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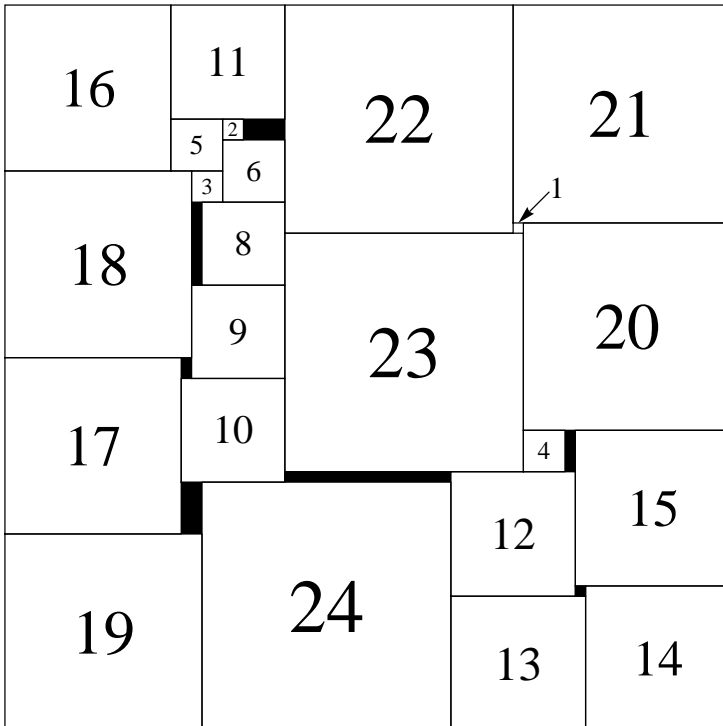
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A number puzzle

Ken Greatrix

$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even,} \\ \text{undefined} & \text{if } x < 2. \end{cases}$$

I first encountered this puzzle in the M335 Logic option (which I studied as M381 in 1992) and have not seen any other comments or documentation until it appeared in M500 **162** by Jane Kerr. Hence some of this account may not be original and I may have 'rediscovered the wheel.'

I believe that I have found a solution which could lead to a proof of the conjecture that $f(f(f(\ldots f(x)\ldots))) = 1$ for all $x \geq 2$. Unfortunately, I do not have the mathematical techniques to evaluate this solution, so here are some of the ideas with which I have been trying in my attempts to solve the puzzle. It is my hope that this does lead to a proof and that someone may be able to properly evaluate it.

As this is not a formal proof, I have not fully justified some of my statements, but I will be able to do so if required.

Using the notation $f_n(x_0) = x_n$ to denote the compositions (or iterations) of the function, I have identified three sets of numbers:

$$\begin{aligned} &\text{(i)} \quad x \in X \quad \text{if } f_n(x) = 1, \\ &\text{(ii)} \quad y \in Y \quad \text{if } f_n(y) \to \infty \text{ as } n \to \infty, \\ &\text{(iii)} \quad z \in Z \quad \text{if } f_n(z) = z_0. \end{aligned}$$

I suppose that like everyone else who has investigated this function, I too compiled a computer program. I started at $x = 2$ (incrementing the tested number by 1), and iterated on x until $f_n(x_0) = x_n < x_0$ (see **EK**'s footnote in M500 **162**).

The pattern of numbers generated by each these iterations has a 'tree-like' structure and I realised that it is possible to 'reverse' the function, as follows:

$$f^{-1}(x) = \begin{cases} (x-1)/3 & \text{if } x = 6k+4, \\ 2x & \text{for all } x. \end{cases}$$

This is not a true inverse because (for example) $f^{-1}(16) = 32$ and also $f^{-1}(16) = 5$; but it is used to illustrate my next point, in which I would like to place the function in $\mathbb{N}^3$.

$$f^{-1}(x, w, v) = \begin{cases} ((x-1)/3, w+1, 1) & \text{if } x = 6k+4, \\ (2x, w, v+1) & \text{for all } x. \end{cases}$$

So, starting from (1, 1, 1) and ignoring $f^{-1}(4) = 1$, all numbers in X are represented only once on this graph. The reason for the w- and v-axes is merely to separate the pathways from each value of x to 1; but having introduced them I can show that they are unbounded. The values of v and w are related to the iterations required to reduce x to 1. Also note that in this reversed format the number 1 generates the whole set X.

In addition, the reversed function can be applied to any values of y or z and again can generate an infinite set. So if only one exception to X can be found then approximately only half of $\mathbb{N}$ can be reduced to 1 by the function.

If we assume for the time being that Y and Z are empty sets, then I can investigate some of the properties of the set X.

When the 'tree' of numbers revealed no further pattern, I looked at the number of steps, n, required for $2x_n > x_0 > x_n$ and also such that $x_i > x_0$ for $i = 1$ to $n - 1$. These step numbers, arranged in groups of 4, revealed an interesting pattern.

1	6	1	3	1	11	1	3	1	8	1	3	1	11	1	3	
1	6	1	3	1	8	1	3	1	96	1	3	1	91	1	3	...

Every other number in this list is a 1, so even numbers require only one step to reduce (what a great surprise!). Every fourth number is a 3, so $f_3(4k+1) = 3k+1$, where $3k+1 < 4k+1$ when $k > 0$. But what about the other numbers in the list, which are associated with numbers of the form $4k + 3$?

I next modified the program to start at 3 and increment by 4 to investigate these numbers. I have arranged the output in columns of 8 to show a further pattern.

6	11	8	11	6	8	96	91	
6	13	8	88	6	8	11	88	
6	88	8	13	6	8	73	13	
6	68	8	50	6	8	13	24	
6	11	8	11	6	8	65	34	
6	47	8	13	6	8	11	21	
6	13	8	21	6	8	13	50	...

The number 6 appears in the 1st and 5th columns and shows that $f_6(16k+3) = 9k + 2$, with $9k + 2 < 16k + 3$. The 8s reveal $f_8(32k + 11) = 27k + 10$

and $f_8(32k + 23) = 27k + 20$. There is a repeating pattern of three 11s, so $x_0 \in \{7, 15, 59\}$ are associated with $f_{11}(128k + x_0) = 81k + x_{11}$.

Similarly, the set $x_0 \in \{39, 79, 95, 123, 175, 199, 219\}$ is associated with $256k + x_0$ and $243k + x_{13}$, showing the repeated 13s, and the set

$$x_0 \in \{287, 347, 367, 423, 507, 575, 583, 735, 815, 923, 975, 999\}$$

give a repeated pattern of 16s, and are associated with $1024k + x_0$ and $729k + x_{16}$.

