M500 NEWSLETTER No. 12 April 1974

EDITOR: (Mrs) Marion Stubbs, Southampton

UNSPEAKABLE NUMBERS - Bob Escolme, M100 (Merstham, Surrey)

Certainly at least two of the transcendentals are extremely useful (Ron Davidson on Transcendentals in M500/9), and certainly there can be no argument over a statement that pi and e are more useful than 100.11378, say. Whether or not pi and e are more useful than 1 is probably not an argument worth pursuing but most people might agree that any number has a potential use of some sort, even if it is only for the purpose of adding it to another to see what the answer is. But that is not so, as will be known to everyone who has come across that number theorem of great utility: "$\exists$ useless numbers".

The full meaning and importance of the breakthrough in number theory which this theorem represents will become apparent only after its proof, which goes something after the following fashion.

Sets A and B are said to be equinumerous if there exists a one–one function (or bijection as it is sometimes called) f: $A \to B$. If there exists a bijection g: $\mathbb{N} \to S$, between a set S and the set $\mathbb{N}$ of natural numbers, then S is said to be equinumerous with $\mathbb{N}$, or countable or denumerable.

First show there exists a bijection h: $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$, so that the Cartesian Product of $\mathbb{N}$ by $\mathbb{N}$, the set $\{(a, b): a, b \in \mathbb{N}\}$ is denumerable. (1)

This follows from the function g: $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ having assignments

$$(a, b) \to b^2 + a, \text{ if } b < a$$

$$b^2 + 2b - a, \text{ if } b > a$$

$$b^2 + b = a^2 + a, \text{ for all } (a, b) \in \mathbb{N} \times \mathbb{N}, \text{ if } b = a.$$

The problem is to show that g is a one–one function.

> From (1) above it is a relatively simple matter to exhibit the necessary bijection which establishes the following piece of mumbo-jumbo: "The disjoint union of a denumerable set of denumerable sets is a denumerable set." (2)

The weaker version of this result sounds even more gobbledy-gook:

> "The union of an at most denumerable set of at most denumerable sets is an at most denumerable set." (At most denumerable means that the set in question has n elements, where $n \in \mathbb{N}$, or the set is denumerable.) (3)

The following results are not difficult to establish: Use (1) to show that S is denumerable $\Rightarrow S \times S$ is denumerable. (4)

Find a bijection to prove that $\mathbb{Z}$, the set of all integers, +ve, –ve or zero, is denumerable. (5)

(4) and (5) $\Rightarrow \mathbb{Z} \times \mathbb{Z}$ is denumerable (6)

$\mathbb{Z} \times \mathbb{Z}$ is denumerable $\Rightarrow \mathbb{Q}$, the set of rationals is denumerable. (7)

("$\Rightarrow$" means "implies that")

Use (2) and (7) to show that the set of all equations of the form

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n = 0 \quad (a_i \in \mathbb{Q}, n \in \mathbb{N})$$

is denumerable. So the algebraic numbers are denumerable. (8)

Next it can be shown that $\mathbb{R}$, the set of all real numbers, is non-finite and non-denumerable. A proof is on the following lines:

Suppose $\mathbb{R}$ is denumerable, then any subset is also. Take [0, 1] as denumerable. It follows that its elements can be indexed by the natural numbers, so that the subset consists of the elements

$$x_1, x_2, x_3, ..., x_n, ... \qquad \text{(a)}$$

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In [0, 1] choose an interval s_1 so that its length is less than 1 and so that it does not contain the element x_1. Next, within s_1 choose an interval s_2 so that its length is less than ½ and so that s_2 does not contain the elements x_1, x_2. Generally, when an interval s_{n-1} has already been chosen, we choose it in an interval s_n so that its length is less that $1/n$ and so that it does not contain the elements $x_1, x_2, x_3, ..., x_n$. In this way we construct an infinite sequence of intervals

$s_1, s_2, s_3, \ldots, s_n, \ldots$

so that each is contained in the preceding one and their lengths tend to zero with increasing n. Then, by an imagination boggling axiom of nested intervals (where female OU maths students are to be found in the Spring?), we can assert that there exists a unique element in [0, 1] that belongs to all the intervals s. Now since by our hypothesis all the elements of [0, 1] are accounted for in (a) above, the element x which is common to all the s coincides with some element x_m of that sequence (i.e. $x = x_m$ for some $m \in \mathbb{N}$). But by our construction s_m does not contain x_m so that $x \neq x_m$ for any $m \in \mathbb{N}$. Thus a contradiction, resolved only by assuming that [0, 1] is not denumerable, with the consequence that $\mathbb{R}$ is not either.

