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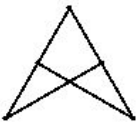

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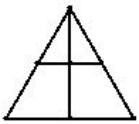

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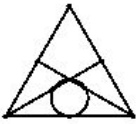

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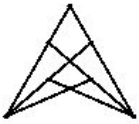

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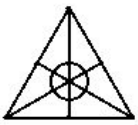

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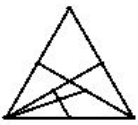

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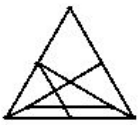

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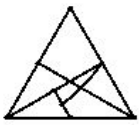

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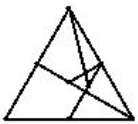

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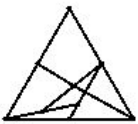

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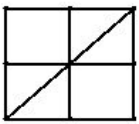

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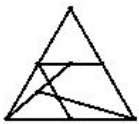

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The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Editor – *Tony Forbes*

Editorial Board – *Eddie Kent*

Editorial Board – *Jeremy Humphries*

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

Two methods for partitions of integers

Tommy Moorhouse

Introduction

A partition of a positive integer n is an expression for n as a sum of positive integers. The number of distinct partitions $p(n)$ of n grows very rapidly with n: for example $p(10) = 42,\ p(100) = 190,569,292$. This article gives two methods for finding a recurrence relation for $p(n)$, one based on an expression for the exponential of a power series, the other employing an integer logarithm function, which we denote by κ.

1 An exponential method for deriving recurrence relations

A recurrence relation is an expression for obtaining terms of a given type from other terms of the same type. The expressions

$$u_{n+1} = u_n - u_{n-1},$$

$$v_{n+1} = v_n + 1/v_{n-2}$$

are both examples of recurrence relations. In order to solve the relations we would need to know some initial values, e.g. u_0, u_1. This subject is well developed and expositions can be found in many elementary texts.

One well-known method of solving recurrence relations involves forming a 'generating function' from the terms as follows. Given a set of elements $u_i, i = 0, \ldots, \infty$, we form the formal power series $S(x) = \sum_{i=0}^{\infty} u_i x^i$, where x is to be treated as a variable when manipulating the function $S(x)$. Inserting the initial values and substituting the other u_i by their expressions in terms of lower elements we find another expression for $S(x)$ which may help us to solve the recurrence relation. The reader may refer to elementary texts for examples. Importantly, not all recurrence relations admit explicit solutions and this is the case for the relation we derive below. In these cases computers can be used to find very many of the terms, and this is the approach we use.

In this article we will be concerned with obtaining a recurrence relation for the terms in a generating function for the function assigning to an integer n the number of partitions of n. Recurrence relations for this function are known (see for example Apostol [1, Section 14.6]), but we will find a simple expression and show how it can be coded into the computer language MAPLE. The idea of a partition is examined further in the next section.

Naturally in order to make use of the available methods for solving recurrence relations we need to find one. In the case of partitions of n we

have a powerful tool in the generating function. We have

$$\prod_{i=1}^{\infty} \frac{1}{(1-x^i)} = S(x) = \sum_{i=0}^{\infty} p(i)x^i.$$

The proof is in Apostol [1]. For the purposes of this example all series are treated as formal series and questions of convergence are not considered. The pitfalls of such an approach are set out in many analysis texts (e.g. Whittaker and Watson [2]), but we will not pursue a rigorous development.

We now examine a special case of a general recurrence relation. In the sequel we will take $H(x)$ to be $\log(S(x))$, because we want to use the trick of exponentiating to get back our original series (i.e. $\exp(\log(S(x)) = S(x)$). Obviously there are some issues around the convergence properties and whether the method is consistent, but we will leave these matters aside. Thus we let $H(x) = \sum_{i=0}^{\infty} h_i x^i$, and take

$$\exp(H(x)) = \sum_{i=0}^{\infty} e_i x^i.$$

We require that $\exp(H(x))$ satisfy the usual exponential differential equation

$$\frac{d}{dx}\exp(H(x)) = \frac{d}{dx}(H(x))\exp(H(x))$$

and expand everything in terms of the h_i and e_i to find that

$$(i+1)e_{i+1} = \sum_{j+k=i+1} e_j h_k.$$

Substituting the particular form of $S(x)$ relevant to the partition function into this gives, after collecting terms,

$$np(n) = \sum_{j+k=n} \sigma(j)p(k).$$

Here $\sigma(n)$ is the sum of the divisors of n and the sum over j and k involves non-negative integers only. The following MAPLE code puts this to use.

```
with(numtheory);
part:= proc(n::integer) option remember;
  if n <=1 then 1 else add(sigma(k)*part(n -‘k‘)/n,‘k‘=1..n)
fi;
end;
```

The code runs reasonably quickly and could be used, for example, to check identities involving partitions. It can also be readily adapted to other languages such as JAVA or PYTHON if desired.