It would be nice to think that this pattern continues and could be fully predictable, but at the moment it seems to be chaotic. To date I have seen every step-value up to 481 (at least once) with the highest so far at 649 (but bear in mind that I have only yet compiled lists of sets of x up to step-values of 26).

It is this supposedly repeating pattern of numbers that led me to formulate my 'pre-tested' theory: If you are testing all numbers in $\mathbb{N}$ (in ascending order) with the conjecture, you can save time by skipping numbers from these sets. If all numbers could be given pre-tested status then the problem is solved. The only drawback to this is that, as we have seen from the reversed function, any number from the incomplete list of pre-tested sets could equally well be from X, Y, or Z.

However, I am currently assuming that the sets Y and Z are non-existent and hence I will now demonstrate that all numbers can be given pre-tested status. I will also explain why only certain step-values exist and why others (e.g. 5, 9, 12, ...) do not. I cannot explain the chaotic order of step-values.

At this point it is convenient to introduce a partial composition of the function; also to count p when the even part of the function is used and q when the odd part is used. Hence $n = p + q$.

Let
$$f^*(x_k) = \frac{3x_k + 1}{2^{a_k}} = x_{k+1},$$

where x_k, x_{k+1} are odd. But note that $f_q^*(x_0) = x_n$, because x_n may be even. If we now fully compose this function in stages:

$$f^*(x_0) = \frac{3x_0 + 1}{2^{a_0}} = x_1, \tag{I}$$

$$f^*(x_1) = \frac{3x_1 + 1}{2^{a_1}} = x_2. \tag{II}$$

Substituting (I) into (II) (missing the intermediate stages) we get

$$f_2^*(x_0) \;=\; f^*(x_1) \;=\; \frac{3^2 x_0 + 3 + 2^{a_0}}{2^{a_0} 2^{a_1}} \;=\; x_2.$$

Continuing this process but missing out the induction step,

$$f_q^*(x_0) \;=$$

$$\frac{3^q x_0 + 3^{q-1} + 3^{q-2} A_0 + 3^{q-3} A_1 + \cdots + 3^{q-2-k} A_k + \cdots + 3 A_{q-3} + A_{q-2}}{A_{q-1} 2^{b_q}},$$

where $f_q^*(x_0) = x_n < x_0$, $A_k = 2^{a_0} 2^{a_1} \ldots 2^{a_k}$ (for $k = 0$ to $q-1$), and 2^{b_q} is a 'stopping' term which will be explained later. Since

$$p = \left(\sum_{i=0}^{q-1} a_i \right) + b_q,$$

we can write the above expression as a Diophantine equation in x_0 and x_n;

$$2^p x_n - 3^q x_0 \;=\; 3^{q-1} + 3^{q-2} A_0 + \cdots + 3^{q-2-k} A_k + \cdots + A_{q-2}$$

in which the RHS is a constant, and since $\gcd(2^p, 3^q) = 1$, the solutions to this equation are $2^p k + x_0$ and $3^q k + x_n$. Different values of x_0 and x_n can be generated from all the different combinations of the a_is and hence the sets of numbers and their associated step-values can be seen to follow. Also note that because $x_0 > x_n$, $2^p > 3^q$. This relationship is completed by $2^p > 3^q > 2^{p-1}$. So far, it is also demonstrated by my computer program up to $q = 186$ ($n = 481$). I use the expression $p > \phi q$ where $2^\phi = 3$, which gives $\phi = (\log 3)/(\log 2) \approx 1.5849625\ldots$, to demonstrate the relationship between p and q. So $p = 1 + [\phi q]$. (Hence step values 5, 9, 12, $\ldots$ do not occur.)

Following on from the Diophantine equations, there is a set of such equations (and hence sets of x_0) for each value of q which I call $X_q \subseteq X$. Since each X_q is contained in an associated modulo set $2p$, each value, q, contributes a proportion of $\mathbb{N}$ to the pre-tested list (i.e. $q = 0$, even numbers, 50%; $q = 1$, $4k + 1$, 25%; $q = 2$, $16k + 3$, 6.25%; $\ldots$).

Now comes the basis of my proof: If the sets Y and Z do not exist then all numbers can be placed in a subset X_q and hence the proportion of $\mathbb{N}$ placed in the pre-tested list is an increasing function on q—limited above by 100%.

Rather than attempt to solve these sets of Diophantine equations I have an easier manipulation. I use binary-coded string sequences (as in the computer sense of a string of characters). They are compiled thus: each time the *odd* part of the function is used, write a '1', and each time the *even* part is used, write a '0' (thus p and q count 0s and 1s respectively). Hence

for $f_1(2k) = k$		write	0,
for $f_3(4k+1) = 3k+1$		write	100,
for $f_6(16k+3) = 9k+2$		write	101000,
for $f_8(32k+11) = 27k+10$		write	10100100,
for $f_8(32k+23) = 27k+20$		write	10101000, etc.

These strings can be classified as follows.

(i) *Complete* (or *valid*): If $f_n(x_0) = x_n$ so that $2x_n > x_0 > x_n$ and $x_i > x_0$ for $i = 1$ to $n-1$.

(ii) *Incomplete*: As in (i) but $x_i > x_0$ for $i = 1$ to n, and can be made *valid* by appending sufficient 0s (until $p > \phi q$).

(iii) *Extended*: As in (i) but $x_i < x_0$ for some $i \in \{1 \text{ to } n-1\}$.

(iv) *Invalid*: A string which does not represent a sequence of compositions of the function. An example of this is given when we consider any odd number, $2k+1$; $f(2k+1) = 6k+4$ which is always an even number. So each 1 must be followed by a 0 (which I call an essential 0) and hence an example of an invalid string is one that contains two or more consecutive 1s.

(To **EK**: Does this correspond with Peter Weir's binary notation?)

Since these strings represent numbers, it should be possible to 'back-calculate' from any non-invalid string to find the associated number; this is done as follows.

