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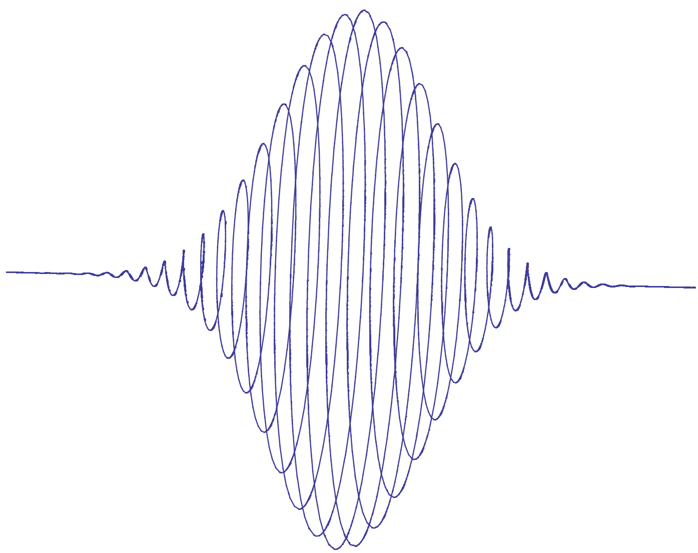
M500 186



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Fun with Fourier

Fourier series approximations to some complex functions

Robin Marks

The MS323 (Introduction to linear dynamics) handbook defines the complex form of Fourier series as follows. For 2π-periodic functions defined on $[-\pi, \pi]$ we can write

$$f(x) \;=\; \sum_{n=-\infty}^{\infty} c_n e^{-inx},$$

where the complex Fourier coefficients are given by the integrals

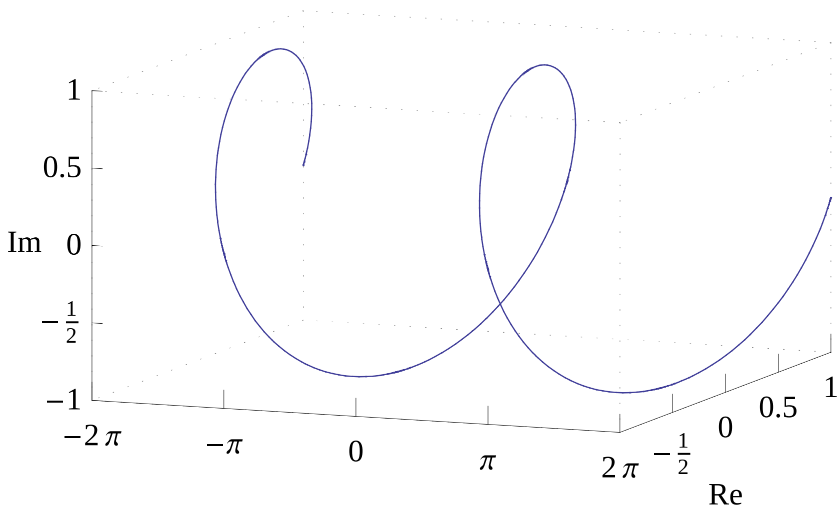
$$c_n \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx.$$

First of all, let us try this in a very simple case, the complex function e^{ix}. Putting $f(x) = e^{ix}$, we get

$$c_n \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix}e^{inx}\,dx \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix(n+1)}\,dx.$$

Thus $c_{-1} = 1/2\pi \int_{-\pi}^{\pi} e^0\,dx = 1$ and for $n \neq -1$, $c_n = \int_{-\pi}^{\pi} e^{ix(n+1)}\,dx = 0$. So we get

$$f(x) \;=\; c_{-1}e^{-i(-1)x} \;=\; e^{ix}.$$



So after considerable effort we have retrieved the original function. This procedure is due to Jean-Baptiste Joseph Fourier (1768–1830), who was clearly a mathematical genius.

Readers may not be impressed so far, but read on! (By the way, whereas e^{ix} gives an anticlockwise helix, e^{-ix} is a clockwise helix.)

Let us look at the function $\frac{1}{2}(e^{imx} + e^{-imx})$, where m is an integer. Putting $f(x) = \frac{1}{2}(e^{imx} + e^{-imx})$, we get

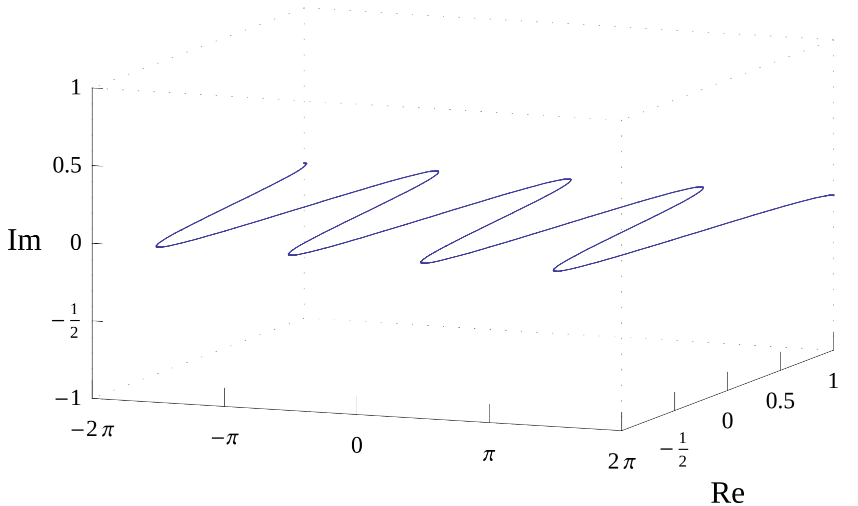
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}(e^{imx}+e^{-imx})e^{inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}(e^{i(n+m)x}+e^{i(n-m)x})\,dx.$$

In the cases $n = m$ and $n = -m$,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}(e^{2inx} + 1)\,dx = \frac{1}{2},$$

whereas $c_n = 0$ for $n \neq m$. Thus

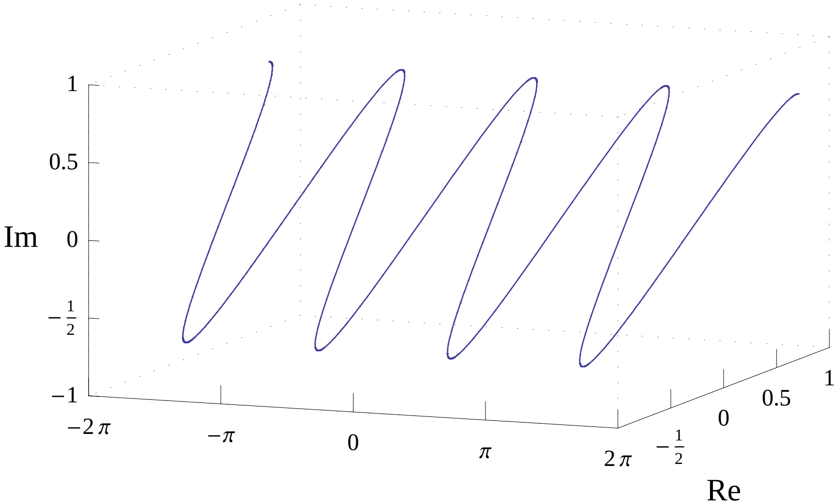
$$f(x) = \frac{1}{2}(e^{imx} + e^{-imx}).$$



Looks familiar?