2 The function κ

We now introduce a function κ from the positive integers to the natural numbers with the property

$$\kappa(nm) = \kappa(n) + \kappa(m)$$

for all pairs of integers n, m. This is an analogue of the logarithm, and can be generalized to give a large family of functions with this property.

We can express any positive integer as a product of prime numbers in an essentially unique way (i.e. unique up to ordering). If we agree to order the prime factors of an integer n by magnitude, and label the smallest p_1 and so on up to the largest p_m say, we have

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}.$$

Now we define $\kappa(n)$ as

$$\kappa(n) = \sum_{i=1}^{m} k_i p_i.$$

This gives a well-defined function with the stated property.

As an aside we note that κ may be extended to a function $\kappa_{\mathbb{Q}}$ on $\mathbb{Q}^+$ as follows: if a, $b \in \mathbb{Z}$ then $\kappa_{\mathbb{Q}}(a/b) = \kappa(a) - \kappa(b)$. This is a group homomorphism from $\mathbb{Q}^{+\star} \to \mathbb{Z}$ as is easily shown. The kernel of this map is therefore a subgroup of $\mathbb{Q}^\star$. We will not pursue this further here.

3 Prime partitions and the function κ

Although not essential to the following argument, the function κ illuminates the main points, and directs us to another tool for investigating partitions. In this section we consider partitions of an integer into prime numbers, for example $20 = 2{+}2{+}2{+}2{+}2{+}2{+}2{+}3{+}3$.

We might ask about solutions to the equation $\kappa(u) = n$ given n. From the definition of κ in terms of the prime factorization of u we immediately see that u corresponds to a unique prime partition of n: namely if $u = \Pi_{i=1}^{r} p_i^{k_i}$ then $n = \kappa(u) = \sum p_i k_i$ which is an expression for the prime partition of n:

$$n = \underbrace{p_1 + p_1 + \cdots + p_1}_{k_1} + \cdots + \underbrace{p_r + p_r + \cdots + p_r}_{k_r}.$$

Each prime partition of n corresponds to an integer u such that $\kappa(u) = n$, so, denoting the number of prime partitions of n by $P(n)$ we have

$$|\kappa^{-1}(n)| = P(n).$$

We have used the set theoretic definition of κ^{-1} as the set of all elements mapped to n by κ. Now consider the product $\Pi_{u\in\kappa^{-1}(n)}u$. We denote this simply by $\Pi(n)$. Then it is immediately clear that

$$\kappa(\Pi(n)) = nP(n).$$

4 An expression for $\Pi(n)$

We can find an expression for $\Pi(n)$ in terms of primes smaller than n. First we note that $2^{P(n-2)}$ divides $\Pi(n)$ because there are $P(n-2)$ partitions of n with at least one 2 appearing. There are $P(n-2\times 2)$ partitions involving 2+2, but these only introduce another power of $2^{P(n-2\times 2)}$ as we have already counted the partitions involving one 2. Continuing in this way we see that the power of 2 dividing $\Pi(n)$ is

$$2^{P(n-2)+P(n-2\times 2)+\cdots+P(n-2j)+\cdots+P(n \pmod 2)}.$$

In general, the power of the prime $p \le n$ dividing $\Pi(n)$ is $P(n-p)+P(n-2p)+\cdots P(n \pmod p)$ and we have

$$\Pi(n) = \Pi_{p\le n} p^{\sum_{i=1}^{[n/p]} P(n-ip)}.$$

Applying κ we find

$$nP(n) = \sum_{p\le n} p \sum_{i=1}^{[n/p]} P(n-ip).$$

We now collect terms in $P(n-r)$ for a given value of r. Clearly if a prime p divides r then there is a term $pP(n-r)$ in the sum, and these are the only terms that arise. Denoting by $a(n)$ the sum of prime divisors (counted once) of n we see that

$$nP(n) = \sum_{j+k=n} a(j)P(k).$$

We must have $P(0) = 1$, $P(1) = 0$ for consistency, although the justification for $P(0) = 1$ is not intuitive: here it derives from the fact that x^0 has

coefficient 1 in the generating function. The recurrence relation then allows us to calculate as many of the $P(n)$ as our stamina or computing power allows.