Take for example 101000 associated with $f_6(16k+3) = 9k+2$. This string starts with a 1, so x_0 must be odd. i.e. $x_0 = 2k_0+1$. Then $f(2k_0+1) = 6k_0 + 4$, which is an even number, demonstrating the use of an essential 0: $f(6k_0 + 4) = 3k_0 + 2$. The next character in the string is a 1, so $3k_0 + 2$ is an odd number, which is the case if k_0 is odd. Hence $k_0 = 2k_1 + 1$. We now have $f_2(4k_1 + 3) = 6k_1 + 5$, giving $f(6k_1 + 5) = 18k_1 + 16$. This is now followed by another essential 0: $f(18k_1 + 16) = 9k_1 + 8$. Continuing, let $k_1 = 2k_2$, so $x_0 = 8k_2 + 3$ and $x_4 = 18k_2 + 8$. There now follows two 0s in the remainder of the string, so $f(18k_2 + 8) = 9k_2 + 4$; substitute $k_2 = 2k_3$

and apply f one last time to arrive at $f_6(16k_3 + 3 = 9k_3 + 2$.

Having done this process, I can now demonstrate another useful manipulation of strings. You will have noticed that at various stages in the above a decision was made for parity, this being solely dependent on the value of k. So, after the last essential 0 of a string, there follow a number of non-essential, replaceable 0s, any of which can be substituted for by a 1 and further 0s appended to give another longer valid string. Let θ represent these non-essential 0s (e.g. $10100\theta\theta$ ends with two non-essential, replaceable 0s). We can replace them in turn to give:

(a) $101010\theta\theta$ which represents $32k + 23$;

(b) 1010010θ which represents $32k+11$ (in which I have shown the next replaceable 0s).

And yet another area of string manipulation brings us full-circle to the Diophantine equations. The number of 0s after each 1 in a valid string is represented by the a-values in the Diophantine equation; hence in the following we have:

e.g.	1	0	1	00	1	00	...	1	0000	1	000	$\theta\theta\theta\theta$
a_i		a_0		a_1		a_2	...		a_{q-2}		a_{q-1}	b_q

Where b_q represents the non-essential 0s at the end of a string and shows how the Diophantine equation and the binary-coded string can be stopped, or continued into the next q-value.

Strings are easier to manipulate, but they can only be used to prove the function if the sets Y and Z do not exist. This is because if values of y or z exist they too will have a string representation and an associated Diophantine equation.

However, I can use strings to demonstrate that the set Y does not exist; by considering what would happen if it did. If there is a value, y, then it will have an associated string possibly of the form $10101010.\ldots$. In terms of the above, this is case (ii)—an incomplete string—and it can be made complete by appending sufficient 0s. But we can also back-calculate to find y (using two stages, 10, each time). Start with an odd number $f_2(2k_0 + 1) = 3k_0 + 2$. This is an odd number also, so $k_0 = 2k_1 + 1$. Also $f_4(4k_1 + 3) = 9k_1 + 8$; again, an odd number so $k_1 = 2k_2 + 1$. Thus $f_6(8k_2 + 7) = 27k_2 + 26$, etc.

By the use of induction, we can see that if this process were to continue, we would arrive at the numbers $y_0 = 2^p k + (2^p - 1)$ and $y_q = 3^q k + (3^q - 1)$, and we can see from this that if $y \in Y$ exists then not only does $f_n(y_0) \to \infty$ as $n \to \infty$, but also $y_0 \to \infty$ as $n \to \infty$; hence Y is empty. In addition,

$y_0 = \infty$ does not have a Diophantine equation. Also note: in this particular case $p = q$; the maximum value y_0 (and also x_0, z_0) is $2^p - 1$ (hence the use of the modulo set $2p$ and not $3q$).

So now we come to the set Z. My idea is based on probability and is (loosely) analogous to the following situation. Take a bucket (with say, 10 units of volume), and put into it a handful each of black and white marbles. Mix them up and remove, by random selection, about a third of the marbles. Double the amount of marbles in the bucket by adding only black ones, mix, and remove about a third of the marbles. Keep repeating this process of mixing, removing and replenishing until the bucket is full of marbles. What is the probability that any white marbles remain in the bucket?

Second step. Take a bucket with 100 units of volume and place therein two handfuls each of black and white marbles. Repeat the above process, and what is now the probability that any white marbles remain when the bucket is full?

Step n. Take a bucket with $10n$ units of volume and put therein n handfuls each of black and white marbles . . . .

This is a very loose analogy, and if you rely on it too much, it fails at the first few steps. Not to be put off by this or by the knowledge that I could have chosen a better analogy, I will press on regardless.

The analogy demonstrates that if a white marble remains in the bucket, this is equivalent to finding a counter-example to the conjecture that $X = \mathbb{N}$. As you can see, it is a decreasing probability function on n.

In order to demonstrate that Z does not exist, I turn to the partially composed function, but give it a slightly different treatment. Let

$$f^*(z_0) \;=\; \frac{3z_0 + 1}{2^{a_0}} \;=\; z_1, \;\; \text{or} \;\; z_1 \;=\; \frac{z_0\,(3 + 1/z_0)}{2^{a_0}}; \tag{III}$$

similarly,

$$f^*(z_1) \;=\; \frac{z_1\,(3 + 1/z_1)}{2^{a_1}} \;=\; z_2. \tag{IV}$$

Substituting (III) into (IV) and applying a full composition,

$$f_2^*(z_0) \;=\; z_0 \, \frac{3 + 1/z_0}{2^{a_0}} \, \frac{3 + 1/z_1}{2^{a_1}} \;=\; z_2.$$

This process continues by induction until $f_q^*(z_0) = z_q = z_0$, which when expanded becomes

$$\frac{z_0\,(3 + 1/z_0)}{2^{a_0}} \, \frac{3 + 1/z_1}{2^{a_1}} \cdots \frac{3 + 1/z_{q-1}}{2^{a_{q-1}}} \;=\; z_0.$$

Now since $p = \sum_{i=0}^{q-1} a_i$ and the z_0s cancel, we arrive at

$$\left(3 + \frac{1}{z_0}\right)\left(3 + \frac{1}{z_1}\right)\ldots\left(3 + \frac{1}{z_{q-1}}\right) = 2^p.$$

I shall not attempt to solve this polynomial, but instead I would like to introduce a sense of a geometric mean, z_r, such that $(3 + 1/z_r)^q = 2^p$, or $3 + 1/z_r = 2^{p/q}$. So, given values of p and q it is possible to calculate z_r, $z_r = 1/(2^{p/q} - 3)$, which shows $2^{p/q} > 3$, or again $2^p > 3^q$.

So, with this in mind, take an example of $p = 7$ and $q = 4$, which gives a value of $z_r \approx 2.75$; with $p = 8$ and $q = 5$: $z_r \approx 31.81$; with $p = 10$ and $q = 6$: $z_r \approx 5.72 \ldots$. Note that it is the lower values of p/q which give the highest values of z_r and not necessarily the values of p and q alone (and in this case, $3^q > 2^{p-1}$ follows automatically).