It is interesting to note that if any complex function as discussed here is multiplied by a complex constant $e^{i\theta}$, this has the effect of rotating the graph around the x axis by an angle θ. For example, multiplying the function above by $e^{i\pi/4}$ gives

$$\frac{1}{2}e^{i\pi/4}(e^{imx} + e^{-imx}).$$



But I digress. Now we look a piecewise continuous function. Let us find the Fourier coefficients for a triangular wave.

We take as the repeating periodic function the two-piece function composed of two straight lines, one between the points $(-\pi, -1)$ and $(0, 1)$, and the other between the points $(0, 1)$ and $(\pi, -1)$. The two pieces have equations $f_1(x) = 1 + 2x/\pi$ over the interval $[-\pi, 0]$, and $f_2(x) = 1 - 2x/\pi$ over the interval $[0, \pi]$, giving the Fourier coefficients

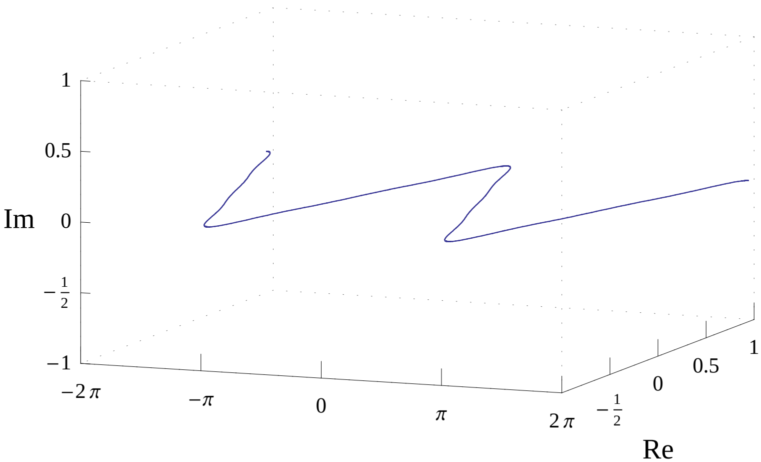
$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{0}\left(1 + \frac{2x}{\pi}\right)e^{inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi}\left(1 - \frac{2x}{\pi}\right)e^{inx}\,dx \\ &= \frac{4 - 2(e^{-in\pi} + e^{in\pi}) + in\pi(-e^{-in\pi} + e^{in\pi})}{2n^2\pi^2} \\ &= \frac{2 - 2e^{in\pi}}{n^2\pi^2} \qquad (\text{since } e^{in\pi} = e^{-in\pi}) \end{aligned}$$

for $n \neq 0$, and $c_0 = 0$.

Here is a list of these Fourier coefficients c_n for $n = -5$ to $= 5$:

$$\frac{4}{25\pi^2},\ 0,\ \frac{4}{9\pi^2},\ 0,\ \frac{4}{\pi^2},\ 0,\ \frac{4}{9\pi^2},\ 0,\ \frac{4}{25\pi^2},\ 0.$$

Let us have a look at the approximation to the triangular wave, synthesized from the Fourier series with just these values of c_n.



As you can see, The graph shows a good approximation to a triangular wave.

Now let us try something a little more complicated. Imagine an infinitely long triangular prism with a straight line wrapped around it in a helical fashion. This gives a piecewise continuous complex function of x. Starting at one edge of the prism at $x = -\pi$, let three consecutive straight lines take us around the prism and back to the initial edge at $x = \pi$. This three-piece element repeats periodically.

Let us find the Fourier coefficients for this 'triangular helical' function. We take as the repeating periodic function a three-piece piecewise continuous function with three straight lines in the 3-space made up from x, $\operatorname{Re} f(x)$ and $\operatorname{Im} f(x)$, beginning at the point $(-\pi,\ \operatorname{Re} e^{-\pi i},\ \operatorname{Im} e^{-\pi i}) = (-\pi,\ -1,\ 0)$ then going to the next point $(-\pi/3,\ \operatorname{Re} e^{-\pi i/3},\ \operatorname{Im} e^{-\pi i/3})$, then to the point $(\pi/3,\ \operatorname{Re} e^{\pi i/3},\ \operatorname{Im} e^{\pi i/3})$ and finishing at $(\pi,\ \operatorname{Re} e^{\pi i},\ \operatorname{Im} e^{\pi i}) = (\pi,\ -1,\ 0)$.

The first piece has equation

$$e^{-\pi i}\left(1 - \frac{\pi + x}{2\pi/3}\right) + e^{-\pi i/3}\ \frac{\pi + x}{2\pi/3}.$$

The Fourier coefficients c_n are given by

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{-\pi/3}\left(e^{-\pi i}\left(1-\frac{\pi+x}{2\pi/3}\right)+\ e^{-\pi i/3}\,\frac{\pi+x}{2\pi/3}\right)e^{inx}\,dx \\
&\quad+\frac{1}{2\pi}\int_{-\pi/3}^{\pi/3}\left(e^{-\pi i/3}\left(1-\frac{\pi/3+x}{2\pi/3}\right)+e^{\pi i/3}\,\frac{\pi/3+x}{2\pi/3}\right)e^{inx}\,dx \\
&\quad+\frac{1}{2\pi}\int_{\pi/3}^{\pi}\left(e^{\pi i/3}\left(1-\frac{-\pi/3+x}{2\pi/3}\right)+e^{\pi i}\,\frac{-\pi/3+x}{2\pi/3}\right)e^{inx}\,dx \\
&= \begin{cases} \dfrac{9}{8n^2\pi^2}\left(-i\sqrt{3}(e^{-in\pi/3}-e^{in\pi/3})\right. \\ \qquad\left.+\,(e^{-in\pi/3}+e^{in\pi/3})-(e^{-in\pi}+e^{in\pi})\right), & n\neq 0, \\ 0, & n=0. \end{cases}
\end{aligned}$$

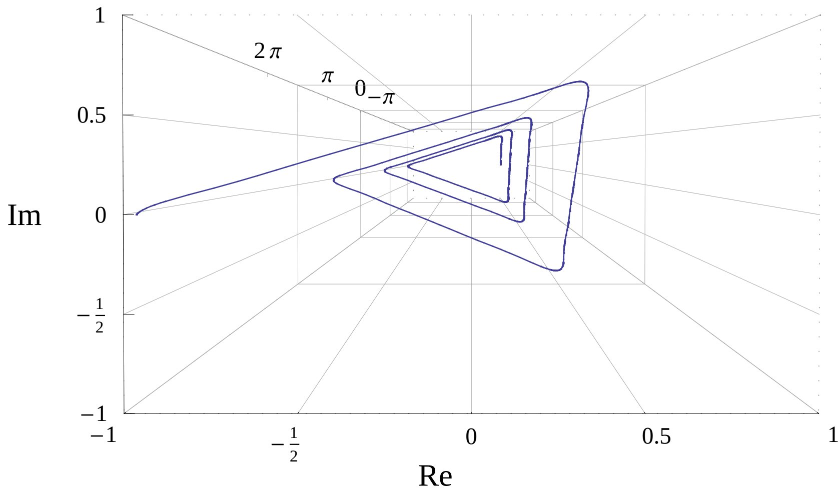
Alternatively, this can be written

$$c_n = \begin{cases} \dfrac{9}{4n^2\pi^2}\left(-\sqrt{3}\sin\dfrac{n\pi}{3}+\cos\dfrac{n\pi}{3}-\cos n\pi\right), & n\neq 0, \\ 0, & n=0. \end{cases}$$

