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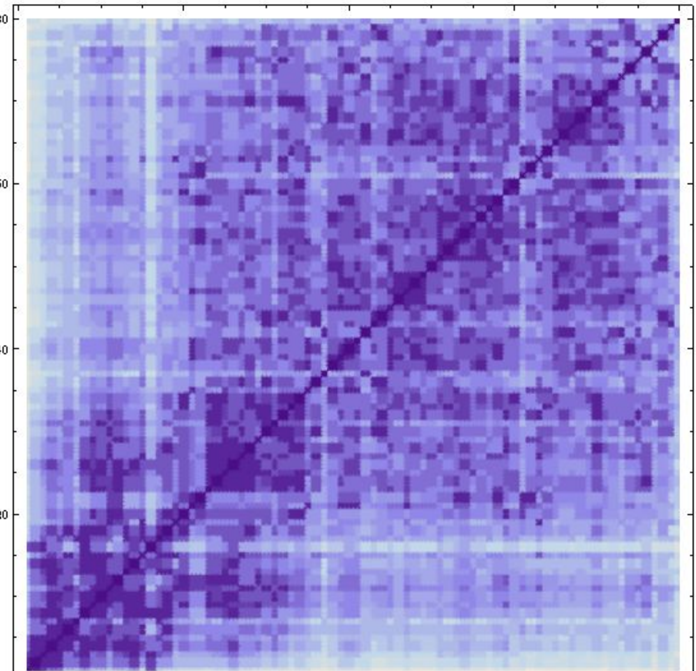
# M500 185



## The M500 Society and Officers

**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

**The magazine** M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**MOUTHS** is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

**The September Weekend** is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

**The Winter Weekend** is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

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**Advice to authors**. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

# Why does calculus work?

## Sebastian Hayes

It is fashionable today, at any rate amongst pure mathematicians, to view mathematics as a 'free creation of the human mind' which neither has nor needs any basis whatsoever in material reality. However true this may be of the more exotic branches of modern mathematics such as the theory of transfinite sets or Banach spaces, it is certainly not true of arithmetic and Euclidean geometry.

Historically, mathematics developed in response to specific social needs and pretty unromantic ones at that. It was the large, centrally controlled empires of the Middle East, Assyria and Babylon in particular, which developed both written language and written arithmetic. The reasons are not hard to see: as administrators of vast domains, these precursors of EEC bureaucrats needed efficient means of recording data. The aborigine or herdsman with his small store of worldly goods at arm's reach or grazing in front of his tent had little use for numbers, and countless tribes managed quite well with no more than the equivalent of *one*, *two* and *many* as number words.

Egyptian scribes invented geometry—the Greek term simply means 'land-measurement'—primarily in order to measure the irregularly shaped plots adjoining the Nile and tax them accordingly. And for all the supposed Platonism of Greek geometry, Euclid always has his eye on the actual construction of figures—the very first proposition of Book I is '[*how*] *to construct an equilateral triangle on a given straight line*'.

What of calculus? Archimedes, a practising civil and military engineer, developed the 'Method of Exhaustion' (which was eventually to become the integral calculus) in order to evaluate the surface areas and volumes of standard shapes. And Newton invented his method of fluxions primarily in order to plot the orbits of heavenly bodies. All this was hardly 'free creation': like arithmetic and geometry before it, calculus was a constrained invention—constrained by the way things are.

Fairly early on in the history of the calculus, a non-mathematician, Bishop Berkeley, made a number of penetrating criticisms of Newton's procedures. The Bishop's motives seem to have been religious rather than scientific. According to the usual story, annoyed by certain sceptical remarks made about Christian dogma by Halley (he of the comet), the Bishop resolved to show that the 'new philosophy' had at its very centre a mystery quite as impenetrable as that of the Trinity. Newton himself, who seems to have had serious doubts about the soundness of his own creation, could do little more than reply testily that his methods 'give the right results'—something that the Bishop did not dispute.

As time went on calculus was applied to a host of technical problems from clock-making to bridge-building, seemingly with great success. But for all that the pragmatic British (and especially Scottish) tradition of engineering and experimental science continued to regard it with suspicion and even hostility well into the late nineteenth century, considering that the 'infinitely small' was best left to the poet or theologian. (I have in my possession a very good book, *The Principles of Structural Mechanics*, published 1912, which not only ignores calculus entirely but recommends students not to bother with it.)

Obviously calculus does work; but why does it? Either its success is a pure fluke, or calculus must share certain important formal features with the observed behaviour of physical bodies. So, does calculus provide a good model of actual physical behaviour? Rather surprisingly, at first sight it seems not.

Suppose we have a machine and we are going to set it to work. Can the input we give to it be arbitrarily decreased? Obviously not: any energy input below a certain level will not be enough to overcome internal friction and so no work will be done whatsoever. (To think otherwise is to quarrel with the second law of thermodynamics.)

Are the roles of energy input and work done interchangeable? No, they are not: output depends on input but input does not depend on output except in sophisticated machines which have feedback devices and even then only to a small degree.

In the mathematical treatment, however, $\delta x$ and $\delta y$, representing the increments in the independent and dependent variables respectively, can always be arbitrarily decreased, at any rate in 'continuous functions'. This means that matter, forces, time, space and so forth can be chopped up into 'infinitely small' segments—can one really believe this? Even if one could, the assumptions of calculus are obviously wrong if we are dealing with phenomena that are known to be discrete. But calculus is used all the time in molecular thermodynamics even though $\delta n$ can, in reality, never be less than 1, that is, a single molecule. The same goes for population studies.

Also, in calculus the independent and dependent variables $x$ and $y$ can be, and frequently are, inverted at will: this means in realistic terms that effects can cause causes, which is fatuous.

Again, what are we to make of a function which is 'equal to its own derivative'? This would seem to imply that certain species or physical systems can grow instantaneously—but even the fastest reproducing viruses take a quarter of an hour before they split in two and a forest fire still needs time to ignite the neighbouring trees.

At the risk of mortally insulting the mathematical reader, I propose to go 'back to basics' and re-examine the main procedures of calculus while bringing to the task the minimum of preconceived ideas, especially geometrical. How does an increment in the dependent variable $\delta y$ change with respect to an increment in the independent variable $\delta x$? This is the main problem to which Newton and Leibnitz addressed themselves. Now, if $y$ is strictly proportional to $x$, $y = Ax + C$, with $A$, $C$ constants, then the rate of change will be $((A(x + \delta x) + C) - (Ax + C))/\delta x = A$ and so will remain the same no matter how large or how small we make $\delta x$. In such a case we do not need calculus. In every other case the so-called derivative can only be determined by discarding non-zero quantities and so does not give the exact rate of change.