Now by definition the transcendentals, $\mathbb{T}$, are the real numbers with the algebraic numbers, $\mathbb{A}$, deleted. The real numbers are non-denumerable. Therefore $\mathbb{T}$ is non-denumerable. If $\mathbb{T}$ were denumerable then its union with $\mathbb{A}$, which is $\mathbb{R}$, would be denumerable by result (2). Indeed we can delete the algebraic numbers from the reals, leaving $\mathbb{T}$ which is not denumerable, and then go on to delete from $\mathbb{T}$ a denumerable set of denumerable subsets of $\mathbb{T}$ and still leave a set which is neither finite nor denumerable. Thus the transcendentals provide an overwhelming majority of the real numbers: Cantor's result, quoted by Ron Davidson in M500/9.

We go on to show that among the transcendentals, which include some of the most useful numbers, there exists a Brobdingnagian set of utterly useless numbers.

First we arm ourselves with an alphabet of denumerable proportions. Let us choose the hieroglyphics a, b, c, ..., x, y, z, –, 28, 29, 30, ... (where 28, 29, 30 are, as you can see, marks on a piece of paper and not numbers). Define a word as any sequence of letters of our alphabet beginning and ending with a single '–' (thus, for example, the following are two words in our language: –number—oooooh–). And let us define a sentence as a finite sequence of words which begins and ends with '——'. Thus –oooooh– is a word while ——oooooh—aaaaah—— is a sentence. We can extend our definitions to paragraphs, to books, to libraries.

Theorem: There exist numbers which cannot be defined by a finite number of sentences.

This follows from the fact that our dictionary is only of denumerable proportions and so is the set of all finite sentences that we can construct with our dictionary, while ℝ is not denumerable. We call these numbers Johnsonian numbers.

Note that exactly the same line of reasoning which established the transcendentals to be a non-finite and non-denumerable set shows the Johnsonians to be non-finite and non-denumerable. At most only a denumerable subset of the Johnsonians are algebraic leaving a non-finite and non-denumerable set of transcendental Johnsonians.

Now the first page of our dictionary will not be very interesting (nor for that matter will the next be, nor the next, nor the next ...). However, let us agree not to use the first word to define a number, at any rate, not for the moment: let us keep –a– in reserve. Now suppose that by some non-

linguistic means we could isolate and identify just one of the huge set of Johnsonians. If we could isolate one then we could assign –a– to it, thus making it non-Johnsonian.

Remember that the transcendentals are non-denumerable, although we could form a denumerable subset of transcendentals by isolating individual numbers such as pi and e. It follows that, apart possibly, from a denumerable subset of algebraic numbers, the Johnsonians consist of the transcendentals that cannot be isolated. They must be the transcendentals we cannot think of - for if we could think of them we could isolate them.

Conclusion: a few transcendentals can be isolated and are useful. As to the unspeakable and unthinkable balance, we have established the existence of a huge set of transcendentals, not one of which we will ever construct, let alone use. And as a definition of useless that takes some beating.

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MATH-QUOTES - Ronald Davidson, M201, MDT241 (Southampton)

"...I was especially delighted with the mathematics, on account of the certitude and evidence of their reasoning; but I had not as yet a precise knowledge of their true use; and thinking that they had but contributed to the advancement of the mechanic arts, I was astonished that foundations so strong and solid should have had no loftier superstructure reared on them..."

Rene Descartes
(These days we have M201)

"An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish..."

James Joseph Sylvester

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THE INTERPOLATION OF PSI, Unit 2 - Michael Gregory

Psi was going to a Cayley... .

