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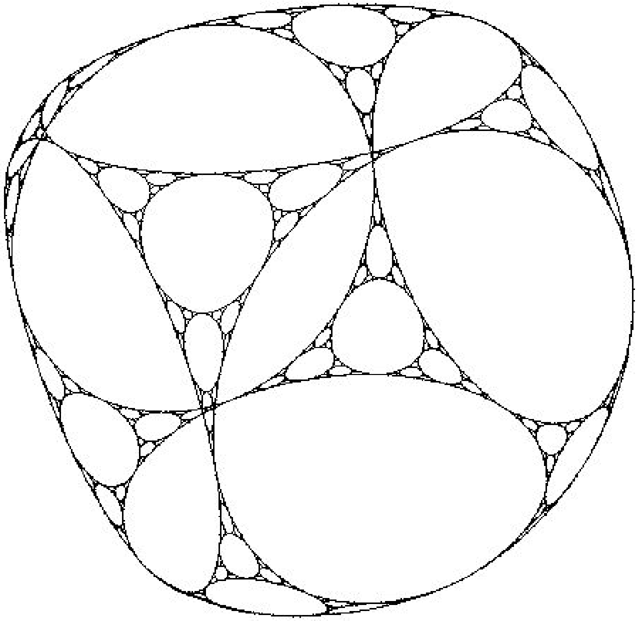


The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details, send a stamped, addressed envelope to Diana Maxwell, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation.

The accounts of the M500 Society for 2010 are available on the Society's web site.

A combinatorial football problem

Garry Green and Terry Griggs

Sport is a rich area for the application of Mathematics at all levels. For combinatorialists, the traditional game of football, towards the end of the season when only one or two rounds of fixtures remain, sometimes offers interesting situations. This was particularly the case at the end of the 2010/2011 season in the Blue Square North League. On 29th April 2011, a date remembered for a certain Royal Wedding, the top of the League was as follows.

		P	W	D	L	F	A	Diff	Pts
1.	Alfreton	39	28	5	6	94	31	+63	89
2.	Telford	39	22	13	4	68	29	+39	79
3.	Boston	39	22	10	7	70	33	+37	76
4.	Eastwood	39	22	6	11	80	48	+32	72
5.	Nuneaton	39	21	9	9	64	41	+23	72
6.	Guiseley	39	20	12	7	54	39	+15	72

For non-soccer aficionados P, W, D, and L are the numbers of games played, won, drawn, and lost respectively; F and A are the numbers of goals scored for and against and Diff is F−A. The final column is the number of points, calculated as 3 for a win, 1 for a draw, and 0 for a loss.

There was one game left for each team to play, on Saturday 30th April. In this division the top team, which must be Alfreton because they cannot be overtaken, is automatically promoted. The next four teams enter a play-off scenario with position 2 against position 5 and position 3 against position 4 with the winners of the two games playing a final game to determine which other team will be promoted. Clearly both Telford and Boston have already achieved play-off status but the other two places are between three teams; Eastwood (E), Nuneaton (N), and Guiseley (G). In the case of equal points, position is decided on goal difference and then if a further criterion is needed, goals scored. So qualification for the play-offs for these three teams depends on the results of their last games. What makes the situation particularly interesting though is these final fixtures.

Alfreton *v.* Nuneaton and Eastwood *v.* Guiseley

To determine the possible outcomes needs a careful case by case analysis. Below is a table showing the final positions of the three teams dependent on the results of the two matches. We have made the assumption that no team will either win or lose so heavily that the goal differences will change the positions. This is not an unreasonable assumption given that they are +32, +23, and +15.

	Alfreton win	Draw	Nuneaton win
Guiseley win	4 G 75 5 E 72 6 N 72	4 G 75 5 N 73 6 E 72	4 N 75 5 G 75 6 E 72
Draw	4 E 73 5 G 73 6 N 72	4 E 73 5 N 73 6 G 73	4 N 75 5 E 73 6 G 73
Eastwood win	4 E 75 5 N 72 6 G 72	4 E 75 5 N 73 6 G 72	4 E 75 5 N 75 6 G 72

As can be seen all of the six order possibilities occur with ENG appearing four times and each other possibility just once. But this does not mean that the probabilities of these orders are in the same proportion. That would only be the case if the probabilities of each of the three outcomes in both of the games is 1/3. So how might these probabilities be assessed? For the Alfreton *v*. Nuneaton game, clearly Alfreton seem to be the better team and have home advantage. But they have nothing to play for whereas Nuneaton do and so might be better motivated. Nevertheless probably Alfreton have the advantage. The other game is more difficult to predict. The teams seem to be evenly matched but Guiseley have home advantage. Looking at the betting odds on the internet on the evening prior to the games gave the following.

