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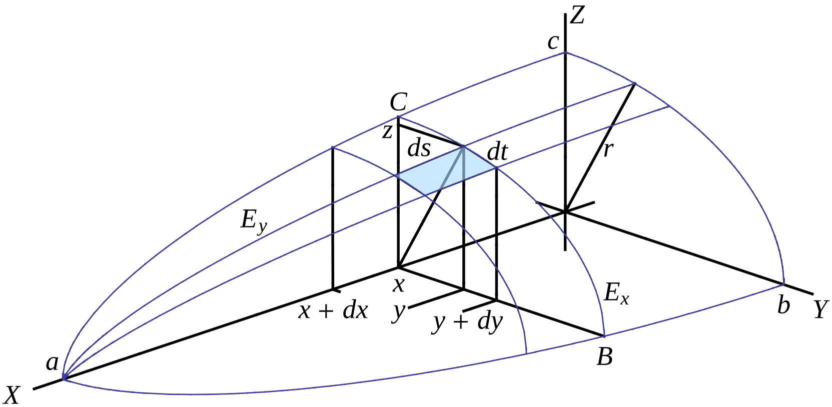
M500 259



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A quantum mechanical treatment of a sloping potential well

Tommy Moorhouse

Introduction Problem 256.5 [see page 6 of this issue] concerned the sum of the energies of two identical capacitors of total charge Q. A simple model for the discharge between two plates can be built by taking the potential energy to vary linearly between the plates. It is interesting that a quantum mechanical version of this problem is hard to find in the literature ([Landau and Lifshitz] considers only an infinitely long well). We will consider here a variant of the infinite potential well, the sloping well, find the energy quantization condition and look at the transition to the 'flat' well solutions. Working through the calculations that are indicated here might be a good way to become familiar with the Airy functions.

The sloping well Consider an infinitely deep potential well, with a particle confined to move in the x-direction between $x = 0$ and $x = L$. Between these points the potential energy is given by αx, where $\alpha = -qV$. The Schrödinger equation for an electron of mass m, charge q and energy E trapped in the well is

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E + \alpha x)\psi \;=\; 0.$$

We define $\tau = -\lambda(E + \alpha x)$ with $\lambda^3 = 2m/(\hbar^2\alpha^2)$ to get the equation

$$\psi'' - \tau\psi \;=\; 0.$$

The general solution of this equation is a linear combination of the two Airy functions, which we denote by $\phi_a(\tau)$ and $\phi_b(\tau)$. Substituting back we get

$$\psi(x) \;=\; A\phi_a\left(-\left(\frac{2m}{\hbar^2\alpha^2}\right)^{1/3}(E + \alpha x)\right) + B\phi_b\left(-\left(\frac{2m}{\hbar^2\alpha^2}\right)^{1/3}(E + \alpha x)\right).$$

The boundary conditions are $\psi(0) = 0$, $\psi(L) = 0$. Writing $k = (2m/\hbar^2)^{1/3}$ and inserting the normalization constant K, these conditions are satisfied for

$$\begin{aligned}\psi(x) \;&=\; K(\phi_a(-k\alpha^{-2/3}(E + \alpha x))\phi_b(-k\alpha^{-2/3}E)\\ &\quad - \phi_b(-k\alpha^{-2/3}(E + \alpha x))\phi_a(-k\alpha^{-2/3}E))\end{aligned}$$

provided that E is chosen to give a zero of the expression on the right when $x = L$. Here K is chosen so that

$$\int_0^L \psi(x)^2 dx \;=\; 1,$$

giving a probability of 1 that the particle is somewhere in the well. Calculating K is a little tedious but not difficult, and is discussed below. If $\rho_n = -|\rho_n|$ is the nth zero of ψ (counting away from zero on the negative axis) we label the energies as

$$E_n = -\rho_n \frac{\alpha^{2/3}}{k} - \alpha L.$$

This expresses the quantization of energy in the sloping well. As in all well-behaved Sturm–Liouville problems the eigenvalues (energies) are bounded below. The solutions $\psi_n(x)$ at each energy level are distorted standing waves, and the standard methods of quantum mechanics can be used to explore the properties of $\psi_n(x)$. We will not pursue this here.

It ought to be the case that as α tends to zero the sloping well tends to the standard infinite well and the solutions should tend to some solution $\psi_n(x) = \sqrt{2/L}\sin(x\sqrt{2mE_n}/\hbar)$. This is indeed the case, and the slightly delicate proof using the asymptotic expansions of ϕ_a and ϕ_b is as follows.