In the garage he tried a crippled leap-frog test onto his cycle, and discovered a physical problem with the saddle point, clearly more damping was needed in the functional unit. Instead he drove off in his Turing machine, an automorphism with a red and white ruled surface, a convex hull and an odd extension behind. Taking the critical path along the highway he saw a Catena, driven by a gentleman of Post-Canonical form, nering his trailing edge. He was prompted to wonder how he could make a Maltese cross - it's trivial - let him see a Möbius strip. They converged at the traffic lights, and the old halting problem recurred and he came to a dead halt only ø from applying the method of repeated bisection and meeting the Supremum.

Trying to restart the car he imagined a queue tending to infinity, while he played with the commutator and the vertical feed to the power set, not knowing one complex part from another. But off he went and soon reached the address. He noticed a youth with a cosh at the centre of a group. An argument about a young "Witch of Agnesi" was in progress, and remembering Galois' death Psi hoped there would not be another dual problem. Since his principal ideal was to get on with it, he ignored a do-nothing instruction, marched in at the front door and listened to a recitation of "The Rising Sun Lemma" - a poem by Milton Keynes.

Look out for the next instalment.

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PROBLEM CORNER No. 12

1. Eddie Kent

Here's a puzzle whose solution is dazzling in its simplicity: Find two integers, neither of which contains any zeros, whose product is 1 000 000 000.

2. Bob Margolis (who says it was given to him by Bob Tunicliffe at W.H.)

Scene: We are talking about finite sets of points, some of which are joined, some not. (Or, equivalently, relations on finite sets ...!)

Defn: Pn to be the following property of a set of points: Of any subset on S containing $n + 1$ points, at least 2 are joined.

Prove: S has property $Pn \Rightarrow S$ is the union of totally connected subsets.

Solution unknown. (See note below)

3. Roger Claxton (This question arises from an A-level paper)

The throwing of 2 unbiased dice gives the following sample space when the resultant face values are summed. (Let the sum = r).

SS =	2	3	4	5	6	7	8	9	10	11	12
$P(r)$ =	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Is there a formula for expressing $P(r)$ in terms of r other than by saying:

$$r \leq 7,\ P(r) = \frac{r-1}{36}\ ; \quad r > 7,\ P(r) = \frac{13-r}{36}\ ?$$

i.e. can a single formula be derived that will deal with all the values of r? (Answer unknown to me and to several other people I have spoken to.)

Ed: If you think you have a solution for No. 2, please send to Bob Margolis at OU South East Regional Office. If correct, I will publish it, but we have no space to waste on lengthy erroneous proofs, and this one looks likely to be lengthy. I certainly cannot check this one, nor indeed, any other solutions, since editorial time is non-existent. Which reminds me...

METRIC TIME - Letter received from the Editorial Mum

I wonder how we shall change to Metric Time here in this house. Never from me. We have seen nothing in our paper or on TV. Why can't things stay as they are?

(Ed: Sorry, Mum!)

LETTERS

Peter Weir

Please confirm that you took more than 2 weeks to cover M202 4/5 (groups) because I really couldn't have managed without the free week!

(Ed: Yes)

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Charles Jackson - M321 (Weymouth)

I am delighted to see that MOUTHS is now getting nationwide publicity via the Mathematics Faculty's course material; I am sure that this will enable many more students to benefit from this valuable extension to the 'self-help group' system.

As the Chairman of the Education Committee of the OUSA Council, I am keen to hear from students on all matters connected with their participation in the OU 'way of life', especially in those cases where the OUSA may be able to bring pressure to bear to improve the service to the student.

A matter which may be of more immediate and practical help to readers who are concerned with science or technology courses - since I discovered that my very elderly slide rule had a 10 inch stock and a 9.95 inch slide I have been collecting details of those slide rules present on the market. If any other reader is thinking of buying a new slide rule I should be happy to share this information with him or her - and even obtain slide rules at a discount for OUSA members.

