested unless the ellipse touches the circle, and this happens when the thing under the square root sign is zero; that is, when

$$b \;=\; \frac{1}{\sqrt{2}} \sqrt{a^2 + 2n - n^2 \pm \sqrt{-4a^2 + (-a^2 - 2n + n^2)^2}}.$$

The plus sign gives silly results; so we adopt the minus sign and define the function

$$b(a) \;=\; \frac{1}{\sqrt{2}} \sqrt{\alpha - \sqrt{\alpha^2 - 4a^2}}, \quad \text{where } \alpha = a^2 + 2n - n^2.$$

The area of the ellipse is $A(a) = ab(a)\pi$ and we determine its minimum in the usual way. Using Mathematica (or otherwise), we compute

$$\frac{dA(a)}{da} \;=\; (\text{something}) \left( \alpha - \sqrt{\alpha^2 - 4a^2} + a^2 + \frac{a^2(2-\alpha)}{\sqrt{\alpha^2 - 4a^2}} \right)$$

and we solve

$$\alpha - \sqrt{\alpha^2 - 4a^2} + a^2 + \frac{a^2(2-\alpha)}{\sqrt{\alpha^2 - 4a^2}} \;=\; 0$$

to get

$$a_0 \;=\; \frac{1}{2}\sqrt{9 - 10n + 5n^2 + 3(n-1)\sqrt{n^2 - 2n + 9}},$$

the value of $a$ that minimizes the area.

This looks like a good time to give up. We could go on to substitute $a_0$ into the area formula and get a monstrous function of $n$. Instead we argue with less precision. A straightforward but tedious computation shows that

$$a_0 \;\geq\; \sqrt{2}\left(n - 1 + \frac{1}{n}\right)$$

and after a somewhat less straightforward and somewhat more tedious computation we obtain a corresponding inequality for $b_0 = b(a_0)$,

$$b_0 \;\geq\; \sqrt{2}\left(1 - \frac{1}{2n^2} - \frac{1}{n^3}\right).$$

Multiplying gives

$$a_0 b_0 \;\geq\; 2n - 2 + \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{2}{n^4} \;\geq\; 2(n-1).$$

| $n$ | $a_0$ | $a_0 b_0$ | $a_0 b_0 - 2(n-1)$ |
|---|---|---|---|
| 2 | $\frac{3\sqrt{2}}{2}$ | $\frac{3\sqrt{3}}{2}$ | 0.598076 |
| 3 | $\sqrt{3\left(2+\sqrt{3}\right)}$ | $\sqrt{9+6\sqrt{3}}$ | 0.403669 |
| 4 | $\frac{9+\sqrt{17}}{2\sqrt{2}}$ | $\sqrt{\frac{107}{8}+\frac{51\sqrt{17}}{8}}$ | 0.297603 |
| 5 | $3\sqrt{2}+\sqrt{3}$ | $\sqrt{9+24\sqrt{6}}$ | 0.233332 |
| 6 | $\frac{1}{2}\sqrt{3\left(43+5\sqrt{33}\right)}$ | $\sqrt{-\frac{117}{8}+\frac{165\sqrt{33}}{8}}$ | 0.191006 |
| 7 | $\sqrt{46+9\sqrt{11}}$ | $\sqrt{-71+66\sqrt{11}}$ | 0.161301 |
| 8 | $\frac{1}{2}\sqrt{3\left(83+7\sqrt{57}\right)}$ | $\frac{1}{4}\sqrt{-2826+798\sqrt{57}}$ | 0.139413 |
| 9 | $3+6\sqrt{2}$ | $3\sqrt{-39+48\sqrt{2}}$ | 0.122663 |
| 10 | $\frac{27+\sqrt{89}}{2\sqrt{2}}$ | $\sqrt{-\frac{4933}{8}+\frac{801\sqrt{89}}{8}}$ | 0.109455 |

# Measuring timber

## Colin Davies

When timber used to be measured in feet and inches (metric started to be used in the UK in the 1970s) there were two measurement systems in use.

The British were apparently the principal buyers of timber from Europe, and they required planks cut to widths and thicknesses in inches, and to lengths in feet. So the Europeans complied, and cut planks with (cross section) sizes and hence names like '2 by 4', or '3 by 9'. And the lengths were cut to the nearest foot.