The 'flat' limit The asymptotic expression for $\phi_a(-|z|)$ as $|z| \to \infty$ is

$$\phi_a(-|z|) \to \frac{1}{\sqrt{\pi}|z|^{1/4}} \sin(\zeta + \pi/4),$$

where $\zeta = 2|z|^{3/2}/3$ (see [Richards]). Similarly

$$\phi_b(-|z|) \to \frac{1}{\sqrt{\pi}|z|^{1/4}} \cos(\zeta + \pi/4).$$

Inserting into $\psi(x)$ and writing $\beta = 1/\alpha$ we let β tend to infinity. Now, let $z_1 = \beta^{2/3}k(E + x/\beta)$ and $z_2 = \beta^{2/3}kE$, and write $\zeta_1 = 2(\beta(k(E + x/\beta))^{3/2})/3$ and $\zeta_2 = 2(\beta(kE)^{3/2})/3$. We now need to look more closely at the normalization constant K. We therefore return to the full expression for the normalized function $\psi(x)$, concentrating on the constant K. As mentioned already, a tedious calculation (section 10.4 of [Abramovitz and Stegun] includes some useful identities that are needed to carry out the calculation, including a differential equation satisfied by $\phi_a(t)^2$, and the Wronskian of ϕ_a and ϕ_b) gives

$$K = \frac{k^{1/2}\alpha^{1/6}\pi}{\sqrt{1 - \left(\dfrac{\phi_a(-k\alpha^{-2/3})}{\phi_a(-k\alpha^{-2/3}(E + \alpha L))}\right)^2}}.$$

The asymptotic expression for $\psi(x)$ becomes

$$\psi(x) \;\sim\; \frac{K}{\pi(|z_1||z_2|)^{1/4}}\sin(\zeta_1-\zeta_2).$$

It is not difficult to show, using the asymptotic expressions and expanding in powers of α, discarding the positive powers, that

$$K \sim k^{1/2}\alpha^{-1/3}\sqrt{2E/L}.$$

As α vanishes we can expand the denominator of the rest of ψ as

$$\alpha^{-1/3}k^{1/2}E^{1/4}(E+\alpha x)^{1/4} \;=\; \alpha^{-1/3}k^{1/2}E^{1/2}\left(1+\frac{\alpha x}{4E}+\cdots\right).$$

The second term here vanishes as $\alpha\to 0$, and most of the other terms (crucially the term in α) cancel with K.

Now we consider the sine term in the asymptotic expansion of ψ, $\sin(\zeta_1-\zeta_2)$. Expanding $(E_n+\alpha x)^{3/2}=E_n^{3/2}+3E_n^{1/2}\alpha x/2+\cdots$ (valid for small α) we see that, as α tends to zero, $\zeta_1-\zeta_2$ tends to $k^{3/2}E_n^{1/2}x = x\sqrt{2mE_n/\hbar^2}$. In summary,

$$\psi(x) \;\to\; \sqrt{\frac{2}{L}}\sin\left(\frac{x\sqrt{2mE_n}}{\hbar}\right).$$

One delicate point here is the dependence of E_n on α. However, since ρ_n also depends on α we must conclude that E_n tends to the 'flat' eigenvalue $n^2\pi^2\hbar^2/2mL^2$.

References

[Abramovitz and Stegun] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1965.

[Landau and Lifshitz] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics*, Third Edition, Pergammon, 1989.

[Richards] D. Richards, *Advanced Mathematical Methods with Maple*, Cambridge, 2002.

Problem 259.1 – Four primes

Find a number n such that n is the product of four distinct primes and every group of order n is Abelian.

Solution 254.2 – Interesting integral

Show that

$$\int_0^{\pi/2} \cos(\tan x)dx \;=\; \frac{\pi}{2e}$$

and hence that $\int_0^a \cos(\tan x)dx = a/e$ if a is an integer multiple of $\pi/2$.

Steve Moon

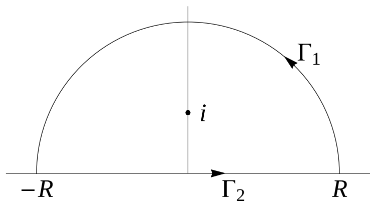
Because of the upper limit $\pi/2$ we will need to check at some stage that the integral converges. First make the substitution $u = \tan x$. The limits become 0 and ∞ and we have $du/dx = \sec^2 x = u^2 + 1$. Hence

$$\int_0^{\pi/2} \cos(\tan x)dx \;=\; \int_0^{\infty} \frac{\cos u}{1+u^2}du. \tag{1}$$

Now $\int_0^\infty 1/(1+u^2)du = \pi/2$ and since $|\cos u| \le 1$ we can be sure that the integral (1) converges. The function $(\cos u)/(1+u^2)$ is even and so we can write

$$\int_0^{\infty} \frac{\cos u}{1+u^2}du \;=\; \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos u}{1+u^2}du \;=\; \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{iu}}{1+u^2}du. \tag{2}$$

The integrand has poles at $\pm i$, where the denominator is zero. We consider integration in the complex plane. We integrate the function $e^{iz}/(1+z^2)$ around a closed semicircle Γ centred on the origin, radius R, so that the pole at i lies inside.