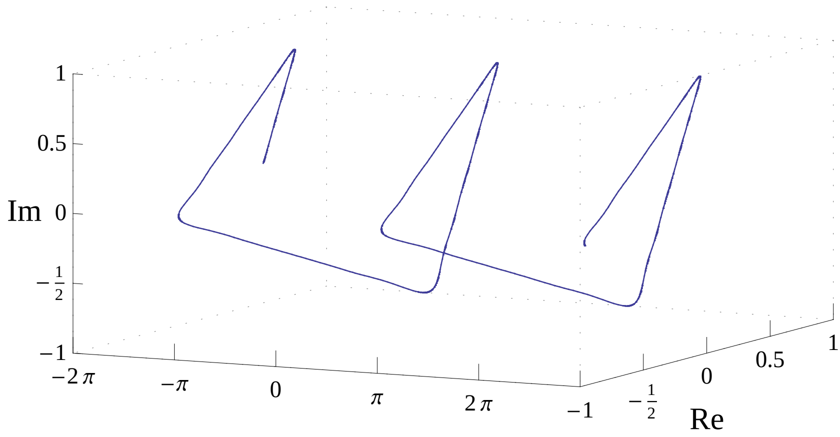
Here is a list of these Fourier coefficients c_n, $n=-7$ to 7:

$$\frac{27}{196\pi^2},\ 0,\ 0,\ \frac{-27}{64\pi^2},\ 0,\ 0,\ \frac{27}{4\pi^2},\ 0,\ 0,\ \frac{-27}{16\pi^2},\ 0,\ 0,\ \frac{27}{100\pi^2},\ 0,\ 0,$$

from which we can reconstruct the triangular helix



or, looked at from a different viewpoint,



quite a good approximation to the piecewise continuous triangular helix.

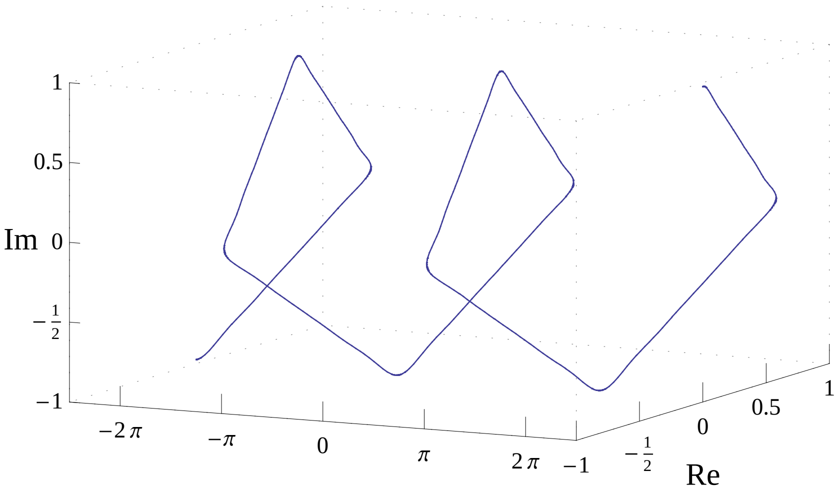
Now let us create a 'square helix'. We take as the repeating periodic function the four-piece continuous function with four straight lines, starting at the point $(-\pi,\ -1,\ 0)$, then going to $(-\pi/2,\ 0,\ -i)$, $(0,\ 1,\ 0)$, $(\pi/2,\ 0,\ i)$ and finishing at the point $(\pi,\ -1,\ 0)$. The first straight line has equation $-(1-2(\pi+x)/\pi) - 2i(\pi+x)/\pi$. The Fourier coefficients are given by

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2}\left(-\left(1-\frac{\pi+x}{\pi/2}\right)-i\,\frac{\pi+x}{\pi/2}\right)e^{inx}\,dx \\
&\quad + \frac{1}{2\pi}\int_{-\pi/2}^{0}\left(-i\left(1-\frac{\pi/2+x}{\pi/2}\right)+\frac{\pi/2+x}{\pi/2}\right)e^{inx}\,dx \\
&\quad + \frac{1}{2\pi}\int_{0}^{\pi/2}\left(\left(1-\frac{x}{\pi/2}\right)+i\,\frac{x}{\pi/2}\right)e^{inx}\,dx \\
&\quad + \frac{1}{2\pi}\int_{\pi/2}^{\pi}\left(i\left(1-\frac{x-\pi/2}{\pi/2}\right)-\frac{x-\pi/2}{\pi/2}\right)e^{inx}\,dx \\
&= \begin{cases} \dfrac{2}{n^2\pi^2}\left(1-2\sin\dfrac{n\pi}{2}-\cos n\pi\right), & n\neq 0,\\ 0, & n=0. \end{cases}
\end{aligned}
$$

Here are the coefficients c_n for $n=-5$ to $n=15$:

$$\frac{8}{25\pi^2},\ 0,\ 0,\ 0,\ \frac{8}{\pi^2},\ 0,\ 0,\ 0,\ \frac{8}{9\pi^2},\ 0,\ 0,\ 0,\ \frac{8}{49\pi^2},\ 0,\ 0,\ 0,\ \frac{8}{121\pi^2},\ 0,\ 0, 0,\ \frac{8}{225\pi^2}.$$

We reconstruct the function from these coefficients and, again, we get a good approximation to the square helix.



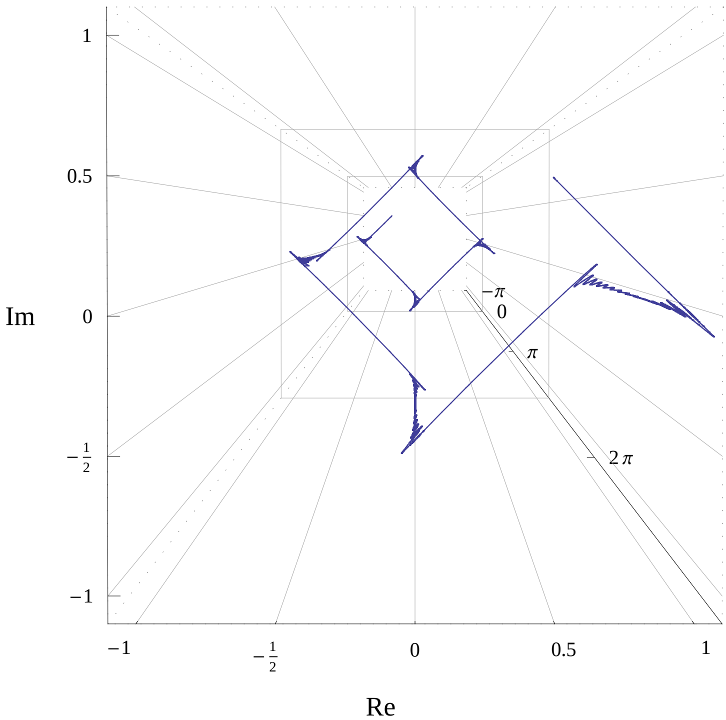
Finally, let us try to approximate a different sort of square helix. We consider the repeating periodic function $f(x)$ defined by $f(x) = -1$ for $-\pi < x < -\pi/2$, $f(x) = -i$ for $-\pi/2 < x < 0$, $f(x) = 1$ for $0 < x < \pi/2$ and $f(x) = i$ for $\pi/2 < x < \pi$. Note that there are discontinuities at integer multiples of $\pi/2$. The equation of the first line segment is $f(x) = -1$. The Fourier coefficients are given by