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Eddie Kent

I have just found this: Why does physical space have three dimensions? Answer (G. J.Whitrow: "*The Structure and Evolution of the Universe*", 1959) - Intelligent life as we know it could not have evolved in a space of more than three dimensions because such spaces do not allow stable planetary orbits. That is from cosmology. And from graph theory: In less than three dimensions the maximum number of brain cells that could be interconnected without short-circuiting is 4. (see note below). (A complete graph can be planar only if it has ≤ 4 points.)

> "Thus we may conclude that the number of dimensions of physical space is necessarily three, no more and no less, because it is the unique natural concomitant of the evolution of the higher forms of terrestrial life, in particular of Man, the formulator of the problem."

Note: According to my BBB text there are approx.10^{10} brain cells each with approx. 10^4 dendrites (possibilities for connection) which gives a product of 10^{14} possible interconnections. To which one can only say "gosh".

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David Francis - Llanelli

The MST282 textbook has suffered from teething troubles with the change of units for the S.I. unit edition. In the example at the top of p. 35 we have one mile associated with 2 km, but the worked-out example ignores the mile completely. One assumes that the mile has been printed in error, this being a more tolerant outlook than the assumption that this is a deliberate attempt to puzzle the student.

However the fundamental error of confusing numbers with quantities is rife throughout the book. On p.34, line –7, we have $\sqrt{10} = 3.16$ km. The left hand side is a number and the right hand side is a length in this equation, and this is a misuse of the equality sign. This type of error occurs another twice before the end of the page is reached, but although the meaning is clear yet this is a slovenly way of writing and could easily lead to error.

It is correct in a formula involving physical quantities to allow the various letters to stand for magnitudes of these quantities. This is desirable, since we can then check whether the formula is dimensionally correct. If, however, we replace one of these letters by a number, then every letter must be defined as being a number. A-level examiners frown on any deviation from this, and it would be interesting to hear the views of our own examiners.

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Professor Ralph Smith replies ...

Return - all is not lost!

(Re M500/11). In truth I had not noticed our 500m high masted ship. Thanks for pointing it out. More galling is the fact that I have often calculated how far away high cirrus clouds could be, etc., and am well aware that the earth is round. (I don't even belong to that group which believed that we lived on the inside of a sphere – and have even forgotten the reference - but that's another story.)

The first point in M500/12 (above) concerning mixed units in "*Mechanics*" is also taken (but see units in letter referred to in para above!) There are two ripostes to the rest of the letter. First, in a perfect world your correspondent is correct; judicious uses of parentheses would have solved the problem, e.g. lines –5, –4, page 34

distance AC = $\left(\sqrt{(10^2 - 10)} = 3\sqrt{10}\right)$ km.

where the equality sign inside the parentheses represents an arithmetic equality and the one outside equality of lengths. Second, I would like to chide your correspondent a little. He has the whole of a book to aim at - I have one letter. Does he really mean ... the mile has been printed in error...? Or is it ... the word 'mile' has been printed ...? Such a slovenly way of writing will seriously confuse ..., or will it? I cannot resist one final point. The offending page contains a diagram with a distance marked as 10 km - clearly nonsense - otherwise the book would be able incorporate physically that darned mast aforementioned.

Return - all is not lost!

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AUTUMN WEEKEND BY THE SEASIDE FOR THE HALVES - Marion Stubbs

The March issue of *Sesame* contains the long-planned ad for our own special weekend at Kingsgate College, Broadstairs, Kent. This will be Sept. 27/29 (Fri. eve. - Sun. aft.) Readers of M500 or *Sesame* during 1973 will know that plans for this were afoot as long ago as last May, when it was first known that there would be no official summer schools for the 1974 Maths half credits. So we are now looking for 30 students taking M231, M251, M321 or MDT241 - regrettably, MST282 showed no interest last year, when asked by me via *Sesame*, and hence have had to be left out. Staff will be from the University of Kent, and everything possible is being done to ensure that students have what they want - final details of the programme will depend on information supplied by those who want to join the weekend. Cost will be £9 all-in.

Anyone want to meet your ed. in the buxom flesh, then? Get in quick, folks, and make this a MOUTHS get-together, not just a weekend crash revision. We can cram more than 30 in, if married couples (or unisex friends?) take double rooms, but singles are available for 30. Write to the Principal, Kingsgate College, Convent Road, Broadstairs, Kent CT10 3PX.