Alfreton win 8/15 Draw 5/2 Nuneaton win 9/2

Guiseley win 13/10 Draw 5/2 Eastwood win 6/4

When converted to probabilities these are as follows.

Alfreton win 0.652 Draw 0.286 Nuneaton win 0.182

Guiseley win 0.435 Draw 0.286 Eastwood win 0.400

Of course the sums of these probabilities for each game is not 1 but approximately 1.12, reflecting the profit margin of the bookmaker. Standardizing the probabilities gives the following results.

Alfreton win 0.583 Draw 0.255 Nuneaton win 0.162

Guiseley win 0.388 Draw 0.255 Eastwood win 0.357

We are now in a position to calculate the probabilities of the six possible outcomes. By multiplication and addition of the probabilities of the outcomes of the two matches we obtain the following table.

Order	ENG	GEN	EGN	GNE	NGE	NEG
Probability	0.422	0.226	0.149	0.099	0.063	0.041

Interpreting these results shows that Eastwood have an 84% chance of reaching the play-offs whilst Nuneaton have a 62% chance and Guiseley a 54% chance. But to finish 4th the chances are Eastwood 57%, Nuneaton 10%, and Guiseley 33%. So Nuneaton have a better chance than Guiseley of making the play-offs but less than a third of the chance of being 4th which seems slightly counter-intuitive.

We embarked on this analysis because as Boston fans we wanted to know which team would be Boston's most likely opponents in the play-off match. But this raises another interesting combinatorial situation. Looking at the League Table shows that if Boston win their last game and Telford lose theirs, then they will both have 79 points and Boston's goal difference will be at least +38 and Telford's at most +38. So the two teams would swap position because of Boston's superior goals scored, unless Boston win by only one goal and Telford lose by only one goal and in doing so score 3 or more goals than Boston. If it is just 2 goals more, for example if Boston win 1-0 and Telford lose 3-4, then both teams would have identical goals for (71) and goals against (33). We are not sure how the positions are then decided. It may be on the results of the matches during the season between the two teams. But Telford won 1-0 at Boston and in return, Boston won 1-0 at Telford. But it might also be on the number of wins and in this case Boston would have 23 to Telford's 22. But until 1981, only 2 points were awarded for a win in which case Telford could not be caught in second position. For the last games both Telford and Boston had what appeared to be easy home games so we assumed that both would win and their positions would not reverse. This makes Eastwood, Boston's most likely play-off opponents.

So what actually happened? As expected both Telford and Boston won so Telford came 2nd and Boston 3rd. Alfreton beat Nuneaton 3-2 and Guiseley and Eastwood drew 2-2. So Eastwood were 4th, Guiseley were 5th, and Nuneaton were 6th. But there was a final twist to the story. Because their ground did not match the requirements for promotion, Eastwood were disqualified from the play-offs, so Boston played Guiseley and Telford played Nuneaton. The final was contested between Guiseley and Telford with Telford winning 3-2 and joining Alfreton in promotion.

Problem 244.1 – Counting graphs

How many simple graphs are there with n vertices, where each vertex has degree 1 or 2?

Counting graphs is in general rather difficult. However, in this case I (TF) think the problem is doable since the graphs in question consist only of collections of paths and cycles. For example if $n = 6$, the answer is eight: (i) 6-path, (ii) 6-cycle, (iii) 4-path and 2-path, (iv) 4-cycle and 2-path, (v) two 3-paths, (vi) 3-path and triangle, (vii) two triangles, (viii) three 2-paths.

Problem 244.2 – A quick number wonder

Martin Hansen

Whilst working on something else I started to suspect that the following might be true.

$$\begin{aligned}
\sqrt{1 \times 2 \times 3 \times 4 + 1} &= 2 \times 3 - 1 \\
\sqrt{2 \times 3 \times 4 \times 5 + 1} &= 3 \times 4 - 1 \\
\sqrt{3 \times 4 \times 5 \times 6 + 1} &= 4 \times 5 - 1
\end{aligned}$$

Is it a fluke or will the pattern continue to hold? Counter-example or a proof, please.

Problem 244.3 – Two sums

Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + 2n)^2} = \frac{4\pi^2 - 33}{48} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(n^3 + 3n^2 + 2n)^2} = \frac{4\pi^2 - 39}{16}.$$

A sliding-block puzzle

Tony Forbes

The diagram represents a sliding-block puzzle consisting of elements moving within the confines of a square tray.

A		1
2	3	4
5	6	7

The objective is to move piece A to the bottom right-hand square (currently occupied by 7) by the usual process of sliding the little squares horizontally and vertically. To make the problem more challenging the pieces are constrained to move as follows.