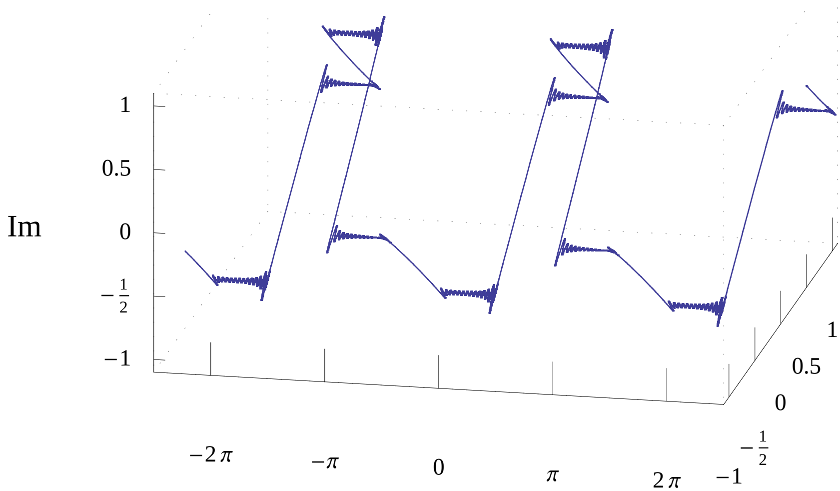
$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\left(-\int_{-\pi}^{-\pi/2} e^{inx}\,dx - i\int_{-\pi/2}^{0} e^{inx}\,dx + \int_{0}^{\pi/2} e^{inx}\,dx + i\int_{\pi/2}^{\pi} e^{inx}\,dx\right) \\
&= \begin{cases} \dfrac{1-i}{2n\pi}\left(-1+\cos n\pi + 2\sin\dfrac{n\pi}{2}\right), & n \neq 0, \\ 0, & n = 0, \end{cases}
\end{aligned}
$$

and here are the coefficients c_n for $n = -5$ to $n = 11$:

$$
\frac{2-2i}{5\pi},\ 0\ ,0,\ 0,\ \frac{2-2i}{\pi},\ 0,\ 0,\ 0,\ \frac{2i-2}{3\pi},\ 0,\ 0,\ 0,\ \frac{2i-2}{7\pi},\ 0,\ 0,\ 0,\ \frac{2i-2}{11\pi}.
$$

Note that we have complex coefficients; also three out of every four are zero and the others have values $2(i-1)/n\pi$. Amazing, don't you think? We reconstruct the square helix from the Fourier series and show it from two different viewpoints.


1
0.5
Im
0
$-\frac{1}{2}$
−1
−1
$-\frac{1}{2}$
0
0.5
1
Re
$-\pi$
0
π
2π


1
0.5
Im
0
$-\frac{1}{2}$
−1
-2π
$-\pi$
0
π
2π
1
0.5
0
$-\frac{1}{2}$
−1

Calculus universalis

Sebastian Hayes

All shapes are measured here, all motions specified,
Awesome arrays of symbols! what is far is near,
The past and future perfectly equivalent;

Magical signs! these curves portray the cry of gulls,
Nightfall, the rocking of an empty rowing-boat,
All shapes are measured here, all motions specified;

What is remote is near: revolving galaxies
Lie captive in this net, each fold contains a sun,
All shapes are measured here, all motions specified;

The hands of unborn creatures leave their imprints here,
Worlds that have never been are listed and compared,
All shapes are measured here, all motions specified;

Designers' plans, the dreams of engineers,
Are judged and sentenced by these ghostly formulae,
To them the lovely and the hideous are equivalent;

All time is always now: one finds no mention here
Of names and dates, conquerors or great discoveries,
All places and all eras are equivalent;

Where is mankind? These letters do not speak
Of what is gay or sad, wistful or intimate,
All persons and all feelings are equivalent;

Dreadful arrays! spellbooks and shibboleths!
No grief or pleasure ever touched these abstract signs,
No children play in them, no lovers sigh or weep,
To them all places and all persons are equivalent,
All shapes are measured here, all motions specified.

'At the same time he [Dugald Stewart, teacher of mathematics at Edinburgh University in 1778] insists that when a man asserts that an act is right he intends to say something which is true. Moral discrimination is a rational operation, just as much as is perception of the fact that the three angles of a triangle together equal one right angle.'

—Frederick Copleston, *History of Philosophy*, vol. 5. [Sent by John Brown]

Time

Sebastian Hayes

'Time' has two basic meanings.

(1) Duration, answering 'How long did it / does it last?'

(2) Sequence, answering 'In what order did things happen?'

Does time exist? Or, rather, need time exist? If nothing ever happened there would be no time because there would be no need for it. This shows that time is not an absolutely essential item: it depends on the prior 'existence' of events. (On the other hand one might argue that 'time' in some sense pre-existed even though it was not experienced: it made it possible for there to be sequences of events even if in point of fact no events had occurrence.)

If everything happened 'at the same time' there would be no time either, because no order of events. What of duration in such a case? 'Duration' is a subjective thing that arises on the comparison of two or more sequences of events. If no sequence, then no duration—or rather *any* duration since one person's duration is as good as anyone else's.

Time then is not completely fundamental. If we posit 'events' and a place for them to happen, call it the 'locality,' we have a 'universe' of sorts already. And if everything that could happen, has already happened—was just there and always had been—then there would be no time, only an eternal present. Is this conceivable? Just about. But though conceivable such a vision would not be liveable which is why in practice no one believes in total predestination though the belief is impossible to disprove.

Is time an illusion as some mystics claim? I think not. Suppose that all strictly physical events had occurrence—were just out there like so many hillocks or footprints across an endless plain. The lightning flash and the forest fire are both out there together, contemporaneously, there is no causality, no sequence. But the fact remains that I do not view things in this light, I perceive a sequence of events. Maybe I am in error but nonetheless my untrustworthy mental perceptions really are sequential and inescapably so whether erroneous or not. So there is time at least in the mental world. Therefore there is time in the real world.

We are told that prior to the 'big bang' there was no space or time, no 'space–time'. But if there was no time then it would seem that nothing could have happened, so there was just a blank. Many theorists, however, speak of a pre-existing quantum vacuum with our universe as just a runaway fluctuation. There are thus events prior to our universe—can we imagine them all occurring non-sequentially? This is hard though perhaps not impossible. In a similar fashion we are invited to envisage the photon in the box as having strictly no precise position prior to an act of measurement. So pre-universe events would happen in no particular order, in any order.

Still, the fact remains that there was a last such event, since this, and only this, fluctuation produced our universe. So there was time after all, or at least there was/is from then on. So time does exist and even must exist—it is like the ontological proof of the existence of God.

What of Einstein and time-travel? If events are 'time-like separated', special relativity tells us we can always find an observer for whom A precedes B and another observer for whom B precedes A. Thus, for me, an Andromedan space-fleet is already on the way to annihilate us, but for you, in the next street or even perhaps in the next room, the decision to send it has not yet been taken. But this is purely academic. When the space-fleet gets to the solar system and so becomes dangerous for the first time, the order of events will be the same for all of us. There is no need to worry about this.

The fashionable term 'space–time' is generally misunderstood and should be dropped. Laymen, and even many scientists, seem to view 'time' as a dimension added on to the three spatial ones and somehow on a par with them whereas it is not, it is different from all three together, orthogonal to space if you like. Better to just speak of the locality, a place where events can and do happen.