As timber tends to swell and shrink a bit, partly according to its variable moisture content, and also to the angles between the directions of sawing and the tangents to the growth rings, the sizes when cut are never very precise. So as an inch is very nearly the same as $2\frac{1}{2}$ centimetres, a plank that the British called a two by four, the Europeans called a five by ten, and everyone was happy. (But in case someone queries that, a British builder would have called it a 'four by two' and not a 'two by four'.)

Then came the question of volume. I suspect that it was someone in a British office in the Russian port of St Petersburg in the late 1800s who concocted The Petrograd Standard. It was the volume in cubic feet of one gross of 5 foot lengths of $3'' \times 11''$ timber. That is 165 cubic feet.

Observe that $144 \times 5 \times 3 \times 11 = 23760$. The 'advantage' of the number 23760 is that can be divided by all integers less than 13 except 7. So timber people in the UK all knew how many feet of each size of sawn timber made a Standard (2970 feet of 2 by 4, or 1980 feet of $1\frac{1}{2}$ by 9, etc.).

So if you wanted to work out how many Standards of timber were in a consignment, all you had to do was to multiply the width (of a piece) in inches by its thickness in inches, by the total length of all the pieces in feet, and then divide by 23760.

The Canadians saw things differently. Their main market was the USA, and everyone in that continent measured the volume of sawn timber in 'Board Feet'. A Board Foot was one foot length of $1''$ by $12''$, or $2''$ by $6''$ or $3''$ by $4''$. So to get the total volume in Board Feet, you multiplied the width in inches by the thickness in inches by the total length in feet, and then divided by 12.

I personally found it much easier to divide a number by 12 than by 23760, and having worked in Canada, I tried to convince the people back home in the UK that we should use the Canadian method. But the British love complication and nobody took my advice.

When the timber trade went metric, I naively assumed that a $2'' \times 4''$ would become a $5 \times 10$ in cm. But the British like to make things complicated when possible, so while the Europeans still call (what used to be) a $2'' \times 4''$ in Imperial, a $5 \times 10$ in metric, the British call it a $50 \times 100$. And when you measure things like building timbers (and especially sheets of plywood that used to be 4 feet $\times$ 8 feet) in square millimetres, you have to cope with big numbers.

Timber makes volume measurement tricky by growing in tapering cylinders instead of rectangular planks, so to calculate the volume of timber in a log 'Hoppus's Measure' was used. Here are two examples explaining how to use Hoppus's Measure.

**Example I** Let the circumference of a tree, or piece of round timber (found by girting it), be 36 inches, one quarter of which is nine inches, the side of the square, and let the length of the piece be 40 feet. What quantity of timber is in this piece?

Find 9 inches, the side of the square, and over against 40 feet stands 22 feet 6 inches, which is the quantity of timber contained in a piece that is 40 feet long and 36 inches round.

**Example II** Let the girt of a piece of timber be 75 inches, a quarter thereof is $18^3/4$, for the side of the square. And let the length of the piece be 45 feet. How much solid timber does this piece contain?

Find $18^3/4$ inches the side of the square, and over against 45 feet, the length of the piece, stands the solid content, viz. 109 feet, 10 inches, and 4 twelfth parts or 1/3 of an inch.

I found these explanations rather difficult to use, and impossible to understand. However, during the 1970s I met Victor Serebriakoff who was also in the timber trade working for Phoenix Timber, where he called himself 'Victor Serry'. It was Victor who explained to me that Hoppus's Measure was based on the mistaken assumption that $\pi = 4$. Since 1970, the internet has been invented and gives this explanation.

> **hoppus foot** (plural **hoppus feet**)
>
> A unit of measurement for the volume of timber in the round, with a value equal to 1.27 cubic feet, where the cross-sectional area is taken as the square of one quarter of the circumference of the round.

I suppose that makes sense if $\pi = 4$ because $4/22 \times 7$ is about 1.27.

# Quiz

## Tony Forbes

Fill the gaps and identify the sources. See how many you can get before you look them up. Answers in the next issue.

(1) “Just there, number ___,” shouted Ford Prefect to the taxi-driver. “Right here!”

(2) “Come in—come in!” roared Long Jack. “It’s wet out yondher, children.” “___, ye said.” This was Uncle Salters.

(3) ___ years old was Ahaziah when he began to reign, and he reigned one year in Jerusalem. His mother’s name also was Athatiah.

(4) “Exactly half my age; I am ___.” “By the way,” said Mr. Swancourt, after some conversation, “you said your whole name was Stephen Fitzmaurice, and that your grandfather came originally from Caxbury.”

(5) Well, to speak with perfect candour, Cecily, I wish that you were fully ___, and more than unusually plain for your age.