So we can split up the integral:

$$\begin{aligned}\int_\Gamma \frac{e^{iz}}{1+z^2}dz \;&=\; \int_{\Gamma_1} \frac{e^{iz}}{1+z^2}dz + \int_{\Gamma_2} \frac{e^{iz}}{1+z^2}dz \\ &=\; \int_{-R}^{R} \frac{\cos z}{1+z^2}dz + i\int_{-R}^{R} \frac{\sin z}{1+z^2}dz + \int_{\Gamma_1} \frac{e^{iz}}{1+z^2}dz.\end{aligned}$$

By Cauchy's residue theorem,

$$\int_\Gamma \frac{e^{iz}}{1+z^2}dz \;=\; \int_\Gamma \frac{e^{iz}}{(z+i)(z-i)}dz \;=\; 2\pi i(\text{residue at } i) \;=\; 2\pi i\,\frac{e^{i\cdot i}}{2i} \;=\; \frac{\pi}{e}.$$

Now consider Γ_1. On Γ_1, $|e^{iz}| \leq 1$ since $z = R\cos\theta + iR\sin\theta$ for some $\theta \in [0, \pi]$. Also by the triangle inequality $|1 + z^2| \geq R^2 - 1$. Furthermore the length of Γ_1 is πR. Therefore

$$\int_\Gamma \frac{e^{iz}}{1+z^2} dz \;\leq\; \frac{\pi R}{R^2 - 1} \;\to\; 0 \quad \text{as } R \to \infty.$$

So we are left with

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos u}{1+u^2} du \;=\; \int_{-\infty}^{\infty} \frac{\cos u}{1+u^2} du \;=\; \frac{\pi}{e} \tag{3}$$

and, as a bonus, $\displaystyle\int_{-\infty}^{\infty} \frac{\sin u}{1+u^2} du = 0$. Hence from (1), (2) and (3)

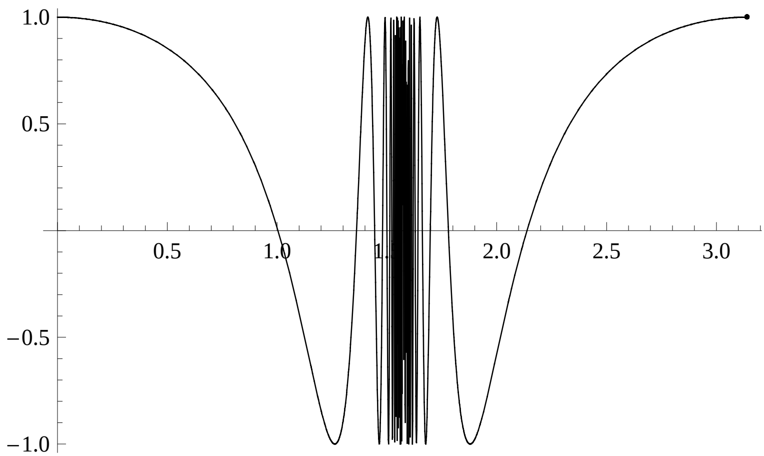
$$\int_0^{\pi/2} \cos(\tan x) dx \;=\; \frac{\pi}{2e}. \tag{4}$$

For the final part, since $\tan x$ has period π we need consider only $[0, \pi]$. In the interval $[\pi/2, \pi]$, $\tan x$ is negative but runs through exactly the same profile of absolute values as in (4). So because $\cos(-x) = \cos x$,

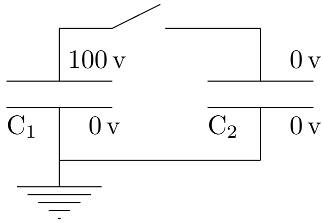
$$\int_{\pi/2}^{\pi} \cos(\tan x) dx \;=\; \int_0^{\pi/2} \cos(\tan x) dx.$$

Hence if $a = k\pi/2$ for some integer k,

$$\int_0^a \cos(\tan x) dx \;=\; \int_0^{k\pi/2} \cos(\tan x) dx \;=\; \frac{k\pi}{2e} \;=\; \frac{a}{e}.$$



Solution 256.5 – Lost energy



The diagram represents the initial state of a circuit containing two capacitors of C farads each, with 100 volts across C_1. When the switch is closed C_1 loses charge to C_2 until they equalize at 50 volts across each capacitor.

Initially the total energy in the system is the $100^2C/2 = 5000C$ joules stored in C_1. But when the circuit has settled down after the switch is closed, the energy is split between the two capacitors at $50^2C/2$ joules each, making a total of $2500C$ joules. What has happened to the other $2500C$ joules?

Tommy Moorhouse

A relativistic treatment

Problem 256.5 asked why the sum of the energies of two identical capacitors of total charge Q is half that of a single capacitor carrying the same charge. It can be shown that getting from the single capacitor state to the two-capacitor state by allowing charge to flow through a resistor always results in the loss of half the stored energy, essentially as heat. However, the transition between the two physical states can occur in any physically allowed manner, and this article considers the relativistic free flow of charged particles between two plates. The motivation is simply that this is a nice way to get a feel for the relativistic treatment of a simple problem.