This suggested the use of the Continued Fraction Algorithm on ϕ, where $2^\phi = 3$, or $\phi = 1.5849625 \ldots$.

k	0	1	2	3	4	5	6	7	8	9
p_k	1	2	3	8	19	65	84	485	1054	24727
a_k	1	1	1	2	2	3	1	5	2	23
q_k	1	1	2	5	12	41	53	306	665	15601
z_{r_k}	-1	1	-5.8	31.8	-295	1192	-8461	99783	-5×10^6	3×10^8

(negative values of z_r are included for completeness).

(To **ADF**: Your values 301994 and 190537 appear later in this table, with a z-value of 9.84×10^{11}. Did you use the CFA to arrive at these values, or do you have another method?)

This tells me that if my computer program runs to about 1200, I know that I shall not find a value of z_0 with less than 791 steps ($p = 485$, $q = 306$), since only the odd values of k in the above table give a positive value of z_r. Similarly, by testing numbers up to about 100,000, then I eliminate the 791 step-value, the next step-value being over 40,000.

Now we come to the interpretation of the 'bucket and marbles' analogy. The bucket is the modulo set 2^{p_k}, the white marbles are all the numbers in this modulo set less than z_{r_k} and the black marbles are the numbers which fall in the subsets X_i, for $i = 0$ to $q - 1$. Hence, as these subsets are filled up, some of the white marbles are replaced by black ones. The process can also be likened to the 'Sieve of Eratosthenes'.

I start my analogy with $z_r = 1192$. Since it can be shown from the Diophantine equations that the maximum value of $z_0 \approx 0.75 z_r$ then I can

use the modulo set 1024 (i.e. 2^{10}) as a convenient starting point. There are 64 white marbles in a bucket which has space for 2^{65} marbles. There are also 960 black marbles in the bucket (hardly equals a handful each, but . . .). From the above, you will see that these black marbles are the numbers less than 1024 which are pre-tested and are arranged in the sets X_q, as in the following table.

q	p	n	members in X_q	available space in 2^p	remaining space
0	1	1	1	2	1
1	2	3	1	2	1
2	4	6	1	4	3
3	5	8	2	6	4
4	7	11	3	16	13
5	8	13	7	26	19
6	10	16	12	76	64

This table indicates that if you remove all the numbers in the modulo set 1024 with step-values up to and including 16 (the black marbles) there are 64 numbers with greater step-values (the white marbles). The table continues.

7	12	19	30	256	226	
8	13	21	85	452	367	
9	15	24	173	1468	1295	. . .

The white marbles are those values which are available for values of z_0 in the associated modulo set 2^{65}, but each set X_i ($i = 0$ to 40) must form a subset of this modulo set and hence some of the white marbles are replaced by black ones.

This step of the analogy fails because two of the white marbles remain in the bucket when it is full; these are 703 and 1055 which take 132 and 130 steps respectively to reduce. Hence my admission that I have chosen a poor analogy! However, it is the following table which indicates to me that I have a solution based on probability

p_k	8	65	485	24727
q_k	5	41	306	15601
z_{r_k}	32	1192	99783	2.9×10^8
steps	96	132	220	644

(The last row indicates the highest step-value observed while testing for the associated z-values.)

Having observed that the associated step-values increase at a much lower rate than the size of the modulo set (bucket), I attempted an evaluation using the estimation from the Continued Fraction Algorithm, as follows.

Let
$$\epsilon \;=\; p/q - \phi;$$
then $2^\epsilon = 2^{p/q-\phi}$ but since $2^\phi = 3$, we have $2^\epsilon = 2^{p/q}/3$. Also $z_r = 1/(2^{p/q} - 3)$, so $2^{p/q} = 3 + 1/z_r$. Hence $2^\epsilon = 1 + z_r/3$. Taking logs to base e,
$$\epsilon \log 2 \;=\; \log\left(1 + \frac{1}{3}z_r\right) \;\approx\; \frac{1}{3}z_r$$
(since $\log(1 + x) \to x$ as $x \to 0$). Using this approximation gives $\epsilon \approx 1/(2.079z_r)$, which can then substituted into the CFA estimation:
$$\frac{1}{q_n q_{n+1}} \;>\; |\epsilon| \;>\; \frac{1}{2q_n q_{n+1}}, \quad \text{or} \quad q_n q_{n+1} \;<\; |2.079 z_{r_n}| \;<\; 2q_n q_{n+1}.$$
Similarly,
$$q_{n+1}q_{n+2} \;<\; |2.079 z_{r_{n+1}}| \;<\; 2q_{n+1}q_{n+2}$$
and by a very open interpretation of these expressions I arrive at $z_{r_{n+1}} \approx z_{r_n} a_{n+1} a_{n+2}$. You will note that this relationship contains the term a_{k+2}, but a similar treatment to find p_{k+1} only contains the term a_{k+1}.

Since
$$|z_{r_n}| \;<\; 2.079 z_{r_n} \;<\; 2q_n q_{n+1} \;<\; 2q_{n+1}^2 \;<\; 3^{q_{n+1}}$$
and $2q_{n+1}^2/3^{q_{n+1}}$ is a null sequence (See *M203*, Analysis A) it follows that $z_{r_n}/3^{q_{n+1}}$ is also null (squeeze rule). If we now choose a_M so that
$$a_M \;>\; \max\{a_1 a_2, a_2 a_3, \ldots, a_{n+1} a_{n+2}\}$$
then $z_{r_{n+1}} \;<\; z_{r_n} a_M$ for all n, giving $z_{r_{n+1}}/3^{q_{n+1}}$ is null (multiple rule).

I apologise for my sloppy presentation—if you have got this far perhaps you should be an OU tutor (perhaps you already are!). It's amazing how quickly I've lost the habit of writing TMAs since I finished studying. Alternatively, the Editorial Board have done a wonderful job in making some sense of my ramblings.

On balance of probability I believe I have solved the problem, but probability has an annoying habit of working two ways. I would like to hear from anyone who may be able to properly evaluate this solution, either through the M500 magazine or by direct contact (see MOUTHS directory).

Problem 166.1 – A geometric theorem

David L. Brown

If, from any point on one side of a given triangle, a line be drawn parallel to
a second side to meet the third side and then, from the same point on the
first side, another line be drawn parallel to the third side to meet the second
side, the parallelogram so formed is equal in area to the parallelogram whose
adjacent sides are respectively equal to the remaining segments of the second
and third sides of the given triangle.