We have—on the assumption that the function can be represented as a power series which is valid for most cases of practical importance—

$$\delta y \;=\; f(x + \delta x) - f(x) \;=\; G(x)\delta x + f_1(\delta x)^2 + f_2(\delta x)^3 + \ldots.$$

Here, $G(x)$ is some function in $x$ not involving $\delta x$, and $f_1(\delta x)^2$, $f_2(\delta x)^3$ and so on are some functions in $(\delta x)^2$, $(\delta x)^3$, etc. What one obviously wants to do, in order to obtain a formula for the 'rate of change', is to divide right through by $\delta x$ and then get rid of all these unwanted expressions in $\delta x$ on the right hand side. We then call $G(x)$ the derivative and write it as $dy/dx$. In the simple case of $f(x) = x^2$ we obtain

$$\delta y/\delta x \;=\; 2x + \delta x, \qquad dy/dx \;=\; 2x.$$

This is precisely what Leibnitz did. The trouble with this procedure is that it involves simultaneously setting $\delta x$ to zero on the right-hand side and non-zero on the left hand side [1]. 'By virtue of a twofold error, you arrive, though not at science, yet at the truth!' as Bishop Berkeley exclaimed in wonderment.

Turning now to the 'integration' of a function, doing it the hard way from first principles, we find that, in like manner, we arrive at the desired answer only by discarding a stream of expressions in $\delta x$ or $1/n$. For example, when looking for the primitive, of $x^r$ we retain only the leading term $x^{r+1}/(r + 1)$ and don't bother about the rest. Rather than look at this in more detail, it is more instructive for us to examine the fundamental theorem of calculus which allows us to use our previous knowledge of derivatives (which are much easier to determine).

Imagine some physical system which changes according to some variable quantity, time, fluid pressure, anything quantifiable, and whose state is known at two points at least. Mathematically, we have a function $f(x)$ which is clearly defined (and finite) for $x = a$, $x = b$. The difference, $f(b) - f(a)$ is just a number. It may be that $f(x)$ is defined at hundreds of thousands of intermediate points $a < x < b$ or at none at all, this is irrelevant. We now divide up the interval $b-a$ into $n$ subintervals which, for the moment at least, we assume to be equal; furthermore we suppose $f(x)$ to be well defined (and finite) at each of these points namely at $a+(b-a)/n$, $a+2(b-a)/n$, $a+3(b-a)/n$, ..., $a+(n-1)(b-a)/n$.

Also, at each one of these points we assume that each increment, the difference between successive values of $f(x)$ (the state of the system), can be treated as a function of the subinterval $(b-a)/n$ and can, much as before, be brought into something like

$$\frac{b-a}{n}\left(g(x)+h\left(\frac{b-a}{n}\right)\right),$$

where $g(x)$ is some function in $x$ not involving $(b-a)/n$, and $h\left((b-a)/n\right)$ is some function in $(b-a)/n$ not involving $x$.

For ease of treatment I shall from now on take $a=0$, $b=1$ but the overall argument is clearly not affected.

Provided the foregoing assumptions hold for increasing values of $n$, $n = 1, 2, \ldots$, we have an exact equivalent of $f(1)-f(0)$ in terms of the sum of successive increments, that is,

$$f(1)-f(0) \;=\; \frac{1}{n}\sum_{x=0}^{1-1/n} g(x) + \frac{1}{n}\sum_{r=0}^{1} h_r\left(\frac{1}{n}\right),$$

where the latter summation concerns the functions in $1/n$, not assumed at this stage to be all the same.

There has, note, been no mention of rectangles, upper and lower limits, nor that the equivalent of the familiar $\delta x = 1/n$ is necessarily small. If now we equate $g(x)$ with the derivative of $f(x)$, we see that the usual formula for a definite integral between $x=a$ and $x=b$ differs from the above in two respects. First, we have got rid of all the functions in $1/n$ and we have added on an extra slice, as it were, since the increments between $a$ and $b$ stop just before $b$ and do not include the value at $b$ itself.

Taking a specific example, if $f(x)=x^2$, we can compare the results of (1) a straight subtraction; (2) integration using the fundamental theorem of calculus.

(1) $f(1)-f(0) \;=\; 1,$

(2) $\displaystyle \frac{1}{n}\left(2\frac{1}{n}+2\frac{2}{n}+2\frac{3}{n}+\cdots+2\frac{n}{n}\right)$

$$= \frac{2}{n^2}(1+2+3+\cdots+n) \;=\; \frac{2}{n^2}\cdot\frac{n(n+1)}{2} \;=\; \frac{n+1}{n} \;\approx\; 1.$$

And so, for any $n$ the formula for ‘integrating via anti-differentiation’ does not give the exact ‘definite integral’ $f(1)-f(0)$.

The mathematical solution, of course, not obtained until the mid-nineteenth century, is to view both the derivative and definite integral as 'limits' since the way a limit is defined cunningly side-steps the issue of whether the limiting value is actually attained or not. Calculus thus becomes a tissue of 'limit equivalences' and not, as one would prefer, numerical equivalences: in a really rigorous treatment this would be signalled to the student by employing the sign '=lim=' meaning 'limit equivalent' instead of the bald '='.

So how did it come about that geniuses like Leibnitz and Newton were incapable of spotting what most sixth formers today absorb in a couple of lessons? There is good reason for this: Newton and Leibnitz did not have that indifference towards the real world that has become the mark of the modern pure mathematician. They wanted to understand reality and believed that a mathematical system, if successful in predicting observed behaviour, must in some sense represent what actually goes on in the real world. Now the analytical treatment requires the wholly unrealistic assumption that $\delta x$—and hence $\delta y$ also—can be made arbitrarily small. Leibnitz in particular, who always dealt in definite ratios between definite quantities, could not rid himself of the conviction that there must be a final ratio between quantities too small to be observed and that this ratio was given by what we now call the derivative. And Newton, hard pressed to explain how a perpetually varying quantity can nonetheless have a specific value at a specific moment, argues in effect that an accelerating body, say, undoubtedly has a specific—and thus not variable—momentum at the precise moment of impact and that in principle any moment can be a moment of impact. It is this 'moment of impact' momentum that is given by the 'fluxion'.

We now know that energy is quantized so there is always a lower limit to all energy transfers. It is Leibnitz and Newton who have been proved right and analytical calculus wrong.

So why does modern-day calculus, the analytical version, nonetheless continue to deliver the goods? The succinct answer is that in practice it doesn't make too much odds so long as the increments are small, but that in a certain number of cases calculus does not deliver the goods.

The so-called infinitesimal calculus was developed to deal with a situation where, typically, we have two widely different scales of values, macroscopic and microscopic, and where moreover the exact values of the microscopic quantities are completely unknown—as they all were, of course, in Newton's day. By 'infinitesimal' we must today simply understand 'quantity that is small by macroscopic standards'. In the mathematical treatment we have to work with the microscopic—since the macroscopic changes generally originate at this level—but we also have to call a halt somewhere if we want to get something tangible to work with. At what point should we stop?