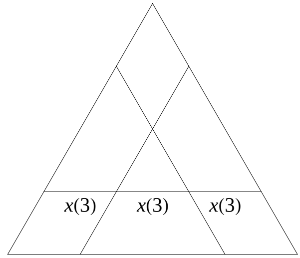
But the twins paradox goes a good deal further than this. Jack the Nimble supposedly has travelled around the universe at a speed close to that of light. He returns to Earth and finds that he has aged less than his twin brother, John the Slow. If by 'being younger' we mean that fewer physiological changes have occurred, that say Jack the Nimble's heart has only beat 10 million times while his brother's heart has beaten 15 million, then this is very difficult to accept.

For it seems to imply that the occurrence or not of a particular event, a heart-beat, in some way depends on one's position in the universe and state of motion. But things either happen or they don't—judgements about how long they take to happen are another matter. This can be made into a law, 'The number of events in an event-chain is absolute'—although in most cases it will be impossible to provide this number. The above rule seems to conflict with relativity.

Problem 186.1 – Polygon division

ADF

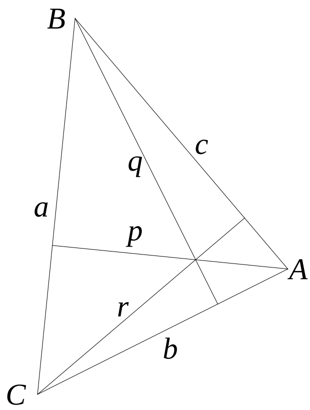
This was suggested by the front cover of M500 **184**. Take a regular polygon of n sides and draw lines in its interior and parallel to the sides such that each line is divided into three equal segments of length $x(n)$ at the points where it intersects with the two adjacent lines. What is $x(n)$?



Solution 183.1 – Three altitudes

Is a triangle defined by its three altitudes?

Ted Gore



Triangle ABC has altitudes p, q, r based on sides a, b, c, respectively. Let Δ be the area of ABC. Then

$$\Delta = \frac{ap}{2} = \frac{bq}{2} = \frac{cr}{2}.$$

The semi-perimeter of ABC is

$$s = \frac{1}{2}(a+b+c)$$

$$= \Delta\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right) = \Delta k,$$

where $k = 1/p + 1/q + 1/r$. Using Heron's formula for the area of a triangle, we have

$$\begin{aligned}\Delta &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{\Delta k\left(\Delta k - \frac{2\Delta}{p}\right)\left(\Delta k - \frac{2\Delta}{q}\right)\left(\Delta k - \frac{2\Delta}{r}\right)}\\ &= \sqrt{\Delta^4 k\left(k-\frac{2}{p}\right)\left(k-\frac{2}{q}\right)\left(k-\frac{2}{r}\right)} = \Delta^2 f,\end{aligned}$$

where $f = \sqrt{k(k-2/p)(k-2/q)(k-2/r)}$. So $\Delta = 1/f$. Then

$$a = \frac{2}{pf}, \quad b = \frac{2}{qf}, \quad c = \frac{2}{rf}.$$

So the three sides of the triangle, which define it uniquely, can be expressed in terms of its altitudes.

David Kerr

From considerations of area, $ap = bq = cr$. Hence $b = ap/q$ and $c = ap/r$. This implies that any other triangle with altitudes $\{p, q, r\}$ must be similar to the triangle with sides $\{a, b, c\}$. This is clearly impossible; hence $\{a, b, c\}$ is unique.

Problem 186.2 – Tennis

ADF

Take the number 236 and expand it in binary notation (11101100). Read the binary digits from right to left interpreting a '0' as a point to Player A and a '1' as a point to Player B in a game of tennis. Then, as you can see, 236 corresponds to the game fifteen love, thirty love, thirty fifteen, thirty all, forty thirty, deuce, advantage Player B, game to Player B. On the other hand, 16 (10000) won't work because Player A wins before the '1' digit is reached.

This suggests a problem: Determine the set of integers that correspond to valid tennis games.

Problem 186.3 – Two hands

Adrian Cox

At what times do the hour and minute hands overlap on a normal analogue clock?

ADF—If that's too easy (or if you have seen it before), you might like to try:

What is the time when the hour, minute and second hands overlap exactly? Apart from that special case, at what times do the three hands of a clock overlap as closely as possible?

Un nombre

This is like 'Deux nombres' (M500 **184**, p. 21) except that only one number is involved.

There are two French mathematicians. One mathematician knows the first letter and the other the last letter of the name of an integer between 1 and 22. They meet.

The first mathematician says to the other, 'Je connais la lettre premiere du nombre mais pas le nombre.'

The other replies, 'Je connais la lettre dernier du nombre mais pas le nombre.'

The first replies, 'Ah! Je connais le nombre!'

Question: What number is it?

'I say, I say. My wife cut the end off one of my trouser legs and sent it to the library.'

'That was a turn-up for the books.'—**JRH**

The creation of irrational numbers

Sebastian Hayes

The relation of the side of a right-angled isosceles triangle to the hypotenuse has a perfectly good *geometrical* existence but a numerical non-existence. We have therefore a choice: number theory or geometry but not both at once.

So matters remained until the late nineteenth century when 'incommensurable ratios' were endowed with a *numerical* existence as well. The first person to have tackled the problem head on seems to have been Dedekind. His solution was to introduce a completely new axiom into mathematics which a modern textbook (Burkill's *First Course in Analysis*) summarizes as follows.

> *Dedekind's axiom.* Suppose that the system of all real numbers is divided into two classes, L, R, every member l of L being less than every member r of R (and neither class being empty). Then there is a dividing number ζ with the properties that every number less than ζ belongs to L and every number greater than ζ belongs to R. The number ζ itself may belong either to L or to R. If it is in L, it is the greatest number of L; if it is in R, it is the least member of R.

Now this procedure undoubtedly does in some sense define such a creature as $\sqrt{2}$—although aesthetically I do not like the bit about ζ belonging to 'either' class.

One feels nonetheless that this image of the number line, which is surely at best a visual crutch, is being overstrained. Apart from this, what Dedekind's procedure actually defines is not a 'quantity' as such but a gap. Instead of talking about 'Dedekind's cut' we should talk about 'Dedekind's gap'. But if we do this we possess no more than we had before since we knew the 'number line was gapped', or the Greeks did anyway.

The fact of the matter is, as I see it anyway, that there is a world of difference between a positive rational number and an irrational. The first represents, or can represent, a specific length or other quantity which, within acceptable limits of technical exactitude, actually does exist while the second represents something which not only does not exist in the real world, but cannot exist there. For there are no irrational lengths in the real world, only in the super-real Platonic world. Dedekind's stratagem amounts to pretending that, in this case at least, the Platonic and the real worlds are one and the same; that is, $\sqrt{2}$ and its companions are 'real numbers'.

There are other ways of creating irrationals but all depend on a similar sleight of hand. 'Every increasing sequence of rational numbers converges to a limit'—the *axiom of completeness*. Who says it does? The author of the textbook and behind him/her the current mathematical establishment. But if I ask to see, hear, read &c. this limit, in the vast majority of cases, I will be fobbed off with something which it is not, namely a so-called rational approximation.

It is possible to sidestep the issue in much the same way as Eudoxus sidestepped the issue of incommensurable ratios. We can say that an irrational number is 'given' by (1) a mathematical formula (of a certain type) and (2) an initial 'store of numbers'—domain if you like—to which the formula is to be applied. Thus ϕ is 'given' by t_{n+1}/t_n with $t_1 = 1$, $t_2 = 1$, $t_{n+1} = t_n + t_{n-1}$. This tells us what to do while making no commitment to the existence or not of the implied limit. However, it is really a coward's way out: we want to know not what you have to do to get ϕ but what ϕ actually 'is'.