5 General partitions

The power of the above method appears to stem from the unique prime factorization of integers. We cannot find a unique factorization of integers in terms of composites to deal with non-prime partitions in the same way. On the face of it we have no chance of repeating the analysis with another version of κ. In fact this is too pessimistic. Let us associate with each partition of n a unique integer as follows:

$$n = \underbrace{1+1+\cdots+1}_{k_1} + \cdots + \underbrace{r+r+\cdots+r}_{k_r} \to p_1^{k_1} \cdots p_r^{k_r},$$

where p_i is the ith prime, counting from $2 \equiv p_1$. This is a well-defined relation with a well-defined inverse. To go further we discuss the generalization of κ alluded to above. For any function f from the positive integers into the integers we define (for n as above)

$$\kappa_f(n) = \sum_{i=1}^{r} k_i f(p_i);$$

κ then corresponds to the identity function, and our new κ corresponds to the function assigning to each prime its order in the list of primes, which we denote by ζ: $\zeta(p_i) = i$. Then if m corresponds to a partition of n as above we have $\kappa_\zeta(m) = n$. Importantly it can be shown that all the functions κ_f behave like logarithms. We can now carry through exactly the same analysis as in the prime case, finding that instead of $a(n)$ the required multiplying function is $\sigma(n)$, as expected.

6 General log-type functions

The functions κ_f are all functions from the positive integers to the positive integers (here we do not consider the generalization to $\mathbb{Q}^\star$ mentioned above). All these functions share the 'logarithm property' $\kappa_f(ab) = \kappa_f(a) + \kappa_f(b)$. In fact any 'integer logarithm' L is of the form κ_f for some f. The proof is based on a recurrence relation for L, namely

$$\sum_{d|n} L(d) = \frac{1}{2}\tau(n)L(n),$$

where $\tau(n)$ is the number of divisors of n. The proof is straightforward because $\sum_{d|n} L(d) = L(\Pi_{d|n} d)$ by the logarithm property. If $\tau(n)$ is even we see that the divisors of n come in pairs so that the product is $n^{\tau(n)/2}$ and applying L gives us the result. The case of $\tau(n)$ odd is only slightly more involved.

We rearrange the relation as

$$L(n) = \frac{1}{\tau(n)/2 - 1} \sum_{d|n, d<n} L(d),$$

which is of the required form. Note that this is undefined if n is prime, so we need to specify the values of L at the primes: but this is just specifying a function f on the primes and hence $L = \kappa_f$.

7 Conclusion

We have examined two routes to the recurrence relation for the prime partitions of an integer. On the way we have introduced a set of integer logarithm functions about which much more could be said. It may be possible to extend the methods to various kinds of partition such as partitions into odd integers and the interested reader may like to pursue this.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1998.

[2] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (fourth edition), Cambridge University Press, 1927.

Problem 218.1 – Wands

Having just returned from a night on the town Harry and his friends cast spells to ward off the effects of excessive drinking. But in the confusion on leaving the club they had each picked up a wand at random from the pile that was given back to them. Also the magic isn't perfect, and when it fails the caster will turn into a toad (albeit one without a hangover). The probability of the spell working properly is p with one's own wand and zero with someone else's wand.

Show that the probability of them all turning into toads is approximately e^{-p}. What is the expected number of toads created by this escapade?

The topologist's dream

Eddie Kent

A topologist, it is said, is someone who can't tell a cup of coffee from a doughnut. Bear that in mind.

This story was published on April 1st, but that doesn't make it necessarily untrue. Dr Robert Bohannon, described in the press release as a molecular scientist (i.e. one made of molecules), has observed a serious gap in the market. He clearly noticed, possibly while driving to work, how difficult it is to eat a doughnut and drink a cup of coffee at the same time. Hence he has presented the world with the caffeinated doughnut. He calls it Buzz Donuts.

His earlier attempts were less than completely successful; in fact he said of them 'They were terrible, absolutely horrid,' and 'It would just make you puke.' But eventually he hit on a process that could mask the bitter taste of the induced alkaloid. The dream becomes reality.

Now, of course, all we are waiting for is doughnut flavored coffee. Something sweet and sticky that deposits sugar and jam down your front as you drink.

If you don't believe me visit www.buzzdonuts.com.

Goldbach's conjecture

Hugh McIntyre

The beautiful simplicity of the Goldbach conjecture has appealed to me for many years and I've doodled with it off and on. It beats counting sheep for sleeplessness. Lately I have come to the conclusion that the conjecture is a special case of a wider conjecture.

Define a set $G(k)$ as containing the even integers which can be represented as the sum of two non-equal odd primes in precisely k ways, k being a positive integer or zero. For example, $G(1) = \{8, 10, 12, 14, 38, \dots\}$.

Goldbach's conjecture is that the set $G(0)$ is empty.

Regardless of the truth or otherwise of this conjecture, does there exist a non-empty set $G(k)$ for every $k > 0$?

And, are all the $G(k)$ finite sets?