Pieces 3 and 6 cannot move into a corner.

Pieces A and 7 can only move horizontally into or out of a corner.

Pieces 2 and 4 can only move vertically into or out of a corner.

There are no restrictions on the movements of pieces 1 and 5.

You could implement this thing with the top-left 3×3 part of a standard Sam Lloyd puzzle, but you must at all times remember to avoid breaking the rules. If you want a purpose-built toy to play with, you might like to know that this game has been marketed under the names *Impossible!!* and *Twice* by Dario Uri of Bologna. The constraints are achieved mechanically by inserting pins in the centres of the corner squares of the tray and cutting horizontal and vertical grooves in the appropriate pieces.

The makers give the best known solution length as 50 moves. However, **Dick Boardman** has managed to improve this figure to 42, and we are delighted to publish his solution in this magazine. (Look away now if you want to try the puzzle for yourself.)

The marketed package actually contains two games (as suggested by the name *Twice*). In the second variant piece 2 has its orientation changed; it is now allowed to move horizontally but not vertically into and out of a corner. Here Dick is in agreement with the stated optimum solution: 70 moves.

Solution to the sliding-block puzzle

Dick Boardman

0	1	2
3	4	5
6	7	8

Number the squares as on the right. My program examines possible positions in a 'breadth first' search. That is, it creates a list of positions as follows.

Starting from a 'parent' position, there are two, three or four possible 'child' positions. If a child position is not already in the list it is added to the list, together with a record of who its parent was and how far it is from the start position, that is, its depth. When all the children have been added, the next child becomes the parent and its children have a depth one greater. Thus the list grows, with the depth either the same, or increasing by one. Whenever the depth changes, all positions with the smaller value are in the list.

The target is to move tile A into the bottom right corner (square 8). If a position meets this target, my program works backwards through the list, parent by parent, to the start position, to find the route to get to it. This procedure finds the shortest possible route (or routes) from the start to the target. The author of the puzzle says that the shortest known is in 50 moves; however, my program finds a solution in 42 moves.

Start with pieces A, 1, 2, ..., 7 in locations 0, 2, 3, ..., 8 respectively. For the nth step, piece p is moved from location s to location d, leaving the array as indicated in the 5th column, zero denoting the empty square.

n	s	d	p	state	n	s	d	p	state	n	s	d	p	state
1	2	1	1	A10234567	15	6	7	1	A64253017	29	2	5	4	250A34167
2	5	2	4	A14230567	16	3	6	2	A64053217	30	1	2	5	205A34167
3	4	5	3	A14203567	17	4	3	5	A64503217	31	4	1	3	235A04167
4	1	4	1	A04213567	18	1	4	6	A04563217	32	7	4	6	235A64107
5	0	1	A	0A4213567	19	0	1	A	0A4563217	33	8	7	7	235A64170
6	3	0	2	2A4013567	20	3	0	5	5A4063217	34	5	8	4	235A60174
7	4	3	1	2A4103567	21	6	3	2	5A4263017	35	4	5	6	235A06174
8	7	4	6	2A4163507	22	7	6	1	5A4263107	36	3	4	A	2350A6174
9	6	7	5	2A4163057	23	4	7	6	5A4203167	37	6	3	1	2351A6074
10	3	6	1	2A4063157	24	1	4	A	5042A3167	38	7	6	7	2351A6704
11	0	3	2	0A4263157	25	0	1	5	0542A3167	39	4	7	A	2351067A4
12	1	0	A	A04263157	26	3	0	2	2540A3167	40	5	4	6	2351607A4
13	4	1	6	A64203157	27	4	3	A	254A03167	41	8	5	4	2351647A0
14	7	4	5	A64253107	28	5	4	3	254A30167	42	7	8	A	23516470A

Solution 241.4 – Product

Obtain an expression (as a function of n) for the product

$$f(n) \;=\; \prod_{k=2}^{n} \frac{k^2}{k^2-1}.$$

Basil Thompson

The numerator of the expression is $(n!)^2$. The denominator is

$$(1\cdot 2)(2\cdot 4)(3\cdot 5)\ldots((n-1)(n+1)) = (n-1)!(n+1)!/2.$$

Thus $f(n) = 2n/(n+1)$.

Richard Gould

Expressing each term as

$$\frac{k}{k-1}\cdot\frac{k}{k+1},$$

we see that all fractions except the first and last cancel, giving $f(n) = \dfrac{2n}{n+1}$.