[**ADF**—It is a tradition that non-trivial problems in Euclidean geom-
etry are almost totally incomprehensible without the benefit of a diagram.
Fortunately for me, David supplied one. The given triangle is ABC and D
is any point on one side of it. Lines that look as though they are parallel to
each other really are parallel to each other, and $FG = AE$.
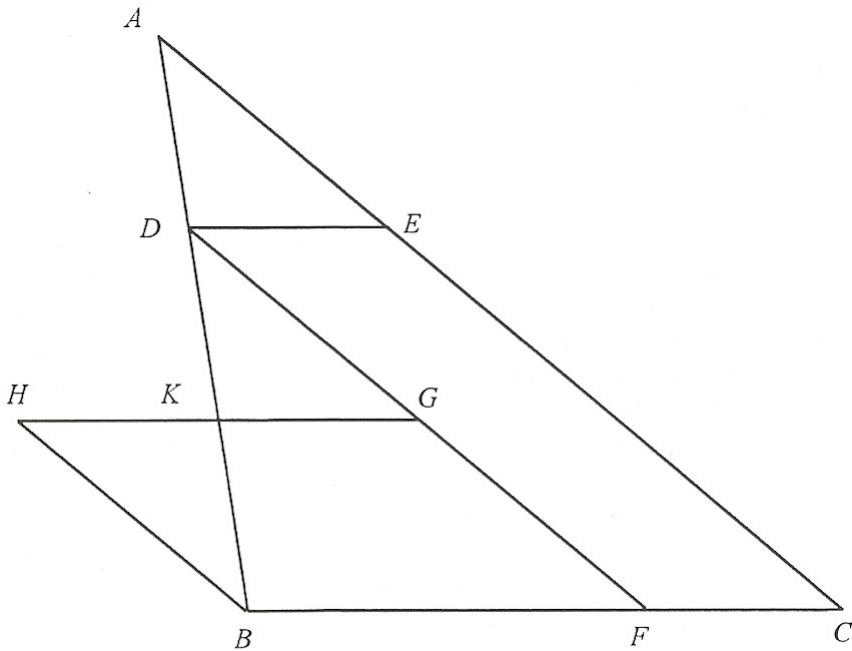
You are required to prove that the two parallelograms $DECF$ and $BHGF$
have equal areas.]

Problem 166.2 – Words
ADF

A *word* is a string of letters, where each letter belongs to a given alphabet of symbols.

Problem i: How many n-letter words in an alphabet of b symbols are there? Hint: The answer is b^n.

Problem ii: Find an expression for $K_2(n, b)$, the number of n-letter words, in an alphabet of b symbols, that do not contain any double letters.

Problem iii: Find an expression for $K_3(n, b)$, the number of n-letter words, in an alphabet of b symbols, that use every symbol at least once.

Problem iv: Find an expression for $K_4(n, b)$, the number of n-letter words, in an alphabet of b symbols, that do not contain any double letters and use every symbol at least once.

Define two words to be *equivalent* if you can transform one into the other by a permutation of the symbols of the alphabet. Thus $aaabbcd \equiv bbbccad$ as can be seen by changing $a \to b$, $b \to c$ and $c \to a$.

Problem v: Find an expression for $K_5(n, b)$, the number of equivalence classes of n-letter words in an alphabet of b symbols that do not contain any double letters and use every symbol at least once.

This is getting complicated; so let's look at a non-trivial example before going any further. There are 81 four-letter words in the alphabet $\{a, b, c\}$. The full set is $S_1 = \{aaaa, aaab, aaac, aaba, aabb, aabc, aaca, aacb, aacc, abaa, abab, abac, abba, abbb, abbc, abca, abcb, abcc, acaa, acab, acac, acba, acbb, acbc, acca, accb, accc, baaa, baab, baac, baba, babb, babc, baca, bacb, bacc, bbaa, bbab, bbac, bbba, bbbb, bbbc, bbca, bbcb, bbcc, bcaa, bcab, bcac, bcba, bcbb, bcbc, bcca, bccb, bccc, caaa, caab, caac, caba, cabb, cabc, caca, cacb, cacc, cbaa, cbab, cbac, cbba, cbbb, cbbc, cbca, cbcb, cbcc, ccaa, ccab, ccac, ccba, ccbb, ccbc, ccca, cccb, cccc\}$.

Removing words from S_1 that have a double letter leaves $S_2 = \{abab, abac, abca, abcb, acab, acac, acba, acbc, baba, babc, baca, bacb, bcab, bcac, bcba, bcbc, caba, cabc, caca, cacb, cbab, cbac, cbca, cbcb\}$; so $K_2(4, 3) = 24$.

The set of words that use all three symbols is $S_3 = \{aabc, aacb, abac, abbc, abca, abcb, abcc, acab, acba, acbb, acbc, accb, baac, babc, baca, bacb, bacc, bbac, bbca, bcaa, bcab, bcac, bcba, bcca, caab, caba, cabb, cabc, cacb, cbaa, cbab, cbac, cbba, cbca, ccab, ccba\}$, giving $K_3(4, 3) = 36$.

After eliminating words with double letters from S_3, or alternatively words that do not use all three symbols from S_2, these 18 remain: $S_4 =$

{*abac, abca, abcb, acab, acba, acbc, babc, baca, bacb, bcab, bcac, bcba, caba, cabc, cacb, cbab, cbac, cbca*} and under the equivalence relation they are grouped into just three classes of six words each, represented by *abac, abca* and *abcb*. Hence $K_4(4,3) = 18$ and $K_5(4,3) = 3$.

Now we go back to **Problem iv** and remove all *palindromes* (words that read the same forwards or backwards) from the set. They can only occur when n is odd (otherwise the centre of the palindrome has a repeated letter) and when $n \geq 2b - 1$ (because all b symbols have to be used).

Problem vi: Find an expression for $K_6(n, b)$, the number of n-letter words (in an alphabet of b symbols) that do not contain any double letters, that use every symbol at least once, and that are not palindromes.

Finally, we extend the equivalence relation in **Problem v** to words that are the reverse of each other. Two words are henceforth considered to be equivalent if there is a permutation of the alphabet that changes the first word to the second word *or its reversal*. Hence *aaabbcd* ≡ *daccbbb* under the new definition, for one can apply the permutation $a \to b$, $b \to c$, $c \to a$ to get *bbbccad* and then reverse the order of the symbols.

Problem vii: Find an expression for $K_7(n, b)$, the number of equivalence classes, under the new definition of equivalence, of n-letter words (in an alphabet of b symbols) that do not contain any double letters, use every symbol at least once and are not palindromes.