'Topology is the study of topological spaces.' [Textbook definition]

Solution 212.3 – 100 seats

Simon Geard

I solved this problem by writing a small program to simulate the situation (below). This showed that the solution—the probability that the last person gets their seat—is 0.5.

Theoretically I suggest considering the probability q that the last person does not get their seat. Suppose there are just two seats in the aeroplane; then $q = 0.5$. Let Q_n be the probability that the nth person takes the last seat. Then $Q_n = 1/n^2$ since they can only choose the last seat if their allocated seat is occupied. Thus

$$\begin{aligned} q &= \frac{1}{n} + \frac{n-1}{n}\frac{1}{(n-1)^2} + \cdots + \frac{2}{3}\cdot\frac{1}{2^2} \\ &= \frac{1}{n} + \frac{1}{(n-1)^2}\frac{1}{(n-1)(n-2)} + \cdots + \frac{1}{3\cdot 2} \\ &= \frac{1}{n} + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \cdots + \left(\frac{1}{2} - \frac{1}{3}\right) \\ &= \frac{1}{2}. \end{aligned}$$

It follows that the probability that the last person does get their seat is also 0.5.

```
! Simulation of the aeroplane problem
!
! N seats on an aeroplane and N people to fill them, each with a
! unique ticket number. Passengers proceed one at a time to their
! seats. The first person ignores their ticket number and chooses
! a seat at random. All subsequent people sit at their seat if it
! is available, otherwise they choose a seat at random. What is
! the probability that the last person can sit in their allocated
! seat.
!
! This simulation program is written in Fortran. To build it
! you'll need a Fortran compiler. If you haven't got one get g95
! from http://www.g95.org.
!
! To run the program with 10 seats type at a command prompt
!
!                          aeroplane 10
```

```
!
! # seats    % availability  % std.dev
!     2          50.035         0.4830
!     3          49.931         0.4910
!     4          50.022         0.4979
!     5          50.026         0.5639
!     6          50.023         0.5229
!     7          50.030         0.5292
!     8          49.962         0.4597
!     9          50.021         0.5095
!    10          50.081         0.4409
!
module simulate
  public :: hasSeat, tidy
  private
  integer               :: nseats = 0
  logical, allocatable :: seatOccupied(:)

contains
  subroutine tidy
    if (allocated(seatOccupied)) deallocate(seatOccupied)
  end subroutine tidy
  logical function hasSeat(ns)
    ! Run the simulation once and return .true. if the final seat
    ! is available
    implicit none
    integer, intent(in) :: ns
    integer :: c
    integer :: i
    nseats = ns
    if (.not. allocated(seatOccupied)) then
       allocate(seatOccupied(ns))
    end if
    seatOccupied = spread(.false.,1,nseats)
    ! Random choice of 1st passenger
    c = selectSeat()
    if (c == 1) then
       hasSeat = .true.  ! Everyone else can sit as allocated
       return
    end if
    if (c == nseats) then
       hasSeat = .false. ! Last seat now occupied
       return
    end if
```

```
    seatOccupied(c) = .true.
    ! Other passengers
    do i=2,nseats-1
       if (seatOccupied(i)) then
          ! Seat occupied so choose another one at random
          c = selectSeat()
          if (c == nseats) then
             hasSeat = .false. ! Last seat now occupied
             return
          end if
          seatOccupied(c) = .true.
       else
          ! Seat available
          seatOccupied(i) = .true.
       end if
    end do
    hasSeat = .not. seatOccupied(nseats) ! Seat available
  end function hasSeat
  integer function selectSeat()
    ! Choose a random unoccupied seat
    implicit none
    real :: h
    do
       call random_number(h)
       selectSeat = 1+nint((nseats-1)*h)
       if (.not. seatOccupied(selectSeat)) exit
    end do
  end function selectSeat
end module simulate

program aeroplane
  use simulate
  implicit none
  integer, parameter :: N = 10000 ! Number of trials in a simulation
  integer, parameter :: L = 100   ! Number simulations
  integer            :: c         ! Count of the number of times in a
                                  ! trial the seat was available
  integer            :: ns        ! Number of seats in the trial
  integer            :: nargs     ! Number of arguments
  character(len=10)  :: arg       ! Holder for the first argument
  real*8             :: mean      ! Mean of results
  real*8             :: stdev     ! Standard-deviation of results
  real*8             :: results(L)! Results vector
  integer            :: i, j, k   ! Dummy loop counters
```

```
  ! Get the number of seats from the command line (default to 100)
  nargs = command_argument_count()
  if (nargs >= 1) then
     call get_command_argument(1,arg)
     read(arg,*) ns
  else
     ns = 100
  end if

  write(*,'(a)') '# seats    % availability  % std.dev'
  if (ns < 0) then
     k = -ns
     call runSimulation
  else
     do k=2,ns
        call runSimulation
     end do
  end if

contains
  subroutine runSimulation
     ! Do the L simulations
     do j=1,L
        c = 0
        !  Do the N trials
        do i=1,N
           if (hasSeat(k)) then
              c = c+1
           end if
        end do
        results(j) = dble(c)/N
     end do
     mean = sum(results)/L
     stdev = sqrt(dot_product(results,results)/L - mean**2)
     write(*,'(3X,i3,8X,f7.3,8X,f6.4)') k,mean*100,stdev*100
     call tidy
  end subroutine runSimulation

end program aeroplane
```