Tony Forbes

On the other hand, $g(n) = \prod_{k=2}^{n} \dfrac{k^2}{k^2+1}$ seems to be more difficult.

n	2	3	4	5	6	7	8	9	10	$\ldots$
$g(n)$	$\frac{4}{5}$	$\frac{18}{25}$	$\frac{288}{425}$	$\frac{144}{221}$	$\frac{5184}{8177}$	$\frac{127008}{204425}$	$\frac{8128512}{13287625}$	$\frac{329204736}{544792625}$	$\frac{1316818944}{2200962205}$	$\ldots$

Can anyone see a pattern in those fractions? If you take the product to infinity, you get a simple answer in each case. Thus $f(\infty) = 2$ but changing the sign from minus to plus produces $g(\infty) = \Gamma(2-i)\Gamma(2+i) \approx 0.544058$. This suggests another problem.

Problem 244.4 – Another product

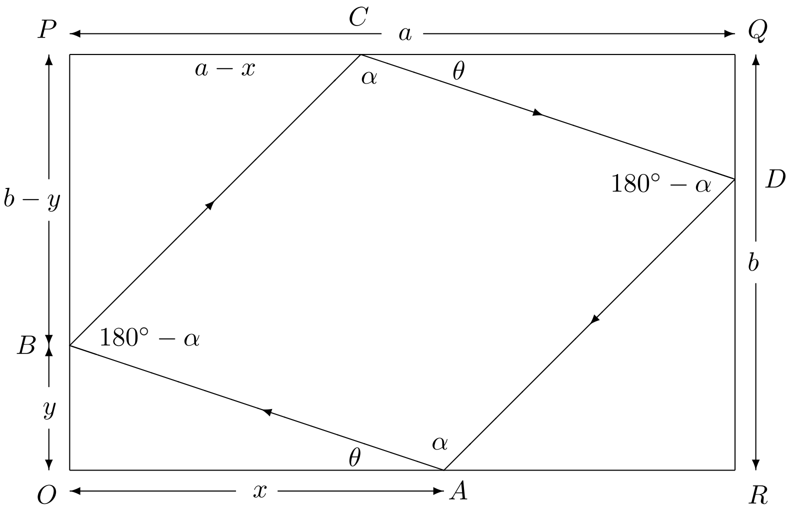
Show that $\displaystyle\prod_{k=1}^{\infty} \frac{k^2}{k^2+1} \;=\; \Gamma(1-i)\Gamma(1+i) \;=\; |i!|^2 \;=\; \frac{\pi}{\sinh \pi} \;\approx\; 0.272029.$

Solution 203.7 – Rhombus

A snooker table has a playing area of sides $a \times b$, $a > b$, and its cushions have coefficient of elasticity e, $0 < e \le 1$. A ball, initially placed in contact with the a side, is struck so that it leaves at angle θ to the side. The ball then follows a rhombus-shaped path and returns to its starting point. Show that $t = \tan\theta$ satisfies the quadratic $2\,a\,b\,t^2 + (a^2 - b^2)(1+e)\,t - 2\,a\,b\,e = 0$.

Steve Moon

The snooker table is $OPQR$ with dimensions $a \times b$, $a > b$. The ball is initially struck at A, follows the path $ABCDA$ describing a rhombus with $\angle OAB = \theta$, $\angle BAD = \alpha$.



Let the initial speed along AB be u. Let $OA = x$, $PC = a - x$, $OB = y$, $BP = b - y$. Therefore

$$y = x\tan\theta \quad \text{and thus} \quad BP = b - x\tan\theta.$$

Note that $x \ge a/2$ because the rebound speed normal to the side hit is reduced by a factor of e each time; so $\angle OBA > \angle PBC$ for the first bounce. If $e = 1$ then $x = a/2$ and $y = b/2$ for a rhombus path.

Resolving, the speed parallel to the direction of AO is $u\cos\theta$; so after hitting the table edge at B, the rebound speed component parallel to PC is $eu\cos\theta$ and the speed component parallel to the direction of BP is $u\sin\theta$.

The ball now travels from B to C. Therefore the time taken to traverse the distance BP at speed $u\sin\theta$ equals the time to traverse the distance PC at speed $eu\cos\theta$. Thus

$$\frac{a-x}{eu\cos\theta} \;=\; \frac{b-x\tan\theta}{u\sin\theta}.$$

Hence

$$(a-x)\tan\theta \;=\; e(b-x\tan\theta)$$

and therefore

$$x \;=\; \frac{a\tan\theta-eb}{\tan\theta(1-e)}, \tag{1}$$

on assuming $0<e<1$. (If $e=1$, go back a line to derive $\tan\theta=b/a$.)