If anyone cannot get a place, please let me know. If there is such demand we can surely arrange another weekend somehow. Anyone wanting advanced science or technology short courses, try the Univ. of Salford Extra-Mural Dept. Their prospectus of weeks and weekends is stunning and unintelligible to a mere mathematician. The ScoA Library now has a huge box full of prospectuses of short courses - on every topic under the sun except mathematics. Write to M500 for information, naming your subject, and enclose s.a.e.

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<u>Lytton Jarman</u>

Golly, at £9 it looks cheaper to live at Kingsgate than at home. How do they do it? I think it costs £5 a day to keep someone in gaol on beans and porridge! I have already booked my place.

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<u>HOOPS - Unit 3 - All Hoops are Finite</u>. Bob Margolis

Still pursuing the idea of a Lagrange-type theorem for subhoops, thoughts inevitably turn to hoop morphisms. This is where things begin to get interesting and even fairly helpful. We begin (as usual) with:

<u>Definition</u>: A hoop morphism is a function f: $(H, \text{o}) \rightarrow (K, *)$ where (H, o) and $(K, *)$ are hoops and $f(h_1 \text{ o } h_2) = f(h_1 * h_2)$ for all $h_1, h_2 \in H$

I'll usually use 'o' for the operation in both H, K.

It would be nice if we could define 'kernel', 'normal subhoop', etc. The kernel we can't manage, because K hasn't an identity. Normal subhoop turns out to be a possibility, after a fashion. With Lagrange in mind, it's more a partition than a kernal that I'm after. This we can get from f, in fact, (shades of M100/19) any function f: $H \rightarrow K$ gives an equivalence relation on A.

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<u>Define</u> a relation R on H by

$x \text{ R } y \Leftrightarrow f(x) = f(y)$

Then R is an equivalence relation. (Proof? Or look up M100/19)

Lemma: If f: $H \rightarrow K$ is a hoop morphism, and R is the equivalence relation on H induced by f, then:

(a) The equivalence classes are subhoops
(b) Every equivalence class is of the form $x \text{ o } S$ for a fixed equivalence class S
(c) R is compatible with o
(d) o(S) divides o(H)

Proof: (a) If S is an equivalence class, $s_1, s_2 \in S$, then we know $f(s_1) = f(s_2) = y$, say. (defn. of R) Then $f(s_1 \text{ o } s_2) = f(s_1) \text{ o } f(s_2) = y \text{ o } y = y$ and $s_1 \text{ o } s_2 \in S$ i.e. S is a subhoop.

(b) Let S, T be two different equivalence classes, $s \in S$, $t \in T$. Now there is some $x \in H$ such that $x \text{ o } s = t$.

(1) $x \text{ o } S \subseteq T$

because $f(x \text{ o } s_1) = f(x) \text{ o } f(s_1)$ (morphism)
$= f(x) \text{ o } f(s)$ ($s, s_1 \in S$)
$= f(t)$ for every $s_1 \in S$.

Thus $x \text{ o } s_1 \text{ R } t$

i.e. $x \text{ o } s_1 \in T$ for every $s_1 \in S$.

(2) If $t_1 \in T$, there is a $y \in H$ (not necessarily in S) such that $t_1 = x \text{ o } y$ $f(t_1) = f(x \text{ o } y) = f(x) \text{ o } f(y)$. But $t_1 \in T$, so $f(t_1) = f(t) = f(x \text{ o } s) = f(x) \text{ o } f(s)$. Combining, $f(x) \text{ o } f(y) = f(x) \text{ o } f(s)$ cancelling, $f(y) = f(s)$ So $y \in S$ after all, i.e. $t_1 \in x \text{ o } S$ for each $t_1 \in T$. Thus $T \subseteq x \text{ o } S$.

Combining (1) and (2) gives $T = x \text{ o } S$. Thus, if we choose a fixed equivalence class for S, all the others are of the form $x \text{ o } S$ for suitable chosen x's.