How do we know where to draw the line? This is a matter to be decided by the practising physicist or engineer: the business of the mathematician is to provide a coherent model which can be adapted to circumstances—or alternatively discarded as inappropriate. We, as human beings, consider that if $\delta n$ is a single person this quantity is not negligible, especially if it is someone we know. But if we are dealing in millions, as in world population studies, such a quantity is negligible and so the methods of calculus give perfectly acceptable results. Not always though. The better textbooks—and this includes the OU course on solids and gases, S272—warn the student about the dangers of using analytic methods blindly beyond a certain level of precision because they do not give the right answers.

Traditional geometry deals with the apparently continuous, while arithmetic, and by implication all numerical calculation, deals with the discrete. The standard forms of Greek geometry, circles, cylinders, ellipses and so on are 'ideal figures': they are not to be found in nature and can only be manufactured within a certain degree of tolerance. The formulae used to evaluate the areas and volumes of these 'ideal' figures are also 'ideal'—ideal in the sense that any actual measurement of an actual cylinder will almost certainly not give us what the textbook says it should. (In effect the Greeks were employing the limit concept without realizing it.) Employing the fundamental theorem of calculus, itself an ideal formula, is in this context perfectly appropriate. However, when dealing with the successive energy states of a physical system, or with the growth of an animal population, we are not dealing with ideal states of affairs even in theory.

Also, an area or volume, even a line, is 'all of a piece'. We draw the curve with a sweep of the pen or cursor and there it is in front of us: we work out the values of individual points and the properties of the figure later. In real life the reverse is the case. The 'whole' is never present to our eyes in the way a curve is: reality has to be built up piecemeal. In practice we only know the state of a system at a few moments in time: we then try to guess a suitable formula and interpolate. All science involves interpolation and always will do.

For these and other reasons the analogy between the successive configurations of a physical system and the 'area beneath a curve' should not be pressed too far.

With this proviso the apparent paradoxes of calculus vanish. There is for example no function whose derivative (or integral) is exactly the same as the function itself. If we actually write out $e^{t+\delta t} - e^t$ to $n$ terms and then divide by $\delta t$ we do not obtain $e^t$ and so there is no need to believe that there are biological or physical systems which increase instantaneously though there are many systems whose rate of growth is very nearly proportional to their size.

The centuries old 'mystery' of calculus amounts to little more than a confusion between the varying requirements of pure and applied mathematics. The pure mathematician seeks consistency, generality and elegance while the engineer and scientist want fidelity to the facts. In the pure-mathematical model $\delta x$ is quite properly left as a free variable without a lower limit [2] even though in practically all applications it will have a precise non-zero value which more often than not these days is actually known. Instead of being 'God's shorthand', calculus simply turns out to be an ingenious method of getting approximately true results when we do not know the values of certain small constants. Today, for really accurate work, the tendency is, increasingly, to slog it out numerically using computers. *Sic transit gloria mundi.*

This is not quite the end of the matter though, at least for those of us who are concerned about the true nature of the world we live in. The strictly mathematical problems of the calculus were successfully dealt with a hundred and thirty or so years ago but only at the cost of sweeping the conceptual problems under the carpet, where they remain. On paper, the independent variable can be left without a lower limit or made as small as we wish, but natural processes cannot be made arbitrarily small at the behest of a mathematician; they are what they are.

In calculus we hear a great deal about so-called 'continuous functions'. Indeed mathematically we probably could not get on without them just as in geometry we could not get on without, for example, lines that touch curves 'at one point only'. But are there in actual fact any 'continuous' processes or phenomena? Matter and energy we now know to be discontinuous—what of 'time and space'? Most physicists today meekly follow the lead of the pure mathematician with respect to space but there is a contemporary school of thought which holds that, at a certain level, 'space is grainy'. And Gerald Whitrow, the author of several books on time, introduces the chronon as a sort of 'atom of time'. He defines it as the 'smallest interval of time during which any observable physical change can occur', and suggests that its value should be given by (diameter of an elementary particle)/(speed of light).

This whole approach, which one might call the Discontinuous Theory of Space and Time, is by no means new though it has never found much favour in the West. During the early centuries of the Christian era, a flourishing school of Indian Buddhists at Nalanda, where there was the equivalent of a university, maintained that the whole of reality was reducible to 'a mass of point-instants' (dharma) along with causal laws which governed their successive appearance and disappearance (karma). There was nothing permanent or continuous (except possibly nirvana, which is a rather special type of entity). A thousand years later Descartes proposed something very similar with his doctrine of Perpetual Creation whereby God creates the universe out of nothing at every instant. So the world dies and is reborn eternally—exactly the Buddhist conception except that, for Buddhism, there is no creative agent, natural or supernatural. Descartes might even have developed this conception in unpublished writings that he suppressed for fear of censure.

The 'Jewel in the Crown' of the 'new mathematics' of the seventeenth and eighteenth centuries was that it could deal with movement: the Greeks formulated the basic concepts of statics correctly but were unable to make the leap into dynamics, probably because they were too logically minded. For at the root of Newtonian dynamics and the calculus itself lies the enigma of motion, something that modern science has tamed but done nothing to elucidate.

In the early twentieth century the French philosopher Bergson made a critique of calculus in its way as pertinent (though not as useful) as that made centuries before by Berkeley. If, Bergson argues, the trajectory of a moving body really were continuous the body would never occupy a precise position between its original and final state of rest. Calculus has its cake and eats it too: the moving body is always in motion and yet always at a particular spot. Philosophically, if not mathematically, we must make our choice. Bergson opts for continuity: the moving body is only really somewhere when it is stopped short. This is an acceptable point of view as far as it goes but it is not a fruitful one mathematically—if a line is truly continuous (and thus not made up of points at all) there is seemingly not a lot to be said about it.

Alternatively, we can adopt the Postulate of Radical Discontinuity. On this view, if by 'motion' we understand continuous change of position (the usual sense) then there is no motion, only a succession of stills that gives the appearance of movement as on a cinema screen. This approach eliminates all the logical objections put forward by Zeno in his famous paradoxes. The closing door does not, for example, have to traverse an 'infinite' (or is it transfinite?) number of spatio-temporal locations before it clicks shut in about three seconds, but only a specific (finite) number of positions. The 'moving' arrow proceeds on its way by jerks until it finally embeds itself in the target; Achilles always manages to overtake the tortoise because time is not infinitely divisible, and so on.

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[1] Mathematically, of course, $dx$ cannot be zero in $dy/dx$ on the l.h.s. because division by zero is not defined. But rather more significantly from a realistic point of view, if $dx$ is a special case of the independent variable $\delta x$—and this is how Leibnitz and everyone else at the time viewed it—it cannot be strictly zero, since if it is, nothing at all is happening and there is no increment and no rate of change.