'Wet hair; apply shampoo; work into a lather; wash out; repeat.'

[Instructions on a bottle of hair shampoo]

Russell's attic

Tony Huntington

Eddie Kent [M500 **214**: Russell's attic is a room containing countably many pairs of shoes and countably many pairs of socks. It is easy to see that there are countably many shoes, for instance by matching the left shoes to the odd numbers and the right shoes to the even numbers. But can you say how many socks there are?] has once again raised a subtle and intriguing problem full of apparent paradoxes. He has, perhaps deliberately, omitted to mention whether we are to assume that the shoes are all the same style, and the socks all identical. This leaves it open for us to consider a number of different cases before arriving at a general solution. So, considering first the shoes ...

As Eddie rightly says, the left shoes can be associated with the odd numbers, and the right shoes with the even numbers. If we assume that the shoes were originally purchased in pairs, then their total must be even. However, if we allow for different shoe styles, it does not follow that the total number of odd numbers and even numbers thus produced is equal. I would challenge his assertion that the number of shoes is thus countable, but would accept that the missing shoes are 'accountable' (one disappeared about three in the morning staggering home from Kevin's stag night, and another was thrown from the window at a wailing cat one night, for example, although these two shoes did not come from the same pair). What we can assert is that the total number of shoes will always be even.

Now considering the socks ...

First assume that all of the socks are identical and have been washed at least once. Then there must be an odd number of socks. Now allow all of the socks to exist in pairs where each pair is uniquely distinguishable from every other pair. If all of the socks have been washed at least once, then not only will the total number of socks be odd, but the subset of unmatched socks will also have an odd number of members.

For a mathematical explanation of what is underlying these phenomena, I would commend to readers: *Sock Dynamics and Other Domestic Mathematics* by Ivor Gudideer. In this treatise, the author explains the significance of 'Tight's Equation', which has a remarkable similarity to Schrödinger's Equation. It is a differential equation, and so has steady-state and transient solutions, and it has an Imaginary part which makes it Real. It also successfully explains the probability of a shoe or sock being in a particular place without any guarantee that when you look there you

will find it. The transient solutions to Tight's Equation are particularly interesting (the assertions above are, of course, the steady-state solutions) as they can be applied to the case where: you have an odd sock, you throw it away, and you still have an odd number of socks. With the current interesting discussions in M500 on mathematics and causality, science and religion, I find Tight's Equation a more satisfying explanation of the Real World than the invocation of a 'Sock Heaven' where missing socks go to during the washing process.

None of this discussion leads us closer to answering the fundamental question: 'Can we count the socks?', but at least we now know that if we can count the socks, and the answer is even, then there will always be one more (probably under the bed unless you have looked there) to make the total odd.

[See page 14 for more about socks and the counting thereof.]

Mutually touching cylinders

John Smith

Here is an old open problem. The Web attributes it to Littlewood in 1968.

> What is the largest number of congruent, infinitely long, circular cylinders that can be arranged in 3-d Euclidean space so that each cylinder is touching each other? Is it 7?

A guess as to the likely answer can be obtained by counting degrees of freedom. Each cylinder has 4 degrees of freedom. For N cylinders to mutually touch, there are constraints on $N(N-1)/2$ degrees of freedom. Any final arrangement can be translated, or rotated to give a new arrangement. So a final arrangement has 6 degrees of freedom.

Thus the likely answer is the maximum integer N for which $4N - N(N-1)/2 \geq 6$, which gives N as 7.

From the internet I see that András Bezdek has shown that the maximum number is at most 24. But as yet there are no known arrangements of seven or more cylinders. Some twenty odd years ago I heard it suggested that the problem might be tackled using computer algebra. This must be even more true now.

A picture or description of 6 cylinders is probably worthy of publication in M500. A picture of seven or more could win a remark in the history of mathematics.

Solution 206.3 – Odd socks

Out of n different pairs of socks in a drawer, r socks are removed at random. What is the probability of obtaining d matched pairs? What if two pairs are identical?