(c) To show compatibility (M100/3) we must show:

($x_1 \text{ R } x_2$ and $y_1 \text{ R } y_2$) implies $(x_1 \text{ o } y_1) \text{ R } (x_2 \text{ o } y_2)$

$x_1 \text{ R } x_2$ means $f(x_1) = f(x_2)$.

Thus $f(x_1 \text{ o } y_1) = f(x_1) \text{ o } f(y_1) = f(x_2) \text{ o } f(y_2) = f(x_2 \text{ o } y_2)$.

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(d) Follows directly from (b) because all the equivalence classes have the same order as S.

One feature of the attack on hoops of order 10, 14, etc. has been the concern with the existence (or non-existence) of proper subhoops of a hoop. The order of a proper subhoop is fairly restricted, and to help in the search for non-special hoops, Sue Ahrens has found a very interesting result. We need (yet another) definition and two remarks:

Definition: A minimal subhoop M of a hoop H is a proper subhoop of H such that M has no non-trivial proper subhoops.

Remarks: (1) If a hoop H has any non-trivial proper subhoops then it must have a minimal subhoop.

(2) The intersection of two subhoops is a subhoop. It's probably worth commenting that 'minimal' doesn't mean 'least order' in this situation. For

example, if H is a hoop of order 3 and k is a hoop of order 5, then $H \times K$ is a hoop of order 15. Suppose $a \in H$, $b \in K$. $M_1 = \{a\} \times K$ is a minimal subhoop of $H \times K$ of order 5, and $M_2 = H \times \{b\}$ is also a minimal subhoop, this time of order 3.

<u>Theorem (S. Ahrens)</u>: If M is a minimal subhoop of H, then o(M) divides o(H)

<u>Proof</u>: The strategy is just like for Lagrange's Theorem: we show that $x \circ M \cap y \circ M \neq \varnothing \Rightarrow x \circ M = y \circ M$.

First note that if M is minimal, so is any coset of M. Now suppose $x \circ M \cap y \circ M$ is non-empty (and thus is a coset of either coset.) Then $x \circ m_1 = y \circ m_2$ for some $m_1, m_2 \in M$. Now there is a $u \in M$ such that $m_1 = u \circ m_2$. Then $(x \circ m_1) \circ u = (y \circ m_2) \circ u$ i.e. $(x \circ u) \circ (m_1 \circ u) = (y \circ u) \circ (m_2 \circ u) = (y \circ (m_2 \circ u)) \circ (u \circ (m_2 \circ u))$. Remember $x \circ (y \circ x) = (x \circ y) \circ x = (y \circ (m_2 \circ u)) \circ ((u \circ m_2) \circ u) = (y \circ (m_2 \circ u)) \circ (m_1 \circ u)$.

Cancellation gives $x \circ u = y \circ (m_2 \circ u)$.

But $u \in M$ so $x \circ u \in x \circ M$ and $m_2 \circ u \in M$ so $y \circ (m_2 \circ u) \in y \circ M$.

Thus we have proved that $x \circ M \cap y \circ M$ contains at least two elements. (You ought to find it fairly easy to convince yourself that this really is a new element of the intersection.)

But now we have $x \circ M \cap y \circ M$ as a non-trivial subhoop of a minimal subhoop, so $x \circ M = y \circ M = x \circ M \cap y \circ M$.

Richard Ahrens has partially solved the "Lagrange" problem by proving:

<u>Theorem (Ahrens 2)</u>: If H is a special hoop, S a subhoop, then o(S) divides o(H).

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<u>Proof</u>: This is subtle because it does not use the same sort of method as the previous proof. What is needed is an equivalence relation on H, which has equivalence classes containing o(S) elements. The 'usual' approach would be to try to show that "belongs to the same coset as" is an equivalence relation. Here we take the opposite view, and start with an equivalence relation. In fact we start with a function and use it to produce the relation:

Define: m: $H \rightarrow \{x \text{ o } S: x \in H\}$ by m: $h \rightarrow h \text{ o } S$, $h \in H$.

Then m produces an equivalence relation on H.