[2] In non-standard analysis, developed by Robinson, there is a lower limit, a single 'infinitesimal', 'smaller than any real', that appears everywhere. But Robinson does not believe in his infinitesimal any more than I do: he is simply concerned to develop an interesting and consistent mathematical theory. He states categorically that he has not invented 'new objects but only new deductive procedures'.

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# Problem 185.1 – Three strings

## Jeremy Humphries

You have three pieces of string and a box of matches. A piece of string takes exactly two hours to burn from end to end. That is, if you set light to one end of the string (using a match), the flame will reach the other end precisely two hours later. However, the flame does not necessarily travel along the string at a constant speed.

Using just the three strings and the matches, how can you time 105 minutes exactly?

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# Problem 185.2 – Two streams

## ADF

Once upon a time I lived in a house with this simple hot-water system. Water from a tank (and therefore at a well-defined constant pressure) flows through a perfectly insulated pipe and splits into two paths. One path goes directly to a tap at the kitchen sink marked 'cold'. The other enters a heating appliance and then proceeds to a tap marked 'hot'. On the other sides of the taps the two streams merge into a single outlet.

If I turn on the hot tap and wait until the system stabilizes, very hot water pours from the outlet. If I now turn on the cold tap as well, the temperature drops to a reasonable level.

*Question*: Why?

The reason I ask is that I am puzzled. Surely this cannot happen. For example, suppose the opening of the cold tap splits the water stream 1:1. Half of the water passes through the heater at half speed and spends twice as long getting hot. Therefore its temperature rises by twice the amount. So when the hot water meets an equal volume of cold water at the common outlet the rise in temperature averages out. No change.

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'Every mathematics master dreads the day when he will have to explain the Theory of Pythagoras to boys who have never met it before.'—H. F. Ellis.

# Solution 183.2 – Fifteen objects

> There are fifteen objects to be painted red, yellow or blue. In each case the colour is chosen at random with probability 1/3.
>
> What is the probability of five red, five yellow and five blue?

## Andrew Pettit

Initially I thought that the title of this problem was a line from the 60's cult classic *The Prisoner*—but I digress!

The following three solutions—or variations on a theme—give the result

$$\frac{28028}{531441} = 0.05273962\ldots.$$

***Solution 1*** Taking the definition of the required probability as

$$\frac{\text{Number of ways of painting 15 balls with 5 red, 5 yellow and 5 blue}}{\text{Number of ways of painting 15 balls any colours}}.$$

There are $\binom{15}{5}$ ways of choosing to paint five of the fifteen balls red, $\binom{10}{5}$ ways of choosing to paint five of the remaining ten balls yellow and $\binom{5}{5}$ ways (that is, one way) of choosing to paint all the remaining five balls blue.

There are $3^{15}$ ways of choosing to paint the fifteen balls any colour, so the solution is

$$\frac{\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}}{3^{15}} = \frac{15!}{5!\cdot 5!\cdot 5!\cdot 3^{15}},$$

which leads to the result shown above.

Note: If we had required six red balls, five yellow balls and four blue balls, the result would have been $15!/(6!\cdot 5!\cdot 4!\cdot 3^{15})$. However, if it had simply been six balls of one colour, five of another with the remainder being painted the third colour, this answer would have needed to be multiplied by 3!, which is the number of permutations of the three different colours.

***Solution 2*** A more elegant way of deducing the numerator is to use $(r+y+b)^{15}$ as a generating function and to determine the coefficient of $r^5y^5b^5$.

By treating $y+b=w$, say, it is possible to use the binomial expansion of $(r+w)^{15}$ to find the coefficient of $r^5w^{10}$, which is $\binom{15}{5}$ and then to find the coefficient of $y^5b^5$ in the expansion of $w^{10} = (y+b)^{10}$. Clearly the latter will be $\binom{10}{5}$, giving the same numerator as shown above.

***Solution 3*** The most satisfying solution to the problem is to proceed as in Solution 2, above, to evaluate the coefficient of $r^5y^5b^5$ and then setting $r = y = b = 1/3$ to evaluate the probability. This approach allows for the situation when the probabilities of each colour are different, and clearly demonstrates that the sum of the probabilities is 1.

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Solutions along the lines of variation 1, above, were received from **John Bull**, **Chris Pile**, **Keith Drever**, **David Kerr** and **Ted Gore**.

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# Solution 181.4 – Four points

Choose two points inside a given circle and draw the line segment joining them. Then randomly select another two points inside the same circle and draw the line segment joining these two points. What is the probability that the two line segments intersect?

## ADF

Not really a solution—just more questions. This is very interesting.

I discovered a similar problem on 'sci.math.research' except that it had a *square* boundary instead of a circle. The solution to the square version was given by Robert Israel as 25/108, being one third of 25/36. This agrees with 'around 0.23', the result of a simulation carried out by Warut Roonguthai. The factor 25/36 is the 'well-known' probability that a quadrilateral drawn at random is convex. The 1/3 arises thus: For a given quadrilateral there are three ways to choose two pairs of vertices. If the quadrilateral is convex, in exactly one case the lines joining the pairs cross. If not, the lines joining vertex pairs never cross.

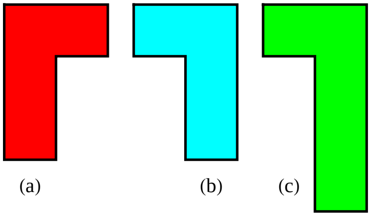
Then I drew a million pairs of lines under the conditions of Problem 181.4 and I was surprised to find more or less exactly the same answer. So two questions:

1. Can someone remind us of the well-known proof that

   $\Pr$ (a randomly drawn quadrilateral is convex) $= 25/36$?

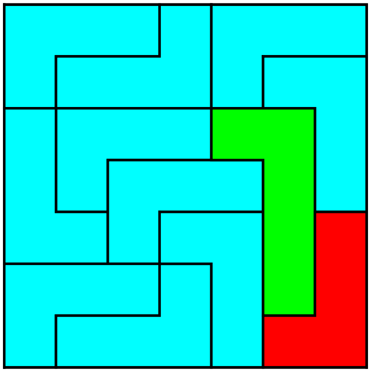
2. Is the shape of the bounding region relevant?

# Seven a side




Arrange one piece of type (a), ten of type (b) and one of type (c) to form a $7 \times 7$ square. You are not permitted to alter the handedness of any piece.