––––––––––––––––––––

At time of writing there was a burst of activity on the Internet concerning this last problem. Ronald Bruck of the University of Southern California posted it on to the news group sci.math.research. I am told it has some connection with products of orthogonal projections in Hilbert space.

I think (**i**) is quite easy, even without the hint. So is (**ii**) and it is not difficult to see how to get from (**iv**) to (**v**). However, as far as I am aware, (**vii**) remains unsolved. The bit about palindromes and reversals seems to be the stumbling-block.

In our little example, $K_6(4,3) = K_4(4,3) = 18$ since there are no palindromes in S_4 and $K_7(4,3) = 2$, for *abac* is equivalent to *abcb* (*abac* $\to$ *bcba* by the permutation $a \to b \to c \to a$ and then *bcba* $\to$ *abcb* by reversal) but neither is equivalent to *abca*.

––––––––––––––––––––

'[Monica was] very very angry with the President, called him lots of four-letter words, the mildest of which was the Big Creep.'

—Lucianne Goldberg. (Spotted by EK.)

Problem 166.3 – Boat

Dick Boardman

A small boat which travels at constant
speed in any direction is racing up a chan-
nel (which runs north - south) against the
tide. The direction of the tide is always due
south but its speed varies across the chan-
nel. At the western edge its speed is zero,
but at a distance x metres from the west-
ern edge, its speed is $x/15$ metres per sec-
ond. The boat starts on the western edge
and must reach a buoy which is 500 me-
tres north and 30 metres east. Obviously,
to reach the buoy, the boat must travel up
the western edge until nearly at the buoy
and then head out and past, eventually al-
lowing the tide to carry it back to the buoy.
The speed of the boat is 2 metres per sec-
ond.

The question is: *How far should it
travel up the edge, and what path should it
follow when heading out in order to min-
imise the time taken to reach the mark?*

I would like an analytic solution, using
techniques from the Calculus of Variations.
However, so far, all I have is computer so-
lution which breaks the path into short sec-
tions and 'hill climbs' towards the mini-
mum. My best solution gets there in 266.13
seconds, leaving the edge after 484.33 me-
tres.

Can anyone find an analytic solution?
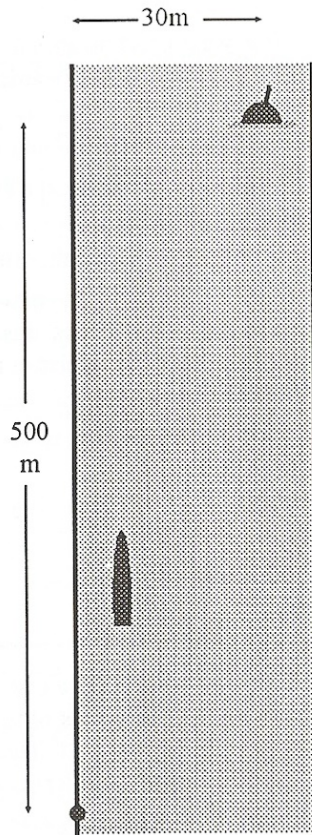
EK writes—The Australian Jose Lopez was given a new heart and lungs
from a young road crash victim. It was then found that his own heart
was sound so that was transplanted into a Tasmanian farmer named Keith
Webb. When Lopez agreed to take part in the Australian 5000 metre Trans-
plant Walk later this year he found he was competing against Webb. "If he
beats me," he said, "I want my heart back."

Solution 163.2 – The Tower of Saigon

The *Tower of Saigon* has n disks of different sizes and four pegs. Initially the disks are threaded in decreasing order of size on one of the pegs to form a conical tower. The object is to transfer the entire tower to one of the other three pegs by moving disks one at a time from peg to peg. The rules are: (a) Only a disk at the top of a pile may be moved; (b) no disk may be placed above a smaller disk. We showed how to do the transfer in

$$S(n) \;=\; \min \left\{ k = 0, 1, \ldots, n-1 : 2S(k) + 2^{n-k} - 1 \right\}$$

moves ($S(0) = 0$) and we asked for (i) a proof that $\Delta S(n) = 1$, 2, 2, 4, 4, 4, 8, 8, 8, 8, ..., and (ii) an explicit formula for $S(n)$.

Peter Fletcher

Extending the table [M500 **163** 19] for $S(n)$ and $\Delta S(n)$ shows that the pattern for $\Delta S(n)$ is 1, 2, 2, 4, 4, 4, 8, 8, 8, 8, ..., 2^p, 2^p, ..., 2^p ($p + 1$ terms), ... (This was to have been proved in part (i).)

After a little playing about with sums and products of powers of 2,

$$
\begin{aligned}
S(1) &= 1 \cdot 2^0 & (n &= 1) \\
S(2) &= 1 \cdot 2^0 + 1 \cdot 2^1 & (n &= 1 + 1) \\
S(3) &= 1 \cdot 2^0 + 2 \cdot 2^1 & (n &= 1 + 2) \\
S(4) &= 1 \cdot 2^0 + 2 \cdot 2^1 + 1 \cdot 2^2 & (n &= 1 + 2 + 1) \\
&\cdots & &\cdots \\
S(12) &= 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + 2 \cdot 2^4 & (n &= 1 + 2 + 3 + 4 + 2) \\
&\cdots & &\cdots \\
S(n) &= 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + t 2^{t-1} + (n - \textstyle\sum_{i=1}^{t} i) 2^t,
\end{aligned}
$$

where $\sum_{i=1}^{t} i \le n$. A standard result is

$$\sum_{j=1}^{m} j \;=\; \frac{m(m+1)}{2}.$$

So

$$\frac{t(t+1)}{2} \;\le\; n, \qquad t^2 + t \;\le\; 2n.$$

Completing the square, $(t + 1/2)^2 - 1/4 \le 2n$; therefore $4(t + 1/2)^2 - 8n + 1$. Taking positive square roots, $2(t + 1/2) \le \sqrt{8n + 1}$ and hence $t \le (\sqrt{8n + 1} - 1)/2$.