As a manageable model we consider charged particles flowing from a charged (high voltage) plate to a low voltage plate. The plates, which are taken to have a high mass, are parallel and separated by a distance δ; and initially the charged plate holds a charge of Q coulombs. The z-axis runs perpendicular to the plates. The mass and charge of the particles makes no essential difference to the outcome, but for now we consider particles of charge q and mass m. We can think of these as electrons generated at the high voltage plate and accelerated until they hit the low voltage plate and are absorbed, releasing energy as heat.

The electric field between the plates is E in the z-direction. This field is constant between the plates and we consider an electron ejected from the charged plate at zero speed. We work in the rest frame of the massive plates, and the motion obeys the relativistic equation

$$\dot{p}_z = qE,$$

which we can write, using the expression for relativistic momentum and noting that v is the speed in the z-direction, as

$$\frac{d}{d\tau}(\gamma v) \;=\; \frac{qE}{m},$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$. The solution with $v(0)=0$ is

$$\gamma v \;=\; \frac{qE}{m}\tau.$$

Solving for v (which just involves unravelling γ), and writing $\kappa = qE/mc$ we find

$$v(\tau) \;=\; \frac{c\kappa\tau}{\sqrt{1+\kappa^2\tau^2}}.$$

Notice—and it is a good idea to check—that the dimensions of κ are T^{-1}. To find an expression for the speed when the electron hits the low voltage plate, at time τ_d, we integrate the speed to get an expression for the distance travelled:

$$x(\tau) \;=\; c\int_0^\tau v(\tau)d\tau \;=\; \frac{c}{\kappa}\sqrt{1+\kappa^2\tau^2}+x_0.$$

Since the electron leaves the plate at $\tau=0$ we have $x_0=-c/\kappa$. This gives for the plate separation δ

$$\tau_d^2 \;=\; \frac{1}{c^2}\left(\left(\delta+\frac{c}{\kappa}\right)^2-\left(\frac{c}{\kappa}\right)^2\right).$$

At the time τ_d we have

$$v(\tau_d)^2 \;=\; \frac{c^2}{(\delta+c/\kappa)^2}\left(\left(\delta+\frac{c}{\kappa}\right)^2-\left(\frac{c}{\kappa}\right)^2\right)$$

and we find a simple result for γ:

$$\gamma \;=\; \frac{\kappa\delta}{c}\left(1+\frac{c}{\kappa\delta}\right) \;=\; 1+\frac{\kappa\delta}{c}.$$

Now we write $E=V/\delta$, where V is the voltage between the plates, and calculate the relativistic energy γmc^2 :

$$\gamma mc^2 \;=\; mc^2+\kappa\delta mc \;=\; mc^2+qV.$$

Happily the energy calculated for the heat released by the relativistic particle is independent of the mass of the carriers and matches exactly the energy calculated by the usual formula for the energy required to move a charge q through a potential V. The rest mass of the electron is not lost from the system.

Now suppose that $Q = Mq$ for some $M \gg 1$, and that one electron at a time passes from the high voltage plate to the low voltage plate as above. The low voltage plate will acquire a charge and eventually the charge on each plate will be $Q/2$. The effective charge difference after n electron transfers will be $Q-2nq$, and the voltage between the plates will be $V_n = (Q-2nq)/C$, where C is the capacitance of the two-plate system. To find the total energy lost (transferred to the low voltage plate by the electrons, which are stopped and heat the plate up) we sum $V_n q$ from $n = 1$ to $M/2$:

$$E \;=\; \frac{q^2}{C}\sum_{n=1}^{M/2}(M-2n) \;=\; \frac{q^2}{C}\left(\frac{M^2}{2}-\frac{M}{2}\left(1+\frac{M}{2}\right)\right) \;=\; \frac{Q^2}{4C},$$

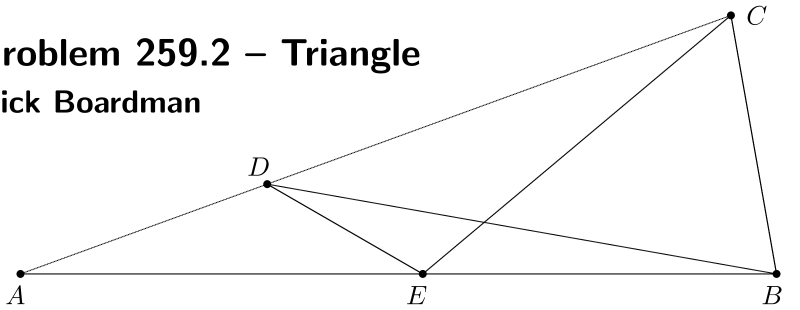
where we have used the fact that $M \gg 1$.

The total energy lost to heat is half that stored in the original system. The remaining energy is stored in the electrostatic field: the plates are fixed, but would 'like' to separate, and this is essentially what it means to say there is 'stored energy'.

Solved non-relativistically in M500 **258** by Mike Lewis, Tony Forbes, Tommy Moorhouse and John Davidson.