Norman Graham

Let e be the number of unmatched socks (removed or remaining) and let f be the number of matched pairs left in the drawer. Then

$$e = r - 2d \quad \text{and} \quad f = n - d - e = n - r + d.$$

The number of selections of r socks from $2n$ is

$$^{2n}C_r \;=\; \frac{(2n)!}{r!(2n-r)!},$$

the number of ways of dividing n pairs into d, e and f is

$$\frac{n!}{d!\,e!\,f!},$$

and the number of ways of choosing one from each of e pairs is 2^e.

Therefore the number of selections with d pairs is

$$\frac{n!\,2^e}{d!\,e!\,f!} \;=\; \frac{n!\,2^{r-2d}}{d!\,(r-2d)!\,(n-r+d)!} \;=\; F(n,r,d),$$

say. Hence the probability required is $F(n,r,d) \,/\, {}^{2n}C_r$.

To prove that the probabilities add up to 1, it is required to show that $\sum_d F(n,r,d) = {}^{2n}C_r$. But ${}^{2n}C_r$ is the coefficient of x^r in $(1+x)^{2n}$ and this is equal to the coefficient of x^r in $(1+2x+x^2)^n$. Using the multinomial expansion

$$(a+b+c)^n \;=\; \sum_{d+e+f=n} \frac{n!}{d!\,e!\,f!}\, a^f\, b^e\, c^d,$$

this is

$$\sum_d \frac{n!}{f!\,e!\,d!}(2x)^e x^{2d} \;=\; \sum_d F(n,r,d)x^r,$$

since $e = r - 2d$.

If two pairs are identical, the solution is obtained by expanding

$$(1+x)^4(1+x)^{2n-4} \;=\; (1+4x+6x^2+4x^3x^4)(1+2x+x^2)^{n-2}.$$

The coefficient of x^r is then the sum over all values of d of

$$F(n-2,r,d) + 4F(n-2,r-1,d) + 6F(n-2,r-2,d-1) \\ + 4F(n-2,r-3,d-1) + F(n-2,r-4,d-2).$$

Similarly, for two sets of pairs use the expansion

$$(1+x)^4(1+x)^4(1+x)^{2n-8}.$$

These results are understood more readily by using a concrete example. For $n = 8$ and $r = 7$, the probabilities are given by the following table.

	$d=0$	$d=1$	$d=2$	$d=3$
Pairs all different	0.090	0.470	0.391	0.049
Two pairs identical	0.022	0.369	0.518	0.091
Two sets of two pairs identical	0.0	0.213	0.621	0.166

Problem 218.2 – Central binomial coefficient

Show that the number of decimal digits in the binomial coefficient

$${}^{2\cdot 10^n}C_{10^n} = \binom{2\cdot 10^n}{10^n} = \frac{(2\cdot 10^n)!}{(10^n!)^2}$$

is approximately equal to the integer formed from the first n digits of $\log_{10} 4$ after the decimal point.

Indeed, for $n = 1, 2, \ldots, \binom{2\cdot 10^n}{10^n}$ has 6, 59, 601, 6019, 60204, 602057, 6020597, ... digits, whereas

$$\log_{10} 4 = 0.6020599913279623904\ldots.$$

The integral of t squared dt
From one to the cube root of three
Times the cosine
Of three π over nine
Is the log of the cube root of e.

[Solution 217.3: $\int_1^{\sqrt[3]{3}} t^2 dt \cdot \frac{1}{2} = \frac{1}{3}$.]

Solution 216.2 – Ramanujan's continued fraction

As it is recounted by Kanigel, *The Man who Knew Infinity* (Abacus, 1991), a Hindu friend of Ramanujan's, Mahalanobis, when he and Ramanujan were both at Cambridge, read out to him a puzzle from *Strand* magazine about an inhabitant of Louvain (which had just been burned by the Germans). This Belgian lived in a house on a long street which was numbered $1, 2, 3, \ldots$ consecutively along his side of the street. The number of his house had a curious property: the sum of all the house numbers before it was the same as the sum of all the house numbers that came after it. The magazine stated that there were more than fifty houses and less than five hundred houses on that side of the street. So what was the Belgian's house number? Ramanujan thought for a moment and then dictated the first few convergents of a continued fraction which included all the solutions to the problem (not just the one falling within the 50–500 range).