## ADF



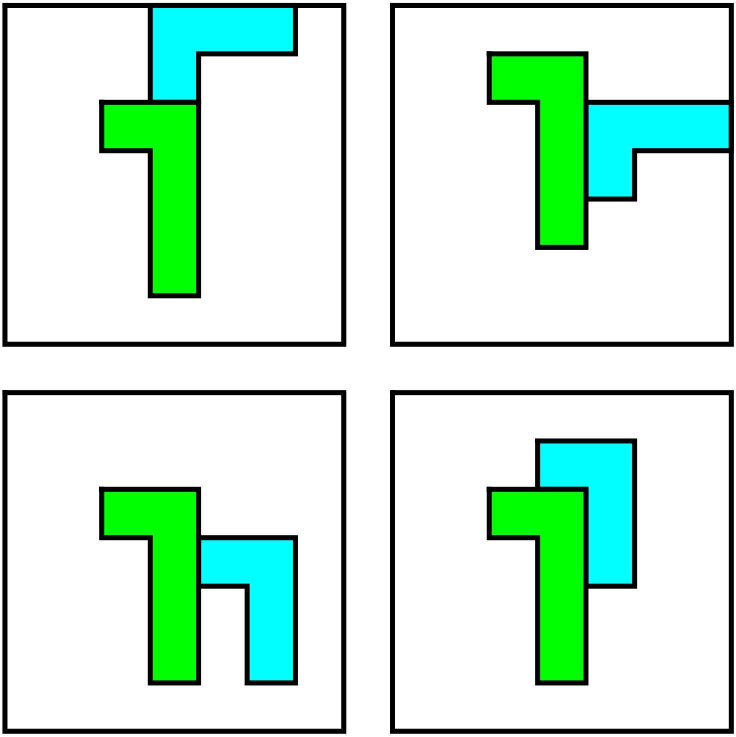
**Dick Boardman**, **John Smith** and **Chris Pile** used computers to perform an exhaustive search by some kind of backtracking algorithm. They found this solution, right, and declared that it is unique (apart from rotations). Dick and John also confirmed that the problem has no solutions if all the 4-square pieces are of the same type, either all (a) or all (b). **John Hulbert** and **Sue Bromley** also found the solution.

Chris points out that an unassisted solution should not be too difficult. He says: ‘The key to this must be the 5-square. If the 49 small squares are coloured alternately with, say, the corner squares black, then the 5-square must cover three black squares. There are thus only 12 possible positions for the 5-square, and four of these leave an area which is not tileable with the available pieces.

Having placed the 5-square, a good strategy is to inspect the remaining area for non-tileable subsets; for example, a $3 \times 4$ rectangle cannot be tiled. Also, placement of subsequent pieces is often forced—to avoid leaving isolated areas.’

John Smith went on to consider other combinations and he discovered that whoever devised the problem had selected the most interesting set of parameters. In each case apart from those which we have already discussed the problem has more than one solution. The results are tabulated opposite. John’s four solutions with ten pieces of type (a) and one piece of type (b) are given on the next page. To preserve some element of mystery the (a)s have been omitted.

| (a) pieces | (b) pieces | solutions |
|---|---|---|
| 0 | 11 | 0 |
| 1 | 10 | 1 |
| 2 | 9 | 11 |
| 3 | 8 | 12 |
| 4 | 7 | 34 |
| 5 | 6 | 31 |
| 6 | 5 | 43 |
| 7 | 4 | 23 |
| 8 | 3 | 21 |
| 9 | 2 | 4 |
| 10 | 1 | 4 |
| 11 | 0 | 0 |



Solutions with ten pieces of type (a), one of type (b)

# Problem 185.3 – 23 numbers

Two things.

(i) Partition the integers 1 to 23 into three sets such that for any three different numbers $x$, $y$, $z$ in the same set, $x + y \neq z$.

(ii) Can you do the same for 24 integers?

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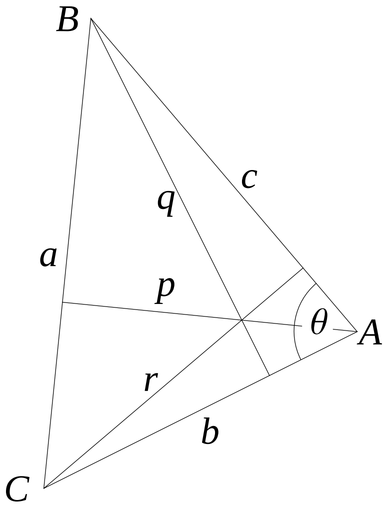
To see how it works, observe that the integers 1 to 17 can be partitioned into 1, 3, 5, 7, 9, 11, 13, 15, 17, 2, 4, 8, 16 and 6, 10, 12, 14, which have the stated property. That was easy; 23 is somewhat harder. The problem appeared on one of the 'sci.math' news groups but as far as I can remember there was no explanation of 23, the main parameter of the problem. Hence what we are really interested in is a definite answer one way or the other to part (ii).

## Solution 183.1 – Three altitudes

Is a triangle defined by its three altitudes?

### Dick Boardman




There is a rather elegant method of constructing a triangle, given its three altitudes.

If you draw a new triangle with sides equal to the three given altitudes, draw its altitudes and make a second triangle from the new altitudes, then the second triangle will be similar to the original (to be constructed) triangle. This depends on being able to draw a triangle with the three altitudes. That is, the sum of any two must be greater than the third.

Unfortunately, the three altitudes of a triangle do not necessarily form a triangle. For example, a triangle of sides $\{10, 11, 3\}$ has altitudes which do not form a triangle. Thus my construction has only limited application.

However, consider a triangle with sides $\{a, b, c\}$, area $w$ and altitudes $p$, $q$, $r$. Using the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos\theta,$$

where $\theta$ is the angle between sides $b$ and $c$.

Now replace $a$, $b$ and $c$ by $2w/p$, $2w/q$ and $2w/r$ in this formula and the area cancels, giving

$$\frac{1}{p^2} = \frac{1}{q^2} + \frac{1}{r^2} - \frac{2}{qr}\cos\theta.$$

Thus given any three altitudes, we can find the angles of the triangle and hence its sides. This triangle is unique. But if the formula gives a value for $|\cos\theta| > 1$, the three lengths cannot be altitudes of a triangle.

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Postulate 1: Knowledge is power. Postulate 2: Time is money. Also power = work divided by time. Therefore work / knowledge = money. Thus as knowledge approaches zero money approaches infinity regardless of the amount of work done—**Lytton Jarman**.

# Letters to the Editors

## Model railways

Dear Tony,

Herewith some observations on model railways (M500 **183**). When Father Christmas brought me my first (Hornby Dublo) train set, I can remember running the engine around an oval track at top speed and discovering that it covered just about one mile in one hour. The scale of 4 mm to 1 ft, or about 76:1, convinced me that this represented a life size speed of 76 mph.

The problem with most model railway layouts is that distance is not scaled accurately, apart from some individualistic scenic items. A reasonable distance between villages of, say, 5 miles, would, in Dublo scale, be well over 100 yards, which is difficult to model convincingly on a 10 ft by 6 ft baseboard. At a speed of 1 mph (as above) or, say, 1.5 ft/sec, the model train would take almost four minutes to travel the scaled down 5 miles, which is consistent with full size.

The smaller size 'N' scale (2mm to 1 ft) allows more realistic modelling of distance, but some enthusiasts restrict modelling to a short length of track or a station yard, etc. At some exhibitions I have seen a speeded up clock in use for 'timetable' operation. Alternatively a length of 'continuous run' track is used to represent a longer journey, or the train can just be hidden in a tunnel siding for a few minutes.