Also, since $ABCD$ is a rhombus, $AB=BC$; so by Pythagoras,

$$x^2+x^2\tan^2\theta \;=\; (b-x\tan\theta)^2+(a-x)^2$$

and the x^2 terms conveniently disappear on multiplying out to give

$$x \;=\; \frac{a^2+b^2}{2(a+b\tan\theta)}. \tag{2}$$

Eliminating x between (1) and (2),

$$\frac{a\tan\theta-eb}{\tan\theta(1-e)} \;=\; \frac{a^2+b^2}{2(a+b\tan\theta)}.$$

With a bit of work this reduces to

$$2ab\tan^2\theta+(a^2-b^2)(1+e)\tan\theta-2abe \;=\; 0$$

and, setting $t=\tan\theta$, the required result follows.

We could have derived the same result by considering triangles CQD and DRA:

(i) the launch angle is $\theta=\angle QCD$, and

(ii) the initial velocity u is then $u_1=eu$ along CD; so the resultant treatment is in terms of velocity components scaled by a factor of e, which cancels out in the derivation of the analogue to (1).

Each time the ball traverses the complete rhombus it is 'relaunched' from A at θ to AO with velocity scaled by a factor of e^2 to that of the previous circuit along AB.

Letter

The three squares problem

Tony,

I have generated a few sets in order that I might find a solution, but unfortunately I can't see any pattern among my numbers. As ever, I put a bit of coding together to generate these sets up to 2000. [This is Problem 237.1 – Three squares: Find three numbers such that the product of any pair plus the square of the third is a square.]

There are two sets of numbers for each entry. The first set is (a, b, c) as described in the problem statement. The second set, which I have designated (x, y, z), are the squares so generated (or rather their roots). Although my original scan was only up to 2000, I tried a few tentative searches up to 5000, but I didn't find any more examples; I wonder if the results are limited. Perhaps you could publish this list in the hopes that the wider readership of M500 may find a proper solution.

Regards,

Ken Greatrix

a	b	c	x	y	z	a	b	c	x	y	z
9	9	40	21	21	41	9	73	328	155	91	329
13	21	136	55	47	137	17	276	1172	569	310	1174
20	48	77	64	62	83	20	81	404	182	121	406
21	68	356	157	110	358	29	36	260	101	94	262
29	141	680	311	199	683	33	89	488	211	155	491
33	185	608	337	233	613	37	69	424	175	143	427
37	85	312	167	137	317	37	240	1108	517	314	1112
45	112	628	269	202	632	48	97	580	242	193	584
53	165	872	383	271	877	56	165	221	199	199	241
57	308	1460	673	422	1466	61	228	1156	517	350	1162
68	165	932	398	301	938	69	80	341	179	173	349
69	301	1480	671	439	1487	77	384	1844	845	538	1852
84	113	788	310	281	794	89	440	969	659	529	989
93	100	772	293	286	778	105	121	904	347	331	911
128	129	1028	386	385	1036	132	205	1348	542	469	1358
137	153	1160	443	427	1169	141	304	1780	749	586	1792
153	209	1448	571	515	1459	161	240	1604	641	562	1616
176	213	1556	602	565	1568	177	308	1940	793	662	1954
189	352	589	493	485	643	224	341	1725	799	709	1747
237	245	1928	727	719	1943	297	320	377	457	463	487
301	1573	1896	1753	1745	2017	320	528	713	692	712	823
340	589	1584	1024	941	1646	400	400	561	620	620	689
536	693	1341	1103	1095	1473						

Solution 241.2 – Irrational numbers

> If πe is irrational, prove that at most one of $\pi+e$, $\pi-e$, π^2+e^2, π^2-e^2 is rational.

Stewart Robertson

Let

$$A=\pi+e, \quad B=\pi-e, \quad C=\pi^2+e^2, \quad D=\pi^2-e^2.$$

Clearly A and B cannot both be rational as this would imply that $\frac{1}{2}(A+B)=\pi$ was rational. Similarly, C and D cannot both be rational as this would imply that $\frac{1}{2}(C+D)=\pi^2$ was rational. Also A and C cannot both be rational as this would imply that

$$\frac{1}{2}(A^2-C)=\frac{1}{2}\Big((\pi+e)^2-\big(\pi^2+e^2\big)\Big)=\pi e$$

was rational. Similarly, B and C cannot both be rational as this would imply that $-\frac{1}{2}(B^2-C)=\pi e$ was rational.

Now, A and D cannot both be rational because this would imply that

$$\frac{D}{A}=\frac{\pi^2-e^2}{\pi+e}=\frac{(\pi+e)(\pi-e)}{(\pi+e)}=\pi-e=B$$

was rational and we showed above that this is not possible if A is rational. Similarly B and D cannot both be rational as this would imply that $D/B=A$ was rational and we showed above that this is not possible if B is rational.