$x \text{ R } y \Leftrightarrow m(x) = m(y) \Leftrightarrow x \text{ o } S = y \text{ o } S$.

$[x]$ will be used to mean "the equivalence class to which x belongs".

In fact the proof shows rather more than that $[x]$ contains o(S) elements. Certainly $x \in k \text{ o } S$ for some $k \in H$. We prove $[x] = k \text{ o } S$:

(1) Suppose $k \text{ o } s_1 = x$, and $y = k \text{ o } s_2$ is another typical element of $k \text{ o } S$. Proving $x \text{ o } S = y \text{ o } S$ will involve two stages

(a) If $u \in x \text{ o } S$, $u = \text{x o } s = (k \text{ o } s_1) \text{ o } s$

$= (k \text{ o } s_1) \text{ o } (s_2 \text{ o } s')$ for suitable s' – S is a hoop

$= (k \text{ o } s_2) \text{ o } (s_1 \text{ o } s')$ special property

$= \text{y o s}''$.

Thus $y \text{ o } S \subseteq x \text{ o } S$.

(b) If $v \in y \text{ o } S$, $v = y \text{ o } s = (k \text{ o } s_2) \text{ o } s$

$= (k \text{ o } s_2) \text{ o } (s_1 \text{ o } s_3)$ for suitable s_3

$= (k \text{ o } s_1) \text{ o } (s_2 \text{ o } s_3)$ special property

$= x \text{ o } s_4$.

Thus $y \text{ o } S \subseteq x \text{ o } S$.

Combining (a), (b) gives $y \text{ o } S = x \text{ o } S$.

(2) Having proved $k \text{ o } s \subseteq [x]$ we must now show:

$y \in [x]$ implies $y = k \text{ o } S$ for some S.

$y \in [x]$ means $x \text{ o } S = y \text{ o } S$, i.e. each $x \text{ o } s \in x \text{ o } S$ is also $y \text{ o } s'$ for some $s' \in S$. We know $x = k \text{ o } s_1$ and $y = k \text{ o } t$ for some $t \in H$ (not necessarily in S).

Thus we can find some $s' \in S$ such that $x \text{ o } s_1 = y \text{ o } s'$

i.e. $(k \text{ o } s_1) \text{ o } s_1 = (k \text{ o } t) \text{ o } s$. Now $s' = s_1 \text{ o } s''$ for suitable s'' so:

$(k \text{ o } s_1) \text{ o } s_1 = (k \text{ o } t) \text{ o } (s_1 \text{ o } s'') = (k \text{ o } s_1) \text{ o } (t \text{ o } s'')$ special property
Cancelling: $s_1 = t \text{ o } s''$ but $s_1, s'' \in S$ so $t \in S$ after all; i.e. $y \in k \text{ o } S$.

Thus $[x] = k \text{ o } S$ and so has o(S) elements

There is an easy corollary to this theorem which says, in effect, that all subhoops of a special hoop are "normal" (term carefully left undefined!)
Corollary: The function m: $H \to \{x \text{ o } S: x \in H\}$ defined above is a hoop morphism.

Proof: (a) If $x, y \in H$ and $s \in S$ then $(x \text{ o } y) \text{ o } s = (x \text{ o } s) \text{ o } (y \text{ o } s)$
i.e. $(x \text{ o } y) \text{ o } S \; (x \text{ o } S) \text{ o } (y \text{ o } S)$

(b) If $(x \text{ o } s_1) \text{ o } (y \text{ o } s_2) \in (x \text{ o } S) \text{ o } (y \text{ o } S)$ then
$(x \text{ o } s_1) \text{ o } (y \text{ o } s_2) = (x \text{ o } y) \text{ o } (s_1 \text{ o } s_2)$ special property
$= (x \text{ o } y) \text{ o } s_3 \in (x \text{ o } y) \text{ o } S$.

Combining: $(x \text{ o } S) \text{ o } (y \text{ o } S) = (x \text{ o } y \text{ o } S$.

Because the equivalence classes produced by m were cosets of x, this says o is compatible with the equivalence relation, and the induced operation on the set of equivalence classes is also o. (Or, to be precise, the extended version of o used to define things like x o S, S o T, etc.)