Norman Graham

Let x be the house number required and z the number of further houses. Then

$$\sum_{i=1}^{x-1} i \;=\; \sum_{i=1}^{z} (x+i).$$

Therefore

$$\begin{aligned}
\tfrac{1}{2}x(x-1) &= zx + \tfrac{1}{2}z(z-1), \\
x^2 - x &= 2zx + z^2 + z, \\
z^2 + z(2x+1) - (x^2 - x) &= 0, \\
z &= \frac{1}{2}\left(-(2x+1) \pm \sqrt{(2x+1)^2 + 4(x^2-x)}\right).
\end{aligned}$$

The negative sign does not apply since z is positive. Therefore

$$z \;=\; \frac{1}{2}\left(-(2x+1)\sqrt{8x^2+1}\right).$$

For integral z, this has a solution if and only if $8x^2+1$ is a perfect square, say y^2. Now $y^2 = 8x^2 = 1$ is a Pell equation whose solutions (y_n, x_n) are (for selected n) the nth convergents of $\sqrt{8}$; i.e.

$$\frac{y_n}{x_n} \;=\; a_1 + \frac{1}{a_2+}\;\frac{1}{a_3+}\;\cdots\;\frac{1}{a_n},$$

where

$$\sqrt{8} \;=\; a_1 + \frac{1}{a_2+}\;\frac{1}{a_3+}\;\ldots \text{ to } \infty.$$

But

$$\sqrt{8} \;=\; 2 + (\sqrt{8}-2) = 2 + 4/(\sqrt{8}+2).$$

Therefore $a_1 = 2$. Also

$$\tfrac{1}{4}(\sqrt{8}+2) \;=\; 1 + \tfrac{1}{4}(\sqrt{8}-2) = 1 + 1/(\sqrt{8}+2);$$

hence $a_2 = 1$. And

$$\sqrt{8}+2 \;=\; 4 + (\sqrt{8}-2) = 4 + 4/(\sqrt{8}+2),$$

giving $a_3 = 4$. Similarly, $a_4 = 1$, $a_5 = 4$, ..., $a_{2m} = 1$, $a_{2m+1} = 4$ for $m \geq 1$.

The simplest way of calculating (y_n, x_n) is to use the formulae

$$(y_0, x_0) = (1, 0), \qquad (y_1, x_1) = (a_1, 1),$$

and for $n > 1$,

$$y_n = a_n y_{n-1} + y_{n-2}, \qquad x_n = a_n x_{n-1} + x_{n-2}.$$

The results for $n = 0$ to 12 are in the following table; $y_n^2 - 8x_n^2 = 1$ for all even values of n. The solutions to the problem are $x = 1$ (trivial), 6, 35, 204, 1189, 6930, ... The answer to the *Strand* problem is $x = 204$. Then $y = 577$ and $z = 84$.

n	a_n	y_n	x_n	$y_n^2 - 8x_n^2$	$\frac{1}{2}(y_n - 1) - x_n$	y_n/x_n
0		1	0	1	0	
1	2	2	1	-4		2
2	1	3	1	1	0	3
3	4	14	5	-4		2.8
4	1	17	6	1	2	2.833 333
5	4	82	29	-4		2.827 586
6	1	99	35	1	14	2.828 571
7	4	478	169	-4		2.828 402
8	1	577	204	1	84	2.828 431
9	4	2786	985	-4		2.828 426 396
10	1	3363	1189	1	492	2.828 427 250
11	4	16238	5741	-4		2.828 427 103
12	1	19601	6930	1	2870	2.828 427 128

Further comments

Since $\sqrt{8} = 2.828\,427\,125$, the last column demonstrates that as $n \to \infty$, $|y_n/x_n - \sqrt{8}| \to 0$ and $y_n/x_n - \sqrt{8}$ are alternately plus and minus.

Both a_n and $y_n^2 - 8x_n^2$ have a cycle of 2 and $y_n^2 - 8x_n^2 = 1$ is only satisfied for n even.

For other values of k instead of 8, there will always be a cycle of r, say, of the values of a_n for $\sqrt{k}$. If r is even, $y_n^2 - kx_n^2 = 1$ is satisfied for $n = ri$ (all values of i). If r is odd, $y_n^2 - kx_n^2 = 1$ for $n = 2ri$, but -1 for $n = (2i+1)r$. Also $r = 1$ for $k =$ square $+ 1$ (e.g. $\sqrt{10} = 3 + \frac{1}{6+}\,\frac{1}{6+}\,\frac{1}{6+}\,\ldots$) but r is even for most other values of k.

A Latin square puzzle

Tony Forbes

No, it's not sudoku.

> *Fill in the blanks such that every row and column contains each of the symbols* $1, 2, 3, 4, 5, 6, 7, 8, 9$.

1	2			4				7
	3		9		6	5		
8						7	5	
		1		8		3		
	5		7		9		4	
		8		1		2		
	8	3						6
		9	5		2		8	
6				2			3	4

I have observed that I find these puzzles quite difficult to do. But the computer says this one is easy—in the sense that no backtracking is required. Anyway, see how you get on. Remember to be on your guard against making illegal inferences according to the familiar sudoku rules. The division of the diagram into 3×3 boxes is just to make the thing look pretty; it has no significance.