It will not have escaped the reader that we have moved a long way from the set-theoretic definition which takes a 'real number' as being the entire 'Cauchy equivalence class' of the relevant rational numbers. I was for a long time puzzled as to why we need to have such a large assembly. Then I remembered that there are, for example, a multitude of different formulas leading to such a 'number' as π for example, as David Singmaster reminds us in M500 **168**. (The most dramatic example of the 'all paths lead to Rome' syndrome is ϕ since it has been proved that the ratio of successive terms in a Fibonacci series converges to ϕ no matter what starting points, t_1, t_2, are used—try it and see.)

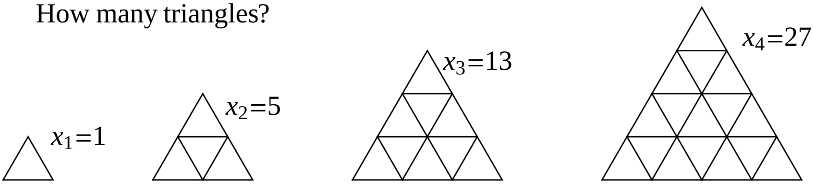
In practice no one ever uses the set-theoretic definition. What we all do is to adopt the analytical definition which views a real number as the limit of an equivalence class of Cauchy sequences. This is shown by our very language—which even I find it impossible to avoid. We speak of 'approximations to' $\sqrt{2}$ or ϕ. Moreover, this language and the way of thinking it embodies imply that the 'limit' in question is actually attained.

What's in a name? Quite a lot apparently. Primitive peoples avoided telling members of other tribes their name, believing that knowing a name gives some sort of power over the person or object. In just such a primitive way we—and that even includes me—feel that the ratio of the radius to half the circumference of a circle really exists because we have a name for it (only since the eighteenth century). And somehow this extends to $1 + 1/2^2 + 1/3^2 + \ldots$ since it sums to $\pi^2/6$.

But this is more a matter of human psychology than scientific fact. All the columns of Pascal's triangle inverted 'converge' and thus 'sum to a limit'. But do you honestly believe that $1 + 1/3 + 1/6 + 1/10 + \ldots$ is a single number?

Even e is to me several shades less real than π.

Solution 184.5 – Triangles



Peter Fletcher

Start with one triangle. Add a row of three and count. Add a row of five and count. Add a row of seven and count. ... This leads to a table.

n	Lengths of sides of triangles										Total
	1	2	3	4	5	6	7	8	9	10	
1	1	0	0	0	0	0	0	0	0	0	1
2	4	1	0	0	0	0	0	0	0	0	5
3	9	3	1	0	0	0	0	0	0	0	13
4	16	7	3	1	0	0	0	0	0	0	27
5	25	13	6	3	1	0	0	0	0	0	48
6	36	21	11	6	3	1	0	0	0	0	78
7	49	31	18	10	6	3	1	0	0	0	118
8	64	43	27	16	10	6	3	1	0	0	170
9	81	57	38	24	15	10	6	3	1	0	235
10	100	73	51	34	22	15	10	6	3	1	315

It is clear that there is a pattern. Write $T_n = 1 + 2 + \cdots + n$. Then

$$x_n \;=\; T_n + T_{n-1} + \cdots + T_1 + T_{n-1} + T_{n-3} + \cdots + \begin{cases} 1 & \text{if } n \text{ is even ,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$x_n \;=\; \sum_{i=1}^{n} T_i + \sum_{k=1}^{m} T_{n-2k+1} \;=\; \sum_{i=1}^{n} \frac{i(i+1)}{2} + \sum_{k=1}^{m} \frac{(n-2k+1)(n-2k+2)}{2},$$

where $m = [n/2]$, the integer part of $n/2$. Using the formula for the sum of the squares, $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$, this can be simplified to

$$\begin{aligned} x_n &= \frac{1}{2}\sum_{i=1}^{n}\left(i^2+i\right)+\frac{1}{2}\sum_{k=1}^{m}\left(n^2-4nk+3n+2+4k^2-6k\right) \\ &= \frac{1}{6}\left((n+1)(n+2)(n+3m)+m(m+1)(4m-6n-7)\right). \end{aligned}$$

ADF

Start with a triangle of size $n-1$ and add a row of triangles. Count the new triangles generated in this way.

There are $n+1-s$ right-way-up triangles of size s. Counting the new upside-down triangles, we have $n-1$ of size 1, $n-3$ of size 2, and so on, until we get to triangles of size $[n/2]$ of which there are two if n is odd and one if n is even. Thus

$$\begin{aligned} x_n &= x_{n-1}+\sum_{i=1}^{n} i+\begin{cases} 1+3+5+\cdots+n-1 & \text{if } n \text{ is even} \\ 2+4+6+\cdots+n-1 & \text{if } n \text{ is odd} \end{cases} \\ &= x_{n-1}+\frac{n(n+1)}{2}+\begin{cases} \dfrac{n^2}{4} & \text{if } n \text{ is even} \\ \dfrac{(n-1)(n+1)}{4} & \text{if } n \text{ is odd} \end{cases} \\ &= x_{n-1}+\frac{3n^2}{4}+\frac{n}{2}-\begin{cases} \dfrac{1}{4} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Combining two steps into one we obtain a single formula,

$$x_n = x_{n-2}+\frac{3n^2}{2}-\frac{n}{2}.$$

Solving this recursion in the two cases corresponding to the starting values $x_1 = 1$ and $x_2 = 3$ leads to

$$x_n = \frac{1}{4}n^3+\frac{5}{8}n^2+\frac{1}{4}n-\begin{cases} \dfrac{1}{8} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This formula was also found by **Martyn Lawrence**.

Solution 184.4 – Three real numbers

Find three real numbers, a, b, c, such that

$$a + b + c = ab \;=\; \frac{70 + 26\sqrt{13}}{27} \quad\text{and}\quad \frac{a}{b} \;=\; \frac{b}{c}.$$

Peter Fletcher

Given that the product and sum include a $\sqrt{13}$, a little experimentation leads to

$$\frac{70 + 26\sqrt{13}}{27} \;=\; \frac{2}{27}\,(2 + \sqrt{13})(11 + \sqrt{13}) \;\approx\; 6.06461.$$

It seems reasonable to assume that a and b are each some fraction, $\frac{1}{27}$, $\frac{2}{27}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{1}{3}$, or $\frac{2}{3}$ of $2 + \sqrt{13}$ and $11 + \sqrt{13}$, such that the product is as above. Eliminating $c = b^2/a$, we have to solve

$$f(a) \;=\; a^4 + a^2(ab) + (ab)^2 - a^3(ab) \;=\; 0. \tag{1}$$

Putting $ab = 6.06461$ and trying various values for a leads to an approximate solution of (1) between $a = 3.73$ and $a = 3.74$. A little more experimentation shows that $\frac{2}{3}(2 + \sqrt{13}) \approx 3.73703$ is an exact solution.