Problem 259.2 – Triangle

Dick Boardman



(i) In the diagram $|AB| = |AC|$, $\angle BAC = \angle ACE = 20°$ and $\angle ABD = 10°$. What is $\angle AED$?

(ii) Let $\zeta = e^{\pi i/18}$. Show that

$$\frac{\zeta^3(\zeta^2-\zeta^{10}+\zeta^{12}-1)}{\zeta^4+2\zeta^8-2\zeta^{10}-\zeta^{14}-3} \;=\; \frac{1}{\sqrt{3}}.$$

Solution 254.6 – Two octics

Solve $x^8 + 4x^5 + 8 = 0$ and $x^8 + 16x^3 + 32 = 0$.

Tony Forbes

Take the first equation. It does not split into polynomials with integer coefficients. So we look for a factorization where some coefficients are irrational algebraic numbers. A sledgehammer is a useful tool. Write

$$x^8 + 4x^5 + 8 \;=\; (x^4 + ax^3 + bx^2 + cx + d)(x^4 + ex^3 + fx^2 + gx + h)$$

and equate coefficients to obtain

$$\begin{gathered} dh = 8, \quad dg + ch = 0, \quad df + cg + bh = 0, \quad de + cf + bg + ah = 0, \\ d+ce+bf+ag+h = 0, \quad c+be+af+g = 4, \quad b+ae+f = 0, \quad a+e = 0. \end{gathered}$$

Putting this lot to Mathematica yields 70 solutions but amongst them is a particularly nice-looking one, $a = 2i,\ b = -2,\ c = 2-2i,\ d = 2+2i,\ e = -2i,\ f = -2,\ g = 2+2i,\ h = 2-2i$, that gives the factorization

$$\begin{aligned} x^8 + 4x^5 + 8 \;=\; & \big(x^4 + 2ix^3 - 2x^2 + (2-2i)x + 2 + 2i\big) \\ & \big(x^4 - 2ix^3 - 2x^2 + (2+2i)x + 2 - 2i\big)\,. \end{aligned}$$

The two quartics, whose coefficients involve integers and nothing more irrational than $\sqrt{-1}$, can now be solved exactly using the method described in M500 **223**, or otherwise. Let

$$D_1 = \Big(64 - 36i + 12\sqrt{39 - 84i}\Big)^{1/3}, \quad D_2 = \Big(64 + 36i + 12\sqrt{39 + 84i}\Big)^{1/3},$$

$$E_1 = \frac{1}{3}\left(1 + \frac{16+12i}{D_1} + D_1\right), \quad E_2 = \frac{1}{3}\left(1 + \frac{16-12i}{D_2} + D_2\right).$$

Then the eight roots of $x^8 + 4x^5 + 8 = 0$ are

$$\frac{1}{2}\left(-i + \epsilon_1\sqrt{E_1} + \epsilon_2\sqrt{1 - E_1 - \frac{\epsilon_1\,(4-2i)}{\sqrt{E_1}}}\right),$$

$$\frac{1}{2}\left(i + \epsilon_1\sqrt{E_2} + \epsilon_2\sqrt{1 - E_2 - \frac{\epsilon_1\,(4+2i)}{\sqrt{E_2}}}\right),$$

where $\epsilon_1, \epsilon_2 = \pm 1$. If you substitute $x \to 2/x$, you obtain the other octic:

$$\begin{aligned} x^8 + 16x^3 + 32 \;=\; & \big(x^4 + 2ix^3 - (2+2i)x^2 + (4-4i)x + 4 + 4i\big) \\ & \big(x^4 - 2ix^3 - (2-2i)x^2 + (4+4i)x + 4 - 4i\big)\,. \end{aligned}$$

Problem 259.3 – Discriminants

Tony Forbes

The discriminant of the cubic

$$f(x) \;=\; x^3 + ax^2 + bx + c$$

is

$$\Delta \;=\; a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2.$$

Show that the discriminant of

$$g(x) \;=\; 3x^4 + 4ax^3 + 6bx^2 + 12cx + 4ac - b^2$$

is $-6912\Delta^2$. What we really want is a simple proof without having to compute the discriminant by brute force. If the coefficient a causes too much trouble, you may assume $a = 0$. And to remind readers, the *discriminant* of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with complex roots $r_1, r_2, \ldots, r_n$ is defined by

$$\Delta_P \;=\; a_n^{2n-2} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (r_i - r_j)^2.$$

Since the product expands to a sum of symmetric functions of the roots, there will always be a rational expression for Δ_P in terms of the coefficients $a_0, a_1, \ldots, a_n$, as in the case of $f(x)$ and $g(x)$, above.

Problem 259.4 – Double integrals

A couple of fiendish-looking double integrals for you to do.