Therefore, having exhausted all possible combinations, we conclude that at most one of $\pi+e, \pi-e, \pi^2+e^2, \pi^2-e^2$ can be rational.

Problem 244.5 – Ten primes

Patrick Walker

1	1	1	7
1	1	5	3
2	2	7	3
9	3	1	1

The numbers in each row, each column and each diagonal are prime. And no two are the same. Is it possible to find a square with the same properties but with all the digits odd?

Solution 236.6 – Products

Compute $\displaystyle\prod_{i=2}^{n}\prod_{j=1}^{i-1}\sin\frac{j\pi}{i}$ and $\displaystyle\prod_{i=2}^{n}\prod_{j=1}^{i-1}\cos\frac{j\pi}{i}$.

Steve Moon

To try to get a feel for a solution we compute the first few products, sometimes resorting to a calculator:

$$\prod_{j=1}^{2-1}\sin\frac{j\pi}{2}=1,\qquad \prod_{j=1}^{3-1}\sin\frac{j\pi}{3}=\frac{3}{4},\qquad \prod_{j=1}^{4-1}\sin\frac{j\pi}{5}=\frac{1}{2},\qquad \prod_{j=1}^{5-1}\sin\frac{j\pi}{5}=\frac{5}{16},$$

$$\prod_{j=1}^{6-1}\sin\frac{j\pi}{2}=\frac{3}{16},\qquad \prod_{j=1}^{7-1}\sin\frac{j\pi}{3}=\frac{7}{64},\qquad \prod_{j=1}^{8-1}\sin\frac{j\pi}{5}=\frac{1}{16},\qquad \prod_{j=1}^{9-1}\sin\frac{j\pi}{5}=\frac{9}{256}.$$

So a pattern has emerged; the inner product for a given i being

$$\prod_{j=1}^{i-1}\sin\frac{j\pi}{i} \;=\; \frac{i}{2^{i-1}}. \tag{1}$$

Therefore

$$\prod_{i=2}^{n}\prod_{j=1}^{i-1}\sin\frac{j\pi}{i} \;=\; \prod_{i=2}^{n}\frac{i}{2^{i-1}} \;=\; \frac{n!}{2^{1+2+\cdots+n-1}} \;=\; \frac{n!}{2^{n(n-1)/2}}.$$

Also $\displaystyle\prod_{i=2}^{n}\prod_{j=1}^{i-1}\cos\frac{j\pi}{i}=0$ since the factor $\cos\pi/2=0$ is always present.

However we still need to prove (1). Making use of Euler's formula for the sine function,

$$\sin\theta \;=\; \frac{\exp(i\theta)-\exp(-i\theta)}{2i} \;=\; \frac{1}{2i}\,\frac{\exp(2i\theta)-1}{\exp(-i\theta)},$$

we have

$$\prod_{j=1}^{k}\sin\frac{j\pi}{k+1} \;=\; \frac{1}{2^k i^k}\left(\frac{\exp\left(\frac{2i\pi}{k+1}\right)-1}{\exp\left(\frac{i\pi}{k+1}\right)}\right)\left(\frac{\exp\left(\frac{4i\pi}{k+1}\right)-1}{\exp\left(\frac{2i\pi}{k+1}\right)}\right)\cdots\left(\frac{\exp\left(\frac{2ki\pi}{k+1}\right)-1}{\exp\left(\frac{ki\pi}{k+1}\right)}\right)$$

$$=\; \frac{1}{2^k}\left(1-\exp\left(\frac{2i\pi}{k+1}\right)\right)\left(1-\exp\left(\frac{4i\pi}{k+1}\right)\right)\cdots\left(1-\exp\left(\frac{2ki\pi}{k+1}\right)\right) \tag{2}$$

since

$$\exp\left(\frac{i\pi}{k+1}\right)\exp\left(\frac{2i\pi}{k+1}\right)\ldots\exp\left(\frac{k\pi}{k+1}\right)$$

$$= \exp\left(\frac{i\pi}{k+1}+\frac{2i\pi}{k+1}+\cdots+\frac{ki\pi}{k+1}\right) = \exp\left(\frac{ki\pi}{2}\right) = i^k.$$

Now consider the identity

$$z^{k+1}-1 = (z-1)(1+z+z^2+\cdots+z^k).$$

The roots of the left-hand side are the $k+1$ $(k+1)^{\text{th}}$ roots of unity. Hence we can factorize $z^{k+1}-1$ as $(z-1)(z-z_1)(z-z_2)\ldots(z-z_k)$, where $z_j = \exp(2\pi ij/(k+1))$. So we can write

$$(z-z_1)(z-z_2)\ldots(z-z_k) = 1+z+z^2+\cdots+z^k.$$

Putting $z=1$, this gives

$$(1-z_1)(1-z_2)\ldots(1-z_k) = k+1,$$

which we can substitute into (2) to get

$$\prod_{j=1}^{k}\sin\frac{j\pi}{k+1} = \frac{1}{2^k}(1-z_1)(1-z_2)\ldots(1-z_k) = \frac{k+1}{2^k}.$$

Solution 242.2 – Quintic

Show that the real root of the cubic x^3-x-1 is also a root of the quintic x^5-x^4-1.