If $a = \frac{2}{3}(2 + \sqrt{13})$ then b must be $\frac{1}{9}(11 + \sqrt{13})$ so that the product has the correct value, $\frac{1}{27}(70 + 26\sqrt{13})$. Then $c = b^2/a = \frac{1}{27}(1 + 5\sqrt{13})$. As a final check, we have

$$a + b + c = \frac{2}{3}(2 + \sqrt{13}) + \frac{1}{9}(11 + \sqrt{13}) + \frac{1}{27}(1 + 5\sqrt{13}) = \frac{70 + 26\sqrt{13}}{27}.$$

Dick Boardman

The solution to this problem is

$$b \;=\; \frac{1}{9}(11 + \sqrt{13}), \quad a \;=\; \frac{1}{3}\,\frac{70 + 26\sqrt{13}}{11 + \sqrt{13}}, \quad c \;=\; \frac{b^2}{a}.$$

The reason that it is a little simpler than the original problem is that the two real roots are equal. I claim no credit for this. All I did was solve it with Mathematica. If I do the same to the original problem (that is, with 25 instead of $(70 + 26\sqrt{13})/27$), I get an immediate answer but it is a horrible expression involving sums of cube roots and things.

ADF

My initial attempt was entirely unimaginative. When I entered the command 'Solve[a + b + c == 25, a/b == b/c, a b == 25, a, b, c]', MATHEMATICA (version 2) responded a few seconds later with over 4000 lines of output. Admittedly the solution reduces to less than 2500 lines if MATHEMATICA's 'Simplify' function is used. Then I noticed that if I ask for the solution (in a, b and c as functions of x) of a more general set of equations,

$$a + b + c = ab \;=\; x, \quad \frac{a}{b} \;=\; \frac{b}{c}, \tag{2}$$

although the output is even worse, the expression $27x^2 - 140x - 144$ appears repeatedly and often. Hence the choice of $x = (70 + 26\sqrt{13})/27$, a root of this quadratic, does indeed simplify the problem.

If we eliminate a and c from (2), we obtain single equation of degree four,

$$b^4 + xb^2 - x^2b + x^2 \;=\; 0.$$

Then, using MATHEMATICA combined with other devices (such as human ingenuity), it is possible to arrive at a reasonably compact solution,

$$b \;=\; \frac{1}{2}\left(f \pm \sqrt{\frac{2x^2}{f} - 2x - f^2}\right),$$

where f is defined by

$$f \;=\; \sqrt{\frac{x}{3}(e_+ + e_- - 2)},$$

$$e_\pm \;=\; \left(\frac{27x}{2} \pm \frac{d}{2} - 35\right)^{1/3}, \quad d \;=\; \sqrt{27(27x^2 - 140x - 144)}.$$

If $x = (70 + 26\sqrt{13})/27$, then $d = 0$, $e_+ = e_- = \sqrt{13}$, $f = (22 + 2\sqrt{13})/9$ and $b = f/2$.

Chris Pile

A simpler answer is obtained if $a + b + c = ab = 24\frac{1}{2}$. Then $a = 3\frac{1}{2}$, $b = 7$, $c = 14$.

In the original 'Dipole' puzzle in *IEE News* the problem was stated as $a + b + c = 21$, $ab = 25$, which has the numerical solution $a = 3.883$, $b = 6.439$, $c = 10.678$, or $a = 19.645$, $b = 1.273$, $c = 0.082$. (A solution in integers can be obtained to the 'Dipole' version if $a + b + c = 19$, $ab = 24$; then $a = 4$, $b = 6$, $c = 9$.)

Solution 184.2 – Monk

> A monk sets out at 6 am from a monastery at the base of a mountain and arrives at the shrine at the top at 6 pm. A few days later he sets out at 6 am, follows the same path back down the mountain and arrives back at the monastery at 6 pm. Prove that on his outward and return journeys there is a point somewhere along the path where he will set foot at the same time of day.

Peter Fletcher

Consider the monk's twin brother, who decides to go up to the shrine to see what the hold-up is with his brother's return to the monastery. He sets off at the same time as his brother leaves the shrine. There will be a point on the path where they will meet.

If the brother happens to have exactly the same rate of progress up the mountain as did the monk, then where and when they meet will be at precisely the same point on the path that the latter had reached a few days before. Therefore on the monk's outward and return journeys there is a point somewhere on the path where he will set foot at the same time of day.

My wife, a non-mathematician but a literature graduate interested in science fiction, could not accept my reasoning regarding the invention of the monk's brother; that I could lay down that he would walk at *exactly* the same speed and with exactly the same stops and starts as his religious brother. However, she would accept that after reaching the shrine the monk might travel back in time a few days and then go down the mountain to meet himself on the way up.

The same solution was also given by **John Bull**, the originator of the problem.

David Tansey

The monk's journey along the path is *continuous*. (He stops to rest or he may even decide to go backwards for a while, but it is reasonable to assume that he does not leap about from place to place.) So his journey may be modelled by a continuous function $f(t)$ which gives the monk's distance from the monastery, along the path, at time of day t. Similarly, there is a continuous function $g(t)$, the path distance from the monastery at time of day t during the monk's return journey. Let $h(t) = f(t) - g(t)$ be the difference function. Then $h(t)$ is continuous. Furthermore, $h(\text{6am}) = -D < 0$, where D is the path distance from the monastery to the shrine, and $h(\text{6pm}) = D > 0$. Hence from the *intermediate value theorem* there is a time $T \in (\text{6am, 6pm})$ such that $h(T) = 0$ and therefore $f(T) = g(T)$.

Solution 184.6 – Limit

$$\text{Show that } \frac{a\sin b - b\sin a}{a\cos b - b\cos a} \to \tan(a - \arctan a) \quad \text{as} \quad b \to a.$$

Jim James

Let $f(b) = a\sin b - b\sin a$; then $f(a) = 0$ and $f'(b) = a\cos b - \sin a$. Let $g(b) = a\cos b - b\cos a$; then $g(a) = 0$ and $g'(b) = -a\sin b - -\cos a$. And now,

$$\lim_{b\to a} \frac{f'(b)}{g'(b)} = \frac{\sin a - a\cos a}{\cos a + a\sin a} = \frac{\tan a - a}{1 + a\tan a} = \tan(a - \arctan a).$$

All conditions of l'Hôpital's rule are satisfied so we can state that

$$\lim_{b\to a} \frac{a\sin b - b\sin a}{a\cos b - b\cos a} = \lim_{b\to a} \frac{f(b)}{g(b)} = \lim_{b\to a} \frac{f'(b)}{g'(b)} = \tan(a - \arctan a).$$

Also solved by **Peter Fletcher**.

Problem 186.4 – Sixteen coins

ADF

There are sixteen £1 coins. One is a counterfeit, identical to the others except for its weight. What is the minimum number of weighings that will guarantee to identify the bogus coin?

By a *weighing* we mean that you select some coins and place them in the pan of a weighing machine. The machine will probably indicate whether or not the set of coins on the pan has the correct total weight. Sometimes the machine malfunctions and gives the wrong answer, but this is rare—and it never happens more than once in a sequence of twelve weighings.

Problem 186.5 – Horse

This comes via **Colin Davies**. He says: 'There have been a number of "grazing at the end of a rope" problems in M500. This one, which [is similar to a problem that] was submitted to *IEE News* by Ray Presnell, gives the area to be grazed and asks for the length of the rope, which is the other way round from usual.'

A horse is tethered to the perimeter of a circular field with radius 1 kilometre. The tether allows the horse to graze all but one π-th the area of the field. How long is the tether?

Letters to the Editors

Recurring decimals

I add a few snippets to the very informative discussion in M500 **184**, pp 16–17.

We have $1/m = 0.(abcd\ldots)(abcd\ldots)\ldots$, where we assume m to be a positive integer which has no factor in common with the assumed base 10. If the period is n digits long we can 'clear' the decimal fraction by first multiplying by 10^n and then subtracting the original reciprocal; that is,

$$(abcd\ldots) \;=\; 10^n\frac{1}{m} - \frac{1}{m} \;=\; \frac{1}{m}(10^n - 1).$$

Hence the appearance of the 9s—n of them in all—when we convert the original infinite series to a proper fraction. Since $(abcd\ldots)$ is an integer, m must divide $9999\ldots$.