(i) Suppose $a^2 + b^2 = 1$. Show that

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{a^2 \cos^2\theta + b^2 \cos^2\phi}{\sqrt{1 - a^2 \sin^2\theta}\sqrt{1 - b^2 \sin^2\phi}}\, d\theta\, d\phi \;=\; \frac{\pi}{2}.$$

(ii) Suppose $a, b, c > 0$. Show that

$$\int_0^a \int_0^b \frac{c\, dx\, dy}{(c^2 + x^2 + y^2)^{3/2}} \;=\; \arctan \frac{ab}{c\sqrt{a^2 + b^2 + c^2}}.$$

Problem 259.5 – Two darts

Two darts are thrown and hit a dartboard at random, whatever that might mean. Show that the probability of their separation exceeding the dartboard radius is $3\sqrt{3}/(4\pi)$.

When I (TF) tested the problem with 100000 trials on a dartboard of radius 1 I got a good approximation to the right answer if for the landing point (x, y) I chose x and y independently and uniformly at random on the interval $[-1, 1]$ subject to $x^2 + y^2 \leq 1$.

Problem 259.6 – Polygon and floorboards

A regular polygon of maximum diameter 1 (that is, 1 is the maximum distance between two vertices) and perimeter p is thrown at random on to floorboards spaced 1 apart. Show that the probability of falling across a crack is p/π.

Observe that this agrees with $2/\pi$, the result of Steve Moon's analysis in M500 **255**, when the polygon has only two sides and degenerates into a thin linear object—like a needle.

Problem 259.7 – Admissible numbers

Let k be a positive integer. A positive integer n is called *admissible* if $n(n-1) \equiv 0 \pmod{k(k-1)}$ and $n-1 \equiv 0 \pmod{k-1}$. Show that when k is a prime power the only admissible numbers are $k(k-1)t+1$ and $k(k-1)t+k$, $t = 0, 1, \ldots$.

Problem 259.8 – Binomial ratio

Let r and s be positive integers. Suppose p is prime and $(p^s - 1)/(p^r - 1)$ is an integer. Is it (i) obviously true, or (ii) true, or (iii) false that s/r is always an integer.

Problem 259.9 – Polynomial

Show that a polynomial $f(x)$ with non-negative integer coefficients is unambiguously determined by its value at $x = f(1) + 1$.

(Hint: try it out on $6x^7 + 45x^6 + 19x^5 + 12x^4 + 7x^3 + 6x + 4$.)

Solution 243.2 – Cosh integral

Let n be a positive integer. Show that

$$I_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\cosh^n x \cosh^{n+1} y} = \frac{2\pi}{n}. \qquad (1)$$

Of course, one can split it up, evaluate each integral separately and then multiply. However I feel that because of the truly elegant nature of (1) there might be an alternative and more enlightening proof of appropriate simplicity. If it makes life easier, you may assume that

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x} = \pi \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x} = 2.$$

Steve Moon

My approach to evaluating I_n uses a reduction formula and still relies on the separability of the double integral. I tried restating this 'all space' integral in terms of polar coordinates (r, θ) and then trying to find a way in, using complex analysis, but I couldn't make contour integration work. So this effort might not meet the 'alternative and more enlightening proof' test; but here goes anyway.

Let

$$I_n = J_n J_{n+1}, \qquad \text{where} \qquad J_n = \int_{-\infty}^{\infty} \frac{dx}{\cosh^n x}$$

and recall that the integral J_n satisfies the well-known reduction formula

$$J_n = \frac{n-2}{n-1} J_{n-2}, \qquad J_1 = \pi, \qquad J_2 = 2.$$

[See, for example, the two tins of biscuits problem in M500 **242**. — TF] Then

$$I_n = J_n J_{n+1} = J_n \frac{n-1}{n} J_{n-1} = \frac{n-1}{n} I_{n-1},$$

giving

$$I_n = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \cdots \cdot \frac{2}{3} \cdot I_1 = \frac{2\pi}{n}$$

since $I_1 = J_1 J_2 = 2\pi$.

A mathematical result is always easy once someone has shown you how to obtain it.

Solution 257.4 – Tracks

> Your portable music player has tracks T_0, T_1, …, T_n, of lengths t_0, t_1, …, t_n respectively. The device selects tracks at random and plays them in full. The probability of track T_i getting selected is proportional to t_i. For what fraction of the time would you expect to be listening to track T_0?

Reinhardt Messerschmidt

Let p_j be the probability that T_j is selected, i.e.

$$p_j \;=\; \frac{t_j}{t_0 + t_1 + \cdots + t_n}.$$

Let X_{jk} be the random variable defined by

$$X_{jk} \;=\; \begin{cases} 1 & \text{if } T_j \text{ is selected for the } k\text{th play} \\ 0 & \text{otherwise.} \end{cases}$$

Let Y_{jk} be the proportion of the first k plays for which T_j is selected, i.e.

$$Y_{jk} \;=\; \frac{1}{k} \sum_{m=1}^{k} X_{jm}.$$

Let Z_{jk} be the proportion of the playing time of the first k plays for which T_j plays, i.e.