Vincent Lynch

I learned how to do polynomial division when I was 12 years old, and that is all that is needed here. Using the usual algorithm we find that

$$x^5-x^4-1 = (x^3-x-1)(x^2-x+1).$$

So all the roots, including the real root, of (x^3-x-1) are roots of x^5-x^4-1.

Solution 242.4 – Two sums

Prove that

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r \;=\; 2^{2n-1}$$

and

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r r \;=\; 2n\binom{2n-1}{n}.$$

Recall that these were the two probability sums associated with the biscuit tins problem, 240.1: There are two tins, each containing $n > 0$ biscuits. Take a biscuit from a tin chosen at random. Keep doing this until one tin is empty. What is the expected number of biscuits that remain in the other tin?

D. Hughes

Consider the first sum. Change the variables from n to m, $m = n - 1$, $n = m + 1$, and from r to k, $k = n - r = m + 1 - r$, $r = m - k + 1$. Then the sum is

$$S_n \;=\; \sum_{m-k+1=1}^{m+1} \binom{2(m+1)-(m-k+1)-1}{k} 2^{m-k+1}.$$

Now

$$\sum_{m-k+1=1}^{m+1} \;=\; \sum_{(m-k+1)-(m+1)=1-(m+1)}^{(m+1)-(m+1)} \;=\; \sum_{-k=-m}^{0} \;=\; \sum_{k=m}^{0} \;=\; \sum_{k=0}^{m}.$$

So

$$S_n \;=\; 2^{m+1}\sum_{k=0}^{m} \binom{m+k}{k} 2^{-k} \;=\; 2^{m+1}F_m,$$

where

$$F_m \;=\; \sum_{k=0}^{m} \binom{m+k}{k} 2^{-k} \;=\; 2^m$$

[see *Concrete Mathematics* by R. L. Graham, D. E. Knuth and O. Patashnik, Addison–Wesley, 1989]. So

$$S_n \;=\; 2^{m+1}2^m = 2^{2m+1} \;=\; 2^{2n-1}.$$

I seemed to recall that my answer to Problem 240.1 was different from equation (2) of M500 **242**, page 9:

$$2^{2-2n} n \binom{2n-1}{n}. \qquad (*)$$

But when I look into it, they are the same in slightly different forms. Furthermore, $(*)$ is probably the most compact form. It is interesting to use Stirling's formula,

$$n! \sim \sqrt{2\pi n}\,(n/e)^n,$$

to approximate the expected value for large n. This gives a nice result: viz. $2\sqrt{n/\pi}$, which is a good approximation for even modest values of n.

It is also easy to show that $\mathrm{prob}[n,1] = \mathrm{prob}[n,2]$, and that $\mathrm{prob}[n,r+1] < \mathrm{prob}[n,r]$ for $r > 1$. [Recall that

$$\mathrm{prob}[n,r] = \frac{1}{2^{2n-r-1}} \binom{2n-r-1}{n-r}$$

is the probability of r biscuits remaining in the non-empty tin after starting with n biscuits in each tin.] In other words, the most likely value for the number of remaining biscuits is 1 or (equally) 2, irrespective of n. This seemed slightly counterintuitive to me. Finally, there is what we might call the chocolate biscuit problem, when the tins are chosen with probabilities p and q $(p > q)$. I've made no progress with this.

PS. One place to look for binomial identities is Henry Gould's web site at http://www.math.wvu.edu/~gould/. The result needed to solve 242.4a is equation (1.16) in Vol.2.pdf.

Problem 244.6 – Flagpole integral

Compute

$$\int_\alpha^{\pi/2} \left(\tan\theta - \sqrt{\tan^2\theta - \alpha^2}\right) d\theta,$$

where $0 < \alpha < \pi/2$ is a constant.