Now, if $m = p$, a prime other than 2 or 5, we know from Fermat's Little Theorem that $10^{p-1} - 1$ is a multiple of p. So the period $(abcd\ldots)$ must seemingly be $p - 1$ digits long for any prime p. What about 3 or 13? If we understand by 'period' a repeating sequence of digits, then the above statement is perfectly correct, but we of course want the shortest such repeating sequence. It follows that the period of every prime must divide $p - 1$ but only *full period primes* like 7 go the whole hilt.

The first few full-period primes are 7, l7, 19, 23, ... and, not finding anything special about them, I was intending to enquire whether an M500 reader could enlighten me as to how to distinguish full-period primes by inspection, or at any rate by factorizing $p-1$. Can anyone help me on this? Well, if you can, you may acquire some mathematical celebrity as I have since read that 'there is no known method of predicting which primes have maximum period' (David Wells, *The Penguin Dictionary of Curious and Interesting Numbers*).

Because of the 'casting out the nines' rule, the sum of the digits of every period of a full-period prime must be a multiple of 9. Curiously though, this seems to be true of *all* primes whereas, by my reckoning, it is only necessary that the period sum should add to 9 when multiplied by the relevant factor of $p - 1$. Can one conclude that if the period sum is not a multiple of 9, then this shows the number is composite?

A topic about which surprisingly little is known even in this computer age is *unique period primes*, i.e. primes whose period length is not shared by any other number. Thus 3 is the only number with period length 1. Unique-period primes are pretty thin on the ground—there are only eighteen below 10^{50} according to Caldwell & Dubner (*Journal of Recreational Mathematics*, Vol. 29 No. 1, 1998). It is (in 1998) not known whether there are infinitely many unique-period primes.

The recommended way to search for them is to examine strings of 9s for factors—as explained by Ken Greatrix and John Reade (M500 **184**). What we want is for a 99999... to produce only *one* new prime factor. In the case of a period of 5 we have $99999 = 9 \cdot 11111 = 3^2 \cdot 41 \cdot 271$. We can forget the 3—it is not a new factor—but we have acquired two numbers with a period of five digits—41 and 271. So neither is a unique-period prime. A string of 1s is apparently called a *repunit*, and it follows that when a repunit is prime it is unique-periodic! Thus $11 = R_2$ is the only number whose reciprocal has a period of length 2. According to Caldwell & Dubner, only five repunit primes are known—what are they? To help you on your way,

(i) for R_n to be prime n must be prime,

(ii) a divisor of R_p ($p > 3$ a prime) has the form $2kp+1$ for some integer k.

Sebastian Hayes

Addition

Dear Tony,

Re: Sebastian Hayes, 'Addition', page 25 of M500 **184**.

Thanks for the discussion on mathematical Platonism. With regards to an alien species having a different physiology, most people assume happily that although they have a different physiology, the laws of physics are the same throughout the universe. The argument then becomes not simply one about Platonism, but whether mathematics is 'descriptive' in that it is rather like language or poetry, or whether it is 'representational' which is more the Platonist idea. If indeed it is representational, the aliens would end up maybe using a different mathematical language, but the underlying mathematical ideas would have to be the same, because we have assumed the laws of physics are the same throughout the universe. Since the laws of physics are the same, they would represent this with the same underlying mathematical ideas.

A good book that discusses this argument is *Philosophy of Mathematics, an introduction to the world of picture proofs and pictures* by James Robert Brown. As you would expect, James Brown (I feel good) veers strongly towards Platonism. He says, 'It must be said that the entire history of mathematics very strongly supports the autonomy of mathematics and hence strongly supports the representational account.'

Sheldon Attridge

Russian roulette

Dear Tony,

Re: your Solution 182.3 – Russian roulette.

Here are two much easier solutions.

1. Remove the bullet. This makes both parties have zero probability of losing.

2. Don't spin the chamber after each turn. This makes both parties have probability 1/2 of losing. We presume that the revolver has an even number of chambers, as is usual.

David Singmaster

John Bull, who originally submitted the problem, points out a subtle flaw in David's solution 2, above. He says: 'One can imagine the following duel:

John goes first. "Click." John lives.
Tony goes second. "Click." Tony lives.
John goes third. "Click."
Tony goes fourth. "Click."
John goes fifth. "Click."
Tony says, "Yah! Boo! Don't want to play any more!"

Or would Tony happily pull the trigger knowing that he has had an equal chance?'

There also seems to be a legal objection. If John's seconds force Tony to play on, it's murder, the avoidance of which was the primary reason for inventing the game in the first place.

Anyway, such frivolities aside, we are still interested in seeing a solution to the genuine mathematical problem of sequencing the turns so that the chances are *exactly* matched. The pattern $ABABAB$ followed by $BBAABBAA\ldots$ gets quite close; A's probability of losing is reduced to $1423111/2846016 \approx 0.500036$ from $6/11$ (corresponding to $ABAB\ldots$). —**ADF**

Sundials

Dear Tony,

I am sure Peter Lord could find the answers to his questions in a lovely little book called *Sundials – Their Theory and Construction*, by Albert E. Waugh. I hope it is still available. It was published in 1973 by Dover Pub. Inc. with ISBN No. 0-486-22947-5 and contains some fascinating mathematics and theoretical explanations as well as practical advice for sundial builders.

Dilwyn Edwards

M500 184

Dear Eddie,

Many thanks for **M500 184**.

The answer to 'Problem 184.2 – Monk' seems so absurdly simple that I feel I must have done something illegal. If you simply superimpose the monk's upward and downward paths, it needs no proof that they will pass each other at some time. If such time travel is barred, the monk's faithful acolyte, equipped with binoculars, could be observing and timing his ascent, and repeating this exactly at 6 am two days later so that he is bound to meet his master coming down.

The Rubik's dodecahedron on the cover led me to produce a model from old egg boxes and sticky-backed plastic. However, I find that I need a five-dimensional universe to allow the pieces to slide. Please supply by first-class post.

For me, the interest of Rubik's cube is not the rather boring and obsessional manoeuvres you can do with it, but the beautiful interlocking spherical mechanism inside it that allows any slice to rotate. I would be prouder to have designed this than to have invented the cube.

Best wishes,

Ralph Hancock

ADF—Three dimensions suffice. The thing exists and I have one sitting on my desk. When you take it apart you can clearly see the 12 face centres attached by spindles to a central sphere. The other pieces are held in place by a cleverly designed system of notches which allows them to slide past each other in the Rubikian manner.

A thought. If the monk proceeds along the path in a step-by-step motion by walking (or running), then surely the most appropriate model for his progress up and down the mountain is a *step function*. Therefore, as step functions are not continuous, the primary assumption of the usual proof does not apply and, indeed, the proposition appears to be false.

M500 Mathematics Revision Weekend 2002

The **28th M500 Society Mathematics Revision Weekend** will be held at **Aston University**, **Birmingham** over **13–15 September 2002**.

Tutorial sessions start at 19.30 on the Friday and finish at 17.00 on the Sunday. The Weekend is designed to help with revision and exam preparation, and is open to all OU students. We plan to present most OU mathematics courses.

For details and an application form, or send a stamped, addressed envelope to **Jeremy Humphries, M500 Weekend 2002.**

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