(6) And among the cities which ye shall give unto the Levites there shall be six cities for refuge, which ye shall appoint for the manslayer, that he may flee thither: and to them ye shall add ___ cities.

(7) “___ Master Legolas!” he cried. “Alas! My axe is notched: the ___ had an iron collar on his neck.”

(8) He had ___ boxes all carefully packed, / With his name painted clearly on each: / But, since he omitted to mention the fact, / They were all left behind on the beach.

(9) “___,” replied the driver, eyeing him askant. “What!” ejaculated Mr. Pickwick, laying his hand upon his note-book.

(10) “A bicycle certainly, but not *the* bicycle,” said he. “I am familiar with ___ different impressions left by tyres. This, as you perceive, is a Dunlop, with a patch upon the outer cover.”

(11) “___!” he said with a malicious grin. “No, doesn’t work. Never mind.”

(12) “No doubt”, said I, “they settles who / Was fittest to be sent: / Yet still to choose a brat like you, / To haunt a man of ___, / Was no great compliment!”

(13) The men of Anathoth, an hundred twenty and eight. The children of Azmaveth, ___.

(14) There are generally ___ teeth in all; in old whales, much worn down, but undecayed; nor filled after our artificial fashion. The jaw is afterwards sawn into slabs, and piled away like joists for building houses.

(15) “I said so! A brave number! My fellow-citizen here would have it ___; ten more heads are worth having. The Guillotine goes handsomely.”

(16) I know thee, I know thy name, I know the names of the ___ Gods who live with thee in this Hall of Maati, who live by keeping word over sinners, and who feed upon their blood on the day when the consciences of men are reckoned up in the presence of the God Un-Nefer.

(17) “There’s more nor three hundred wanting; it’ll be a fine while before *I* can save that. Losing that ___ pound wi’ the corn was a sore job.”

(18) It was in 1862 that, in spite of the Southern Members of Congress, who wished a more southerly route, it was decided to lay the road between the forty-first and ___ parallels.

(19) “He’s got to go to London.” The man went to the telephone and rang up the bottom office. “Walter Morel’s wanted, number ___, Hard. Summat’s amiss; there’s his lad here.”

(20) Of all the things we mean to do / When Anne and I are ___.

(21) “Christ Awmighty! We all need stuff!” Ma said, “How much’d we make today?” “Dollar ___.”

(22) At the same moment the long ___-pounders upon Punchbowl Hill opened their iron throats in triumphant reply to the thunders of the five men-of-war in the harbour;

(23) And then he said, Take them alive. And they took them alive, and slew them at the pit of the shearing house, even ___ men; neither left he any of them.

(24) “Great Jerusalem, they keep turnin’ up every ten minutes or so! We thought we’d lost ___ men by straight count, but if they keep on a-comin’ this way, we’ll git th’ comp’ny all back by mornin’ yit. Where was yeh?”

(25) And there was given unto him a mouth speaking great things and blasphemies; and power was given unto him to continue ___ months.

(26) The work done was exceptionally great for ___ men. They had cut the whole of the big meadow, which had, in the years of serf labour, taken thirty scythes two days to mow.

(27) At this moment the King, who had been for some time busily writing in his note-book, called out “Silence!”, and read out from his book, “Rule ___. All persons more than a mile high to leave the court.”

(28) Mr. Chitling wound up his observations by stating that he had not touched a drop of anything for ___ mortal long hard-working days; and that he “wished he might be busted if he warn’t as dry as a lime-basket.”

(29) Each part, deprived of supple government, / Shall, stiff and stark and cold, appear like death; / And in this borrow’d likeness of shrunk death / Thou shalt continue ___ hours, / And then awake as from a pleasant sleep.

(30) “___,” said Deep Thought, with infinite majesty and calm.

04.01

## Problem 282.5 – Determinant

This came up during an interesting talk on counting trees, by Carrie Rutherford, at the London South Bank Maths Study Group (http://www.maths.qmul.ac.uk/~whitty/LSBU/MathsStudyGroup). Compute

$$\Lambda(n,\lambda) \;=\; \det \begin{bmatrix} \lambda & -1 & \ldots & -1 & -1 \\ -1 & \lambda & \ldots & -1 & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & \lambda & -1 \\ -1 & -1 & \ldots & -1 & \lambda \end{bmatrix},$$

where $n$ is the number of rows in the matrix. Hence or otherwise prove that $\Lambda(n-1, n-1) = n^{n-2}$.

**Front cover** The 29 pentacubes.