I now have a garden railway (LGB narrow gauge) with a scale of approximately 22:1. At two-thirds throttle the train runs at a satisfying speed of about 3 ft/sec. If this is scaled up to 66 ft/sec, it is equivalent to 45 mph. Therefore I suggest that speed scales correctly with distance and time does not pass any quicker for Lilliputians.

**Chris Pile**

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## Cats

Issue **182**, page 25, 'Cats'. Heisenberg would point out that if it is Schrödinger's cat that has been firmly fixed to a stationary vehicle (by which I mean that the velocity of the car and cat combination is exactly zero), then the position of the car and cat must be completely unknown. If it is shown on a wide open road somewhere, that is pure speculation of course, but the lack of other traffic does confirm there were no witnesses.

It had occurred to me long ago that if a traffic warden could be persuaded to agree that the car that he was alleging was parked illegally was in fact stationary at the time of the offence, then it would follow that he would have no idea where the car actually was, so could not prove that it was parked in the wrong place.

**Colin Davies**

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## Dodecahedra

Dear Eddie,

Many thanks for M500 **183**. Something in it has stuck in my unmathematical mind and won't get out, which is Tony's offhand remark about why joining together regular pentagons results in a three-dimensional figure that closes up exactly. So I started counting sides and edges. The first thing that strikes one is that in two dimensions, 3-, 4- and 6-sided regular polygons will tile on a flat surface. But in three dimensions, 3-sided figures will make closed polyhedra in three ways, 4-sided figures in one way, and 5-sided figures in one way, but not 6-sided figures. The numbers of bits don't look at all promising:

| | |
|---|---|
| tetrahedron: | 3-sided face, 4 faces, 6 edges |
| cube: | 4-sided face, 6 faces, 12 edges |
| octahedron: | 3-sided face, 8 faces, 12 edges |
| dodecahedron: | 5-sided face, 12 faces, 30 edges |
| icosahedron: | 3-sided face, 20 faces, 30 edges |

I can really see no pattern in this.

In four dimensions, can there be hyper versions of all the 3-D figures, as there certainly is for a cube? For instance, if you erect a dodecahedron on each facet of a three-dimensional dodecahedron (or however one expresses this), will this figure close in 4-D?

The only other bit of counting I did is equally unpromising: the number of steps it takes to complete each figure starting with one face flat on the table and attaching symmetrical rings of faces (or one final horizontal top face) to all the exposed edges. Taking the starting face as step 1, this gives:

| | |
|---|---|
| tetrahedron: | 2 |
| cube: | 3 |
| octahedron: | 4 |
| dodecahedron: | 4 |
| icosahedron: | 6 |

No joy there. And why does a tetrahedron lie on a flat surface with an apex uppermost, when all the other polyhedra have a side uppermost? All have an even number of sides.

Best wishes,

**Ralph Hancock**

**JRH** writes—Here's an argument to show why there is no Platonic solid with six-sided (or more) faces. It doesn't say why those that exist do work, but it does say why everything else doesn't work.

You need three or more faces to make a vertex. And the internal angles of the individual faces at the vertex must add up to 360° or less, so that the vertex can form. If the sum of the angles is exactly 360° then the vertex is planar, so no associated 3-D figure can form.

Let the 'defect' be the amount by which the sum of the face angles falls short of 360°. Then we have:

| Triangles – internal angle 60° | | | | |
|---|---|---|---|---|
| Number | 3 | 4 | 5 | 6 or more |
| Defect | 180° | 120° | 60° | zero or less |
| Figure | tetrahedron | octahedron | icosahedron | cannot form |

| Quadrilaterals – internal angle 90° | | |
|---|---|---|
| Number | 3 | 4 or more |
| Defect | 90° | zero or less |
| Figure | cube | cannot form |

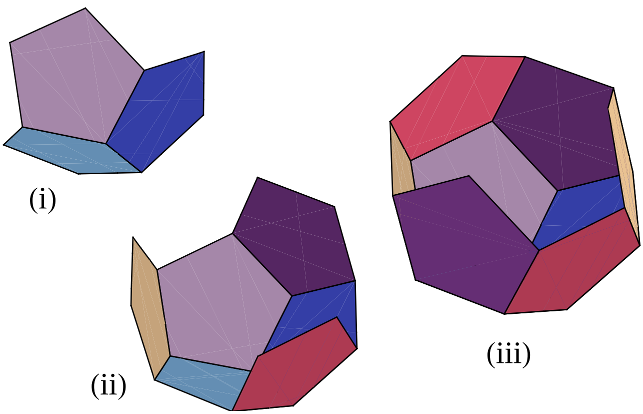
| Pentagons – internal angle 108° | | |
|---|---|---|
| Number | 3 | 4 or more |
| Defect | 36° | zero or less |
| Figure | dodecahedron | cannot form |

| Hexagons (or more) – internal angle 120° (or more) | |
|---|---|
| Number | 3 or more |
| Defect | zero or less |
| Figure | cannot form |

**ADF** adds—Even after the appearance of Barbara Lee's illuminating dissection of the regular dodecahedron [M500 **184** 22] I still have difficulty persuading myself that this object really does exist.

Take three regular pentagons. From the above, it is clear that we can join them together to form an object with three ∨-shaped pairs of edges (i). By symmetry, each ∨ is of the correct size and shape to accommodate a regular pentagon. Adding three such pentagons creates an object with three three-sided slots (ii).

Again, symmetry considerations show that a regular pentagon will fit exactly into each slot. So let's do that (iii). Once more we have three three-sided slots to fill and, as before, each will take three edges of a regular pentagon. But to make the thing close up, the other edges of these pentagons must coincide in pairs. I would like to be extremely convinced that this is what actually happens.

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## Re: Chimps

Dear Ed(s),

Clearly, it is always going to take more effort to change from CHIMP to WOMAN than from APE to MAN, as all female readers must have already observed.

Regards,

**Sue Bromley**

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## Old age

Dear Tony,

The recent short note in M500 **182** (page 25) has reminded me of two articles in early editions of the *Encyclopaedia Britannica.* These are short extracts from each.

From the second edition (1777–83):

> **Dating the Creation**
>
> The compilers of the Universal History determine it (the creation) to have taken place in the year 4305 B.C. so that, according to them, the world is now in its 6096th year of age [in 1783]. ... the whole account of the creation rests on the truth of the Mosaic history. ... Some historians and philosophers are inclined to discredit the Mosaic accounts, from the appearance of volcanoes and other natural phenomena: but their objections are by no means sufficient to invalidate the authority of the sacred writings; not to mention that every one of their own systems is liable to insuperable objections.