The first t terms in the sum for $S(n)$ can be expanded as

$$
\left\{
\begin{array}{l}
(1 + 2 + 4 + 8 + \cdots + 2^{t-1}) \\
+ (\quad 2 + 4 + 8 + \cdots + 2^{t-1}) \\
+ (\qquad\quad 4 + 8 + \cdots + 2^{t-1}) \\
\qquad\qquad \cdots \\
+ (\qquad\qquad\qquad\qquad\quad 2^{t-1})
\end{array}
\right\}
=
\left\{
\begin{array}{l}
2^t - 1 \\
2^t - 2 \\
2^t - 4 \\
\cdots \\
2^t - 2^{t-1}
\end{array}
\right\}
$$

$$
= \; t2^t - (2^t - 1) \; = \; (t-1)2^t + 1.
$$

The first line for this expansion uses a standard result

$$
\sum_{j=0}^{m-1} r^j \;=\; \frac{r^m - 1}{r - 1} \quad \text{for } r > 1,
$$

with $m = t$ and $r = 2$. The subsequent lines follow by inspection. But t must be an integer, so using square brackets to indicate the integer part of a number, an explicit formula for $S(n)$ is therefore

$$
\begin{aligned}
S(n) \;&=\; \big((t-1)2^t + 1\big) + \left(n - \frac{t(t+1)}{2}\right) 2^t \\
&=\; \left(n - 1 - \frac{t(t-1)}{2}\right) 2^t + 1,
\end{aligned}
$$

where $t = \big[(\sqrt{8n+1} - 1)/2\big]$.

Solution 163.3 – Prime multiplication

Solve

```
      P  P                        P  P  P
      P  P                           P  P
   ─────────                  ──────────────
   P  P  P                    P  P  P  P
 P  P  P                    P  P  P  P
 ─────────────             ─────────────────
 P  P  P  P                 P  P  P  P  P
```

where each P has to be replaced with a prime number: 2, 3, 5 or 7.

Peter Fletcher

This was solved basically through trial and error, although some combinations could be easily discounted: E.g., nothing can end in 2; $3 \times 3 = 9$ and $7 \times 7 = 49$. So neither of the pairs of numbers being multiplied can both

end in 3 or both end in 7. Several other excluded combinations reduced the number of trials further. There are two answers to the first part and one to the second:

$$
\begin{array}{rrrr}
 & 7 & 5 \\
\times & 7 & 7 \\
\hline
 5 & 2 & 5 \\
5 & 2 & 5 \\
\hline
5 & 7 & 7 & 5 \\
\end{array}
\qquad
\begin{array}{rrrr}
 & 7 & 5 \\
\times & 3 & 7 \\
\hline
 5 & 2 & 5 \\
2 & 2 & 5 \\
\hline
2 & 7 & 7 & 5 \\
\end{array}
\qquad
\begin{array}{rrrrr}
 & 7 & 7 & 5 \\
\times & & 3 & 3 \\
\hline
 2 & 3 & 2 & 5 \\
2 & 3 & 2 & 5 \\
\hline
2 & 5 & 5 & 7 & 5 \\
\end{array}
$$

Solution 164.2 – ABCD

Given that a, b, c, d are all between 0 and 1, prove that

$$(1-a)(1-b)(1-c)(1-d) \; > \; 1 - a - b - c - d.$$

Martyn Lawrence

I believe the solution to this puzzle is dependent on how you interpret that the values of a, b, c and d are 'all between 0 and 1'. If this means that the variables can take values in the interval $[0, 1]$ then, should all variables be equal to zero, the LHS is equal to the RHS, thus proving the conjecture false by counter-example.

If, however, it means that the variables are all in the open interval $(0, 1)$ then I offer the following proof.

Expansion of the LHS initially produces

$$(1 - a - b + ab)(1 - c - d + cd),$$

which expands further to

$$1 - c - d + cd - a + ac + ad - acd - b + bc + bd - bcd + ab - abc - abd + abcd$$

From the above we can rearrange the terms thus:

$$(1-a-b-c-d)+[(ac-acd)+(bc-bcd)+(ab-abc)+(ad-abd)+cd+bd+abcd]$$

Now, if a, b, c and d lie in the interval $(0, 1)$, then $(ac-acd) > 0$, $(bc-bcd) > 0$, $(ab - abc) > 0$ and $(ad - abd) > 0$.

Thus, all the terms in the square bracket are positive, and their sum is therefore also positive. As we are adding this positive sum to an expression equal to the RHS, i.e. $(1 - a - b - c - d)$, we have proved that the LHS > RHS as required.

Grant Curry

We have

$$
\begin{aligned}
(1-a)(1-b)(1-c)(1-d) &= (1-a-b+ab)(1-c-d+cd) \\
&> (1-a-b)(1-c-d) &&\text{(since } 0 < a,b,c,d < 1) \\
&= 1-c-d-a+ac+ad-b+bc+bd \\
&> 1-a-b-c-d &&\text{(since } 0 < a,b,c,d < 1),
\end{aligned}
$$

as required.

Also solved by **Peter Fletcher** with a proof similar to Martyn's.

Solution 164.3 – 24 squares

> What is the best packing of the 24 squares of side 1 to 24 onto
> the square of side 70?

Barbara Lee

We have $70^2 = 4900$, which is $1^2 + 2^2 + \cdots + 23^2 + 24^2$, and which is the
only instance when the sum of consecutive squares, starting with 1, is itself
a square.

Since the square pyramidal numbers are the partial sums of $1^2 + 2^2 + \cdots + n^2$, we see that 4900 is the first (and only) solution to the old problem
of the cannon balls: What is the smallest number of balls that can first be
arranged on the ground as a square, then piled in a square pyramid?

In 1974 it was proved that the best solution to Chris's tiling problem is
the one in the sketch on the cover, which omits only the 7×7 square.

JRH writes—The fact that 4900 is the only square pyramidal square was
proved in 1918. Martin Gardner says the proof is difficult. Here's an easier
pyramid problem. Tetrahedral pyramids have tetrahedral numbers of balls,
the tetrahedral numbers being the partial sums of the triangular numbers 1,
3, 6, 10, 15, ... A cannon ball enthusiast makes two tetrahedral pyramids.
When he combines the balls in his two pyramids he finds he can exactly
make a third pyramid. What is the smallest number of balls he can have;
first, if his two pyramids are the same size, second, if they are not the same
size?

Infinite product

John Bull

The 'proof' concerning the sum of the reciprocals of the squares of the integers, as offered by Martin Hansen in M500 **164**, was first published by Euler in 1734. Unfortunately it is not a proof. It is only conjecture that $\sin x = 0$ can be represented as a product of linear factors, that the infinite product converges and can be manipulated, and that there are not imaginary roots in addition to the real ones. Despite compelling evidence from numerical experimentation, Euler himself realised that the intuitive step to an infinite polynomial would need to be rigorously underpinned, and he spent the next ten years trying to do this.