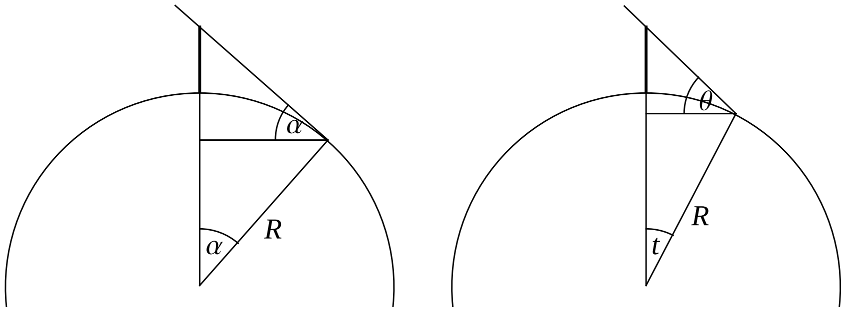
When you get to page 17 you will see that this integral occurs in Vincent Lynch's discussion of the flagpole problem. Ideally an exact solution is desired. However, we would also be interested in a good approximation on the assumption that $2/\alpha^2$ is the Earth's radius in metres. Beware: removing the $\tan\theta$ term might cause trouble near $\theta = \pi/2$.

Solution 241.6 – Flagpole

> Denote the radius of the (perfectly spherical) Earth by R. A flagpole of height 1 is observed at a time chosen at random on a sunny day. What is the expected length of its shadow? Assume that this takes place near the Equator on a day when the sun is directly overhead at midday.

Vincent Lynch

When I first tackled this problem, I found it intractable. But lying on the beach in Zakynthos reading Kumar's excellent *Quantum* and observing the shadows of sunshades, I realized that the key to a solution was to use approximations. As soon as we left the beach I went to a store and bought an A4 exercise book. In half an hour I had a solution.



By symmetry, we only need to calculate the mean from dawn to midday. Just after dawn only the lowest part of the pole casts a shadow. We need to calculate the angle of inclination, θ, of the sun measured from the flagpole foot when the whole of the flagpole first casts a shadow. Let this be α. The sun's rays through the top of the pole are then tangent to the earth's surface. Then we have (see left-hand diagram, above):

$$\cos\alpha \;=\; \frac{R}{R+1}.$$

Using $\cos\alpha \approx 1 - \alpha^2/2$ and $R + 1 \approx R$, we arrive at $\alpha^2 \approx 2/R$ and at this point the length of the shadow is $R\alpha$. For $0 < \theta < \alpha$, the shadow length is $R\theta$.

We now need a new diagram to show what happens when $\theta > \alpha$. We apply the sine rule to the obtuse triangle (see right-hand diagram). This is

$$\frac{R+1}{\sin\left(\theta+\dfrac{\pi}{2}-t\right)} \;=\; \frac{R}{\sin\left(\dfrac{\pi}{2}-\theta\right)}. \qquad (*)$$

If we expand and use the approximations

$$\cos t \;\approx\; 1 \quad\text{and}\quad \sin t \;\approx\; t,$$

we get $t \approx (\cot\theta)/R$ and the shadow length is $Rt = \cot\theta$.

Now that we have expressions for the length of shadow for both phases, we can calculate the mean. It is

$$\frac{2}{\pi}\left[\int_0^{\alpha} R\theta\,d\theta + \int_{\alpha}^{\pi/2}\cot\theta\,d\theta\right] \;=\; \frac{2}{\pi}\left[\frac{R\alpha^2}{2} - \log\sin\alpha\right] \;\approx\; \frac{1}{\pi}\left[R\alpha^2 - \log\alpha^2\right],$$

using $\sin\alpha \approx \alpha$. We now only have to substitute $\alpha^2 \approx 2/R$ to get a result. The mean length of shadow is approximately $(2 + \log R/2)/\pi$. *Wikipedia* gives R as approximately 6378137m; so using that figure I get 5.40339m for the mean shadow length.

However, numerical integration using the exact formula

$$t \;=\; \theta - \arccos\left(\frac{R+1}{R}\cos\theta\right)$$

for the second interval yields a somewhat larger figure, 5.52635m. It turns out that $t \approx (\cot\theta)/R$ is too crude for small θ. If we use $\cos t \approx 1 - t^2/2$ instead of $\cos t \approx 1$ in the simplification of $(*)$, we get a better approximation:

$$t \;\approx\; \tan\theta - \sqrt{\tan^2\theta - 2/R}.$$

It is actually possible to integrate this function of θ. The details are horrendous but the answer that arises for the mean shadow length does agree with the 'exact' value to 5 decimal places.

While looking at the *Wiki* site I found a very good quiz question. 'in which country is the point on land nearest to outer space.' Because of flattening at the poles it is not Everest but the peak of a mountain in Ecuador, not far from the equator; so the equatorial bulge gets it nearer to outer space!

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Front cover: A truncated truncated truncated truncated truncated cube.