From the third edition (1788–97):

> **Longevity**
>
> Immediately after the creation, when the world was to be peopled by one man and one woman, the ordinary age was 900 and upwards. Immediately after the flood, when there were three persons to stock the world, their age was cut shorter, and none of those patriarchs but Shem arrived at 500. In the second century we find none that reached 240; in the third, none but Terah that came to 200 years. ... By degrees as the number of people increased, their longevity dwindled, till it came down at length to 70 or 80 years: and there it stood and has continued to stand since the time of Moses. This has found a good medium, and by means thereof the world is neither overstocked, nor kept too thin.

The article is followed by examples of persons living up to 175 years in the 17th and 18th centuries.

The information from these should enable one to put on his/her 1790s hat and, given the growth in the world population, determine our current longevity in 2001 or, better still, what it will be in the year 3001!

**Ron Potkin**

# Solution 182.5 – $n$ balls

There are $n$ balls in an urn, all of different colours. Two balls are removed at random. The second of the pair is painted to match the first and the two balls are replaced. The balls are thoroughly mixed and again two balls are removed at random. Again the second of the pair is painted to match the first and the balls are replaced. This process is continued. What is the expected number of turns required before all the balls are the same colour?

## David Kerr

The answer, I suspect, is $(n-1)^2$. It is trivially true for $n = 1$ or 2. I have proved it for specific values of $n$ from 3 to 7 with increasing difficulty and, by the Theorem of Confident Assertion, it is therefore true for all $n$.

The proofs for specific cases of $n$ are quite interesting. To explain the method I have shown below the proof for $n = 4$.

The various possible states of the colours of the balls are given by the partitions of 4:

| | |
|---|---|
| State $A = 1, 1, 1, 1$ | all four balls are of different colours |
| State $B = 1, 1, 2$ | two of one colour and two of different colours |
| State $C = 2, 2$ | two of one colour and two of another colour |
| State $D = 1, 3$ | three of one colour and one of another colour |
| State $E = 4$ | all four are the same colour |

At each turn it is not difficult to calculate the various probabilities of staying in the same state or moving to a different state; for example, $\Pr(A \to B) = 1$, $\Pr(B \to B) = 1/2$, $\Pr(B \to C) = 1/6$, $\Pr(B \to D) = 1/3$, etc. We can assemble these probabilities in a transition matrix as follows.

| | $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|---|
| $A$ | 0 | 1 | 0 | 0 | 0 |
| $B$ | 0 | 1/2 | 1/6 | 1/3 | 0 |
| $C$ | 0 | 0 | 1/3 | 2/3 | 0 |
| $D$ | 0 | 0 | 1/4 | 1/2 | 1/4 |
| $E$ | 0 | 0 | 0 | 0 | 1 |

This transition defines a Markov chain. However, to investigate the long run or average behaviour of a Markov chain it must be irreducible, which this chain clearly is not ($E$ is a closed state; that is, when the chain gets into state $E$ it cannot get out). This problem can be resolved quite easily; simply equate states $A$ and $E$. The chain then becomes

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $A$ | 0 | 1 | 0 | 0 |
| $B$ | 0 | 1/2 | 1/6 | 1/3 |
| $C$ | 0 | 0 | 1/3 | 2/3 |
| $D$ | 1/4 | 0 | 1/4 | 1/2 |

This chain is irreducible and in order to calculate the long-run probabilities of the chain being in the various states it is only necessary to solve the matrix equation:

$$[a \;\; b \;\; c \;\; d] \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1/6 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \\ 1/4 & 0 & 1/4 & 1/2 \end{bmatrix} = [a \;\; b \;\; c \;\; d],$$

where $a$ is the probability of the chain being in state $A$ and $b$ is the probability of it being in state $B$ etc. All we have to do is solve the simultaneous equations

$$\begin{aligned} d/4 &= a \\ a + b/2 &= b \\ b/6 + c/3 + d/4 &= c \\ b/3 + 2c/3 + d/2 &= d \\ a + b + c + d &= 1. \end{aligned}$$

After some algebra we get

$$[a \;\; b \;\; c \;\; d] = [1/9 \;\; 2/9 \;\; 2/9 \;\; 4/9].$$

This means that the probability of the chain being in state $A$ (or state $E$) is 1/9 and hence that the average time (in this case, number of turns) to return to state $A$ is 9.

It therefore follows that the average number of turns to get from the state where all balls are different colours to a state where all balls are the same colour is 9.

I have repeated this for $n = 5$, 6 and 7, with increasing difficulty, and the average lengths are respectively 16, 25 and 36. When $n = 7$ the matrix size is $14 \times 14$. I have little doubt that the general answer is $(n-1)^2$ but I can see no prospect of proving it.

# Solution 182.5 – Two £10 notes

> A first ten pound note is laid flat on a table. A second ten pound note is crumpled and placed on top of the first so that none of it protrudes beyond the edges of the first. Prove that there is at least one point of the second note that is directly above a corresponding point of the first.

## David Kerr

Smooth out the second note so that it lies exactly on top of the first. Colour green, all points on the second note that moved to the left; blue, all points that moved to the right, and black, all points that moved neither left nor right. Clearly, there must be a continuous black line running from the top edge to the bottom edge of the note. Repeating this process for up and down movements will give a second black line running from the left edge to the right edge of the note. As both these lines are continuous there must be at least one point where they intersect. All such points will be stationary.

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# Problem 185.4 – Two sins

## David Singmaster

You are all familiar with school students who show that $\frac{16}{64} = \frac{1}{4}$ by cancelling 6 from top and bottom. However, I had a student who showed that $\frac{\sin a}{\sin b} = \frac{a}{b}$ by cancelling 'sin'. This would have been all right if it had been 'sign', but are there other cases where this is true? A little examination shows that it works if $a = 0$ or $b = 0$ or $a = \pm b$, but these cases are really trivial and obviously we are only interested in other solutions.

---

**Errata**—Sue Bromley's fine contributions to M500 **183** both contain minor errors. The last set of equalities in 'Solution 181.2 – Six secs' (p. 13) should have read

$$\sec\frac{\pi}{7}\sec\frac{2\pi}{7}\sec\frac{3\pi}{7}\sec\frac{4\pi}{7}\sec\frac{5\pi}{7}\sec\frac{6\pi}{7}$$
$$= \frac{-1}{(\cos(\pi/7)\cos(2\pi/7)\cos(3\pi/7))^2} = -\left(\frac{1}{4}\left(-\frac{1}{2}+1\right)\right)^{-2} = -64,$$

and the second occurrence of $(-1)^3/2^6$ near the end of 'Solution 181.7 – Five cots' (p.15) should have been omitted (or turned upside-down). Editor's fault. He hopes you will forgive him for these two sins.

# The Abel prize

## Eddie Kent

Urban myths are utterly believable and fill a clear and obvious need. Each one tells of an incident that should have happened. The fact that it might not should never be allowed to spoil the fun. It is essential that they exist without evidence since otherwise they wouldn't be myths. Their necessity is shown in the way they never quite die.