$$\begin{aligned} Z_{jk} &= \frac{t_j(kY_{jk})}{t_0(kY_{0k}) + t_1(kY_{1k}) + \cdots + t_n(kY_{nk})} \\ &= \frac{t_j Y_{jk}}{t_0 Y_{0k} + t_1 Y_{1k} + \cdots + t_n Y_{nk}}. \end{aligned}$$

Note that $(X_{jk})_{k \in \mathbb{N}}$ is a sequence of independent identically distributed random variables with common expected value p_j. By the strong law of large numbers, $Y_{jk} \longrightarrow p_j$ as $k \longrightarrow \infty$, with probability 1. It follows that

$$Z_{jk} \longrightarrow \frac{t_j p_j}{t_0 p_0 + t_1 p_1 + \cdots + t_n p_n} = \frac{t_j^2}{t_0^2 + t_1^2 + \cdots + t_n^2} \quad \text{as } k \longrightarrow \infty,$$

with probability 1.

For example, if $t_0 = 20$ and $t_1 = t_2 = 5$ then $Z_{0k} \longrightarrow 8/9$.

Acknowledgement This solution was greatly improved as a result of an observation made by Tony Forbes.

Solution 258.1 – Battersea Power Station

> From what points in London will the chimneys of Battersea Power Station appear regularly spaced along the skyline? For consistency, we label the four chimneys as A, B, C, D, going clockwise from A at the SW corner, which we take as the origin of a coordinate system, and assign to them coordinates $(0, 0)$, $(0, 160)$, $(50, 160)$, $(50, 0)$ respectively.

Ken Greatrix

If we take a simple attitude to the situation, then by aligning chimney C with the mid-point of the line between chimneys A and B we have a line of slope 80/50 or 1.6, or similarly in the other direction a slope of -1.6 when chimney B is aligned with the mid-point between chimneys C and D.

However, if the chimneys are arranged on a perfect rectangle, then it is impossible to see the required alignment since viewing lines from the chimneys to the opposite mid-points would be parallel.

That's the maths done!

Next I looked at a street-map of London which shows the outline of the building. Towards the North-East, I think Rampayne Street or St John's Gardens would give the required view. In the North-Westerly direction perhaps the junction of Ebury Street and Eaton Terrace; ... if it wasn't for the 'ouses in between (anyone who remembers that old song is probably older than me).

As the Station is on the Southerly edge of my street-map, I am unable to suggest view-points in the South-Easterly and South-Westerly directions.

But it is possible to get a reasonable view on 'Google Street View' by accessing Grosvenor Road as it emerges on the Eastward side of Victoria Bridge.

As a follow-up problem: Four points are arranged in a quadrilateral which is very nearly a rectangle. By choosing ideal points, there is at least one external point whereby the points subtend a set of three equal angles. In reverse, if we than take any angle and construct a quadrilateral between its lines, is it then possible to trisect this angle?

PS. During Wimbledon tennis (Friday I think, during Andy Murray's match) the overhead camera was pointed across London and showed a view of Battersea power station—the chimneys were almost in equal alignment. Has David Singmaster got a helicopter?

Solution 250.3 – Ellipsoid

Show that

$$S_M(a,b,c) \;=\; 4\pi\left(\frac{(ab)^{8/5}+(bc)^{8/5}+(ca)^{8/5}}{3}\right)^{5/8} \qquad (1)$$

is quite a good approximation to the surface area of an ellipsoid with radii a, b and c. For instance, if $a = 10$ and $b = c = 15$, the formula gives 2225.5 whereas the true value is about 2225.0.

Tony Forbes

Write $S(a,b,c)$ for the true surface area. We attempted to do the ellipsoid in M500 **197** and—with the help of the diagram on the front cover of this issue—produced the formula

$$S(a,b,c) \;=\; 8\int_{x=0}^{a}\int_{y=0}^{B} ds\; dt \;=\; 8\int_0^a\int_0^B f(x,y)\; dy\; dx,$$

where $B = b/a\sqrt{a^2-x^2}$ and $f(x,y)$ was some diabolical expression, which I now believe to be less than entirely correct when $b \neq c$. And that's as far as we got without the simplification $b = c$

Then **Dick Boardman** contributed the (correct!) formula

$$S(a,b,b) \;=\; 2b^2\pi + \frac{2a^2b\,\pi}{\sqrt{b^2-a^2}}\;\log\frac{b+\sqrt{b^2-a^2}}{a}. \qquad (2)$$

Putting $b = a+\epsilon$ and developing (2) as a power series in ϵ yields

$$S(a,a+\epsilon,a+\epsilon) \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{8\pi\epsilon^2}{5} + \frac{16\pi\epsilon^3}{105a} + \ldots. \qquad (3)$$