Letters to the Editor

Constants

Dear Tony,

I enjoyed Issue **216** of the M500 magazine, especially as 216 is the only cube which is the sum of three consecutive cubes:

$$216 \;=\; 6^3 \;=\; 3^3 + 4^3 + 5^3.$$

Problem 216.5 [Solve $x = 3\ e^{x^2/214}$] was a neat way to approximate π. A good approximation to another constant is

$$2^{(5/2)^{2/5}}.$$

Yours sincerely,

Patrick Walker

Dear Editor,

In issue **216**, page 19, you refer to 'the fundamental mathematical constants' $\{\pi, e, i, \phi\}$. It seems to me that the constants $\{\pi, i\}$ are of no great significance in algebraic fields other than the euclidean complex numbers. I allow that π appears in the sums of many real number series. Outside of euclidean space, ϕ cannot be any geometric constant, but it is associated with Fibonacci numbers. And e is fundamental in all all algebraic fields. Are the other mathematical constants as fundamental as e?

Dennis Morris

Solution 216.5 – Equation

Solve the equation $x = 3\ e^{x^2/214}$.

John Spencer

Equation $x = 3e^{x^2/214}$ returns a value for x very close to π. The Newton–Raphson method can be used to find a solution, or one can note that

$$3e^{x^2/214} \;=\; \sum_{n=0}^{\infty} \frac{3 \cdot x^{2n}}{n! \cdot 214^n}.$$

If π is substituted for x in the right-hand expression, the series sums to $3.1415990924\ldots$, which is π to five decimal places.

Farewell Norma

Judith Furner

Back in the dark mists of time, flushed with their success at running the September Revision Weekend, the M500 committee decided to run a Fun Weekend in January. At that very first Winter Weekend, which took place in Retford, the committee were joined at the dinner table by one Norma Rosier, who provided us with oranges. It was immediately apparent that she would be a useful committee member and she was duly cultivated. Before long she had offered assistance to Eddie with indexing the magazine. On the 16th June 1984 Norma was co-opted on to the committee. She was a valuable and helpful member and after two years the committee accepted her resignation with great regret. She was moving to the wilds of Lewis, and felt that she could not usefully remain on the committee.

We struggled along without her, and a year later begged her to return. In due course she was again co-opted and Judith was immensely grateful for her assistance with the Revision Weekend. Indeed, she was so efficient that in 1989, when pressure of her own work forced Judith to tender her resignation as Weekend Organizer, it was agreed that Norma would be appointed, with a handover time of two years. Norma organized the Weekend, with one year's break, until 1997, when pressure of her work, in turn, forced her to resign. However she was already organizing the Winter Weekend, which she continued to do until 2004. Norma was also the OUSA representative and ensured that the M500 Society's interests were heard.

It was with the greatest regret that this year the committee once again accepted Norma's resignation. We had to agree that although Lewis was within reach of committee meetings in the UK, Canada really was too far. Norma was an industrious and conscientious member of the committee. Her sharp mind and wit were much appreciated, as were her integrity and common sense. She is much loved and her absence is keenly felt.

Problem 218.3 – Nearly an integer

ADF

Let

$$\alpha = \sqrt[3]{\frac{1}{2}\left(27 + 3\sqrt{69}\right)}, \qquad \beta = \left(\frac{\alpha}{3} + \frac{1}{\alpha}\right)^{2000}.$$

Show that β is within 10^{-120} of an integer.

M500 Winter Weekend 2008

A Weekend of Mathematics and Socializing

Join with fellow mathematicians for a weekend of fun and a look at some interesting, unusual and recreational mathematics. The traditional Winter Weekend will be held at **Florence Boot Hall** (largely in the bar) of **Nottingham University** from the evening of **4th January 2008** to Sunday afternoon **6th January 2008**. Scheduled subjects are:

Mel Starkings:	**Expect to be Entertained**
Rob:	**Rob's (esoteric) Quiz**
Glynn:	**Probably Something on Probability**
Tom Roper:	**Non-Euclidean Spaces**
Dick Boardman:	**Cells and Patterns**
Dennis Morris:	**The Higher-Dimensional Natural Spaces**

Cost: £180 to M500 members, £185 to non-members.

For a booking form, send a stamped addressed envelope to

Diana Maxwell.

Problem 218.4 – Repeated differentiation

Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) \;=\; \frac{(-1)^n\, n!}{x^{n+1}}\left(\log x - \sum_{r=1}^{n}\left(\frac{1}{r}\right)\right).$$

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