Some time ago I mentioned Alfred Nobel, the man responsible for loud bangs and hefty prizes; probably because we were close to a significant anniversary. While doing so I made a brief incursion into the other thing that everyone knows about him—but Jeremy Humphries insisted on cutting it.

Clearly there is no Nobel prize for mathematics. If that is not because Mrs Nobel had an affair with a mathematician, some better reason needs to be found. Mathematics is big; it is the rock on which every science is built. In mathematics you know what is true; in every other field the best you can know is what has not been shown not to be true.

After all there are prizes for peace, literature, physics, chemistry and medicine. Even economics is recognized. All that mathematicians have had at this level is the Fields medal, a Canadian artefact. Very nice, and we are grateful, but it is restricted to candidates under forty, and worth only about £7000.

Now, according to *Physics Today*, there is a mathematical prize to rival Nobel. The Norwegian government has set up an endowment of around £14 million to fund an annual prize in the six-figure range.

It will be called the Abel prize, named after the Norwegian Niels Henrik Abel, the algebraist who died in 1829 aged just 26. It follows from a proposal by Oscar II, king of Norway and Sweden; just about 100 years ago.

Isn't it good, Norwegian generosity?

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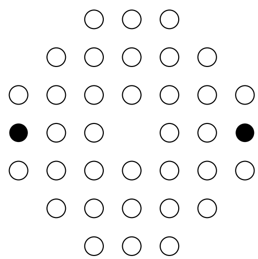
# Problem 185.5 – Two pegs

## John Beasley

In the classical 37-hole French solitaire game, vacate the central hole, mark the two pegs at opposite ends of a centre line and play to leave just these two pegs on the board *having interchanged their initial positions.*



# Solution 183.3 – Seven real numbers

> Suppose $a$, $b$, $c$, $d$, $e$, $f$, $g$ are seven non-negative real numbers that total 1. If $M$ is the maximum of the five sums, $a+b+c$, $b+c+d$, $c+d+e$, $d+e+f$, $e+f+g$, what is the minimum value that $M$ can take?

## Ted Gore

Let $y$ be the average of the five sums. Then

$$a+b+c = y+d_1, \quad b+c+d = y+d_2, \quad c+d+e = y+d_3,$$
$$d+e+f = y+d_4, \quad e+f+g = y+d_5, \tag{1}$$

say. If any of the $d_i$ are non-zero then at least one of them must be positive, so the minimum value of $M$ occurs when they are all zero, giving.

$$a+b+c \;=\; b+c+d \;=\; c+d+e \;=\; d+e+f \;=\; e+f+g \;=\; y.$$

Hence $a=d$, $b=e$, $c=f$ and $d=g$. Now let $a=d=g=x$, where $x \in [0,1/3]$, $b=e=p$ and $c=f=q$, so that the equations (1) are reduced to

$$y \;=\; x+p+q \;=\; x+(1-3x)/2 \;=\; (1-x)/2,$$

which takes its minimum value of 1/3 on $[0,1/3]$ when $x=1/3$. Hence the minimum value of $M$ is 1/3, when $a=d=g=1/3$, $b=c=e=f=0$.

---

## Jim James

By definition, all the five partial sums, $a+b+c$, $b+c+d$, etc. must be less than or equal to $M$. In particular, $a+b+c \le M$ and $e+f+g \le M$, giving

$$a+b+c+e+f+g \;\le\; 2M.$$

But $a+b+c+d+e+f+g=1$, so $d \ge 1-2M$.

Also, $b+c+d \le M$ and $d+e+f \le M$, which give $b+c \le 3M-1$ and $e+f \le 3M-1$. But all the variables are non-negative, so we have

$$0 \;\le\; b+c+e+f \;\le\; 6M-2,$$

for which the minimum value of $M$ is clearly 1/3.

Note that this minimum value of $M$ is only achieved if $b=c=e=f=0$, whence $c+d+e \le M$ implies $d \le 1/3$. But $d \ge 1-2M$, so $d \ge 1/3$, and $d$ must be equal to 1/3. Furthermore, $a+b+c \le M$ and $e+f+g \le M$ imply $a \le 1/3$ and $g \le 1/3$, and since our total sum constraint now reduces to $a+g=2/3$, we must also have $a=1/3$ and $g=1/3$.

---

Also solved by **John Bull**.

# Solution 183.4 – Two real numbers

> If $x$ and $t$ are real numbers, find $t$ such that $\cosh x \le \exp(tx^2)$ for all $x$.

## Basil Thompson

Experimenting with a few values of $x$, we have (to three decimal places): $x = 10$, $t \ge 0.093$, $x = 5$, $t \ge 0.172$, $x = 1$, $t \ge 0.434$, $x = 0.1$, $t \ge 0.499$, .... As $x$ increases, $e^{x^2}$ increases faster than the $e^x$ of $\cosh x$; hence $t$ can be small for $x > 1$, but not for $|x| < 1$. To find a $t$ that works for all $x$ (looks like $t = 0.5$), expand as a series:

$$\cosh x \;=\; 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \;\le\; e^{tx^2} \;=\; 1 + tx^2 + \frac{t^2x^4}{2!} + \frac{t^3x^6}{3!} + \ldots.$$

Dividing by $x^2$ and rearranging,

$$t + \frac{t^2x^2}{2!} + \frac{t^3x^4}{3!} + \ldots \;\ge\; \frac{1}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \ldots.$$

For this to hold for all $x$ we require $t \ge 1/2$.

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## London Mathematical Society

Popular Lectures 2002

**Dr Helen Mason** (Cambridge)

*Our Dynamic Sun*

**Dr John Sylvester** (Kings College, London)

*Geometry Ancient and Modern*

**STRATHCLYDE UNIVERSITY, Thursday 20 June:** Commences at 2.00 pm, refreshments at 3.00 pm, ends at 4.30 pm. Enquiries to Professor A. McBride or Dr A. Ramage, Department of Mathematics, Strathclyde University, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH (tel: 0141 548 3647/3801, e-mails: a.c.mcbride@strath.ac.uk, a.ramage@strath.ac.uk).

**LEEDS UNIVERSITY, Thursday 27 June:** Commences at 6.30 pm, refreshments at 7.30 pm, ends at 9.00 pm. Enquires to Dr R.B.J.T. Allenby, School of Mathematics, University of Leeds, Leeds LS2 9JT (tel: 0113 233 5122, e-mail: pmt6ra@leeds.ac.uk).

**INSTITUTE OF EDUCATION, LONDON, Wednesday 3 July:** Commences at 7.00 pm, refreshments at 8.00 pm, ends at 9.30 pm. Admission with ticket. Apply by 28 June to Miss L. Taylor, London Mathematical Society, De Morgan House, 57-58 Russell Square, London WC1B 4HS (tel: 020 7637 3686, e-mail: taylor@lms.ac.uk). A stamped addressed envelope would be appreciated.