Now do the same for (1) with $b = c = a+\epsilon$ to get

$$S_M(a,b,c) \;=\; 4\pi a^2 + \frac{16\pi a\epsilon}{3} + \frac{8\pi\epsilon^2}{5} + \frac{112\pi\epsilon^3}{675a} + \ldots,$$

which not only agrees with (3) up to the ϵ^2 term but also the difference between the coefficients of $\pi\epsilon^3/a$ is quite small: $112/675 - 16/105 = 64/4725 < 0.014$. Observe that the mysterious exponent 8/5 in (1) arises from the coefficient of ϵ^2 in (3). As you can see, (1) is better than a formula based on the arithmetic mean of the radii,

$$S_A(a,b,c) \;=\; 4\pi\left(\frac{a+b+c}{3}\right)^2 \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{16\pi\epsilon^2}{9},$$

or the geometric mean,

$$S_G(a,b,c) \;=\; 4\pi\,(abc)^{2/3} \;=\; 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{8\pi\epsilon^2}{9} + \ldots,$$

or the 4/5 power mean (which also agrees with (3) up to ϵ^2),

$$S_B(a,b,c) = 4\pi\left(\frac{a^{4/5}+b^{4/5}+c^{4/5}}{3}\right)^{5/2} = 4a^2\pi + \frac{16a\pi\epsilon}{3} + \frac{8\pi\epsilon^2}{5} - \frac{16\pi\epsilon^3}{675a} + \ldots.$$

I do not know if (1) is ever actually used in the field. Unless I have missed something obvious, examples of real-life non-degenerate ellipsoids seem to be quite rare. As for a general formula, all I can do for the present is quote from *Wikipedia*:

$$S(a,b,c) \;=\; 2\pi c^2 + \frac{2\pi ab}{\sin\phi}\big(E(\phi,m)\sin^2\phi + F(\phi,m)\cos^2\phi\big),$$

where $c \le b \le a$, $\phi = \arccos(c/a)$, $m = a^2(b^2-c^2)/(b^2(a^2-c^2))$, and where $F(\phi,m)$ and $E(\phi,m)$ are the incomplete elliptic integrals of the first and second kind respectively, denoted by EllipticF[phi,m] and EllipticE[phi,m] in MATHEMATICA.

Also I am still interested if anyone can find numbers $a > b > 1$ such that $S(a,b,1)$ can be computed exactly using only elementary functions. I am reminded that I asked this question in 2004 as 'Problem 199.1 – Ellipsoid again.' Since then nobody has submitted a solution; so I am inclined to believe that it is not as trivial as it looks.

M500 Winter Weekend 2015

The **thirty-fourth M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 9th – Sunday 11th January 2015.

Cost: £205 to M500 members, £210 to non-members. You can obtain a booking form either from the M500 web site,

http://www.m500.org.uk/winter/booking.pdf,

or by emailing me.

As well as a complete programme of mathematical entertainments, on Saturday we will be running a pub quiz with Valuable Prizes.

Dr Urban Panic's later adventures

Ralph Hancock

I am sorry to have to inform readers that Dr Panic's novel food company TopoSnax is not enjoying the success it perhaps deserves.

He has been having problems with the development of the Flav-R-Act, a snack intended to provide all the sensations of a six-course meal. This is a piece of toast in the form of a tesseract, each of whose faces is coated with a different flavour. As it rotates, our three-dimensional perception of it registers six faces at a time, flavoured with, say, roast beef, gravy, roast potatoes, Yorkshire pudding, cauliflower and carrots. Everything worked fine unless consumers turned their head while enjoying the Flav-R-Act, altering its direction of spin. In one unfortunate orientation it tasted simultaneously of strawberry, anchovy, curry, coffee, vinegar and Stilton cheese.

He has had little success with a food marketed as a slimming aid, the Schröding-R-Snak. This is a cardboard box which, owing to quantum uncertainty, may or not contain a bacon sandwich. Since you only get a sandwich 50 per cent of the time, with repeated purchasing you consume only half as much as is provided by conventional sandwiches. However, when buyers picked up the box, they could tell by its weight whether there was a sandwich in it or not, an act of measurement that caused the superposition to collapse before they reached the checkout; and of course, if they found it was empty they put it back and took another one. Critics complained that he was simply selling boxes half of which contained sandwiches, and no quantum effect was involved at all. "But," said the good Doctor, "how could you tell?"

His Slim-R-Choc confectionery was equally unsuccessful. Marketed under the slogan 'Only 10 Calories in a Bar', it had a window in the wrapper through which you could see one square of chocolate, with others adjacent to it extending under the wrapper in all directions. On opening it proved to be a single square of chocolate surrounded by four little acrylic mirrors. The public was not amused.

His Ev-R-Hot Coffee Warmer worked perfectly, but only made it as far as preliminary trials. This was a metal cube which was permanently at a temperature of 70°C, and could be dropped into any hot drink. However, when it was discovered that the metal was plutonium, the device was hastily withdrawn and has not gone on sale.

Dr Panic has been intrigued by M-Theory, which proposes that space-time has 11 dimensions, seven of which are not observed because they are rolled up very small. "If they can be rolled up," he said, "they can be filled with raspberry jam." His Swiss-R-Oll is still in development, but sadly all workable examples so far have been created in alternative universes.

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Front cover: One eighth of an ellipsoid—see page 15.