There are some remarks about minimal subhoops which arise out of all this - but see Unit 4 which will follow later.

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Math-quotes - Ron Davidson "When I considered what people generally want in calculations. I found that it was always a number!"

Mohammed Ben Musa, c. 825

SOLUTION TO PROBLEM CORNER No. 11 - varied - take your pick!

1. Best ratio = $\sqrt{3}/2$ = approx 0.866 (Bob Margolis)

2. Let height of box/outhouse = a, and x, y not as in diagram, then $(x + a)^2 + (y + a)^2 = L^2$ (and same for Problem 3) (R. Tombs)

$x^2 + 10x + 100 = x\sqrt{1000}$ (George Russell)

Possible sides of triangle are 24.92 and 16.70 (Tom Dale)

('Easy' method involves sim. eqns.: $(x + 10)^2 + (y + 10)^2 = 900$, and $xy = 100$. (Half-page calculations)) Distance of ladder from wall = 16.7 ft or 24.92 ft (M. Stubbs) ('Easy' method involves sim. eqns.: $x^2 + y^2 = 900$, and $xy - 10x - 10y=0$, with large graphs for bit of each, and read off values from one to the other.)

3. Height of ladder = 7.55 ft. or 29.035 ft. (M. Stubbs)

4a. $(x + r)^2 + (y + r)^2 = L^2$ (r = radius of cylinder) (R. Tombs)
$y^4 - 875y^2 - 10y^3 + 900y = 576000 - 22500$ (or something) (Eddie Kent)

NEXT ISSUE: M500/13: The Hoops Course Team are taking a well-earned break, promising Unit 4 later on. John Peters writes on Determinants. We have a pattern for a knitted Klein-bottle, a few problems, a couple of letters, but very little else so far. We need your piece *now*! Typing seems to start around the 15th of the month, and ends with the MOUTHS list two weeks later - which may explain the following:

STOP PRESS: The Kingsgate prospectuses suddenly arrived here from the printers, and many have been mailed direct to you. It is hoped that this gave interested MOUTHS a head start over the *Sesame* readers. Hope no-one was surprised or aggrieved at the arrival of 'unsolicited information'? Better than *Readers Digest* or 4p-off coupons, anyway.

ROOM FOR ONE MORE LETTER - from George Russell, M231, M251 (Warwick) When you listed my previous courses in MOUTHS you omitted A100. Not that I was bothered as I took this only as I needed a second Foundation Course. After passing this, I must disagree with 'ANON' who wanted maths courses rated at 1.2 credits. This would be an insult to M100 students, and a more realistic figure would be 4 or 5, which is hardly practical. The feedback questionnaire with M231 will get some strange answers from me, for if I only spend 6 hours a week on this it will still be more than double the time spent on A100.

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Datta Gumaste I do very much share Dorothy Craggs' hope (M500/11) that students who have been exposed to M202 will given an opportunity of carrying studies of certain topics in M202 further in later years. Some future courses that immediately come to mind are:

1) Group Theory - without doubt the beauty queen of mathematics - at least a full-credit course

2) Number Theory via Algebra - just recall some glorious results in Herstein - a half-credit course

3) Galois Theory - "Elegance, thy name is Galois!" - at least a ½-credit

4) Metric Spaces - they certainly deserve a course on their own - ½-cr.

5) Algebraic Topology - remember fundamental groups? - ½-credit

6) Set Theory - who could forget that intriguing Axiom of Choice - half credit

7) Theory of Categories - the land of ultra-fantasy - at least half credit

8) Theory of Proof - which could form part of mathematical logic, and also include Gődel's proof and ancillary ideas - half credit.

Could we induce someone at Walton Hall to speak out what the tentative plans for the future are?

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FINAL NOTE: We now have 211 subscribers, still rising daily although the major 'rush' seems to be over. If anyone feels like starting a 'Fifth column' to criticise M500 it would not be unwelcome. (If you think it is a load of rubbish say so, but at the same time, do write something yourself!) The Duplicator Fund now at