A discussion of the problem, together with quotes by Polya and John Bernoulli, and lots of references, can be found in: *Excursions in Calculus*, by Robert M Young, MAA 1992, ISBN 0-88385-317-5. I highly recommend this book as a 'must have' for anyone interested in popular mathematics of the style of M500.

Brain dead

JRH

Allegedly, it says in the recent issue of *The Lawyer* that this is a real extract from a court case.

Lawyer: Doctor, before you performed the post-mortem, did you check for a pulse, blood pressure, or breathing?

Witness: No.

Lawyer: So it is possible that the patient was alive when you performed the autopsy?

Witness: No.

Lawyer: How can you be so sure?

Witness: Because his brain was sitting on my desk in a jar.

Lawyer: But could the patient have still been alive nevertheless?

Witness: Possibly. Maybe he was practising law somewhere.

Special Issue 1999

We are about to compile the 1999 Special Issue. Please send your reports of courses which are still current to **Eddie Kent** as soon as possible.

Power of ten

In M500 **164** we meant to ask you to explain why

$$\lim_{n\to\infty} (10^{10^{-n}} - 1)10^n \;=\; \log 10.$$

Unfortunately we omitted a vital minus sign. Also we should have said that the problem appeared in *IEE News*, not *IEEE News*. Apologies.

Edward Stansfield

The fact that this limit is true can be shown as follows:

Let $f(n) = (10^{10^{-n}} - 1)10^n$. Rearrange the expression for $f(n)$ to obtain

$$10^{10^{-n}} \;=\; 1 + 10^{-n}f(n)$$

and then take natural logarithms of both sides to get

$$10^{-n}\log 10 \;=\; \log(1 + 10^{-n}f(n)) \;=\; \log(1 + x(n))$$
$$=\; x(n) - \frac{x(n)^2}{2} + \frac{x(n)^3}{3} - \dots,$$

where the Taylor series expansion on the right hand side is valid if $|x(n)| < 1$. Since the left hand side is asymptotic to zero for large n, the term $x(n) = 10^{-n}f(n)$ must also be asymptotic to zero, which validates the Taylor series expansion. Moreover, when we take the limit to infinity we get

$$\lim_{n\to\infty} 10^{-n}\log 10 \;=\; \lim_{n\to\infty} x(n) \;=\; \lim_{n\to\infty} 10^{-n}f(n).$$

Multiplying through by 10^n then gives $\lim_{n\to\infty} f(n) = \log 10$ as was to be shown. This completes the proof.

Alternatively one can use the Taylor series for $\exp(x)$:

$$10^{10^{-n}} - 1 \;=\; \exp(10^{-n}\log 10) - 1$$
$$=\; 10^{-n}\log 10 + \frac{1}{2!}(10^{-n}\log 10)^2 + \dots.$$

Now multiply by 10^n and let $n \to \infty$. —Eds.

Balls

JRH

Colin Davies sent me the 'Dipole *Micromatters*' column from *IEE News*, 3 September 1998. It has Lewis Carroll's joke 'problem' of the balls in the bag, which goes as follows.

> A bag contains two balls, each either black or white. Ascertain their colours without looking. Carroll states that one is black, the other white, and his 'reasoning' goes like this.

If a bag contains BBW, then the chance of drawing B is 2/3, and no other state gives this chance. Now the bag with two balls has BB with probability 1/4, BW with probability 1/2, and WW with probability 1/4. Now add a black ball to the bag. The probability of drawing a black ball is now easily seen to be $\dfrac{2}{3} = 1 \times \dfrac{1}{4} + \dfrac{2}{3} \times \dfrac{1}{2} + \dfrac{1}{3} \times \dfrac{1}{4}$. So, by the original premise, the bag contains BBW.

The puzzle is to spot the flaw in the reasoning. It states that if you have a bag with two black balls and one white ball and you draw a ball, the chance is 2/3 that it's black. That's true.

It goes on to say that any other state of things would not give this chance. That's false. Indeed, it immediately offers a different state of things which also gives the 2/3 chance of drawing a black ball—namely a bag which has BBB with probability 1/4, BBW with probability 1/2 and BWW with probability 1/4.

In fact there are infinitely many of these states (three balls in a bag, each either B or W) which give a 2/3 chance of drawing a black ball. They are the states where the probability of BBB is q, the probability of BWB is p, and the probability of BWW is q, where $p + 2q = 1$.

The two cases quoted are those where

1) $p = 1$, $q = 0$,

2) $p = 1/2$, $q = 1/4$.

You can construct these states at will; as follows, for example. Assemble three bags containing BB, three bags containing WW and one bag containing BW. Get somebody to select a bag at random, add a black ball to it, and draw a ball from it. The chance is 2/3 that the drawn ball is black; and $p = 1/7$, $q = 3/7$.

But obviously you don't know what's left in the bag.

Sums of odd integers

Sebastian Hayes

Clearly, $4 = 1 + 3$. Also $57 = 17 + 19 + 21$.

This led on to the following question.

What integers can be expressed as a sum of successive odd integers, and in how many ways? (For my purposes 1 counts as an 'odd' integer, a single odd number counts as a 'sum'.)

Well, to start off with, since

$$1 + 3 + 5 + \cdots + (2n - 1) \;=\; n^2,$$

the question boils down to what numbers can be expressed as the difference between two squares,

$$m^2 - n^2 \;=\; (m + n)(m - n),$$

$m, n \in \mathbb{Z}^+$? If $(m + n)$ is even, so is $(m - n)$ and vice versa. This means we may not have $N = 2p$ where p is odd if we want to have N as difference of two (perfect) squares. So the doubles of odd numbers have no expression in terms of successive odd integers.

More usefully, if r and q are factors of N,

$$N \;=\; rq \;=\; \left(\frac{r + q}{2}\right)^2 - \left(\frac{r - q}{2}\right)^2,$$

$r, q \in \mathbb{Z}^+$, r, q of same parity, $r \geq q$, which in terms of Galileo's series is

$$(1 + 3 + 5 + \cdots + (r + q - 1)) - (1 + 3 + 5 + \cdots + (r - q - 1))$$

i.e.,

$$(r - q + 1) + (r - q + 3) + \cdots + (r + q - 1).$$

Thus, if $N = 45 = 9 \times 5$ we take $r = 9$, $q = 5$. Then

$$\begin{aligned} N \;&=\; \left(\frac{9 + 5}{2}\right)^2 - \left(\frac{9 - 5}{2}\right)^2 \;=\; 7^2 - 2^2 \\ &=\; (1 + 3 + \cdots + 13) - (1 + 3) \\ &=\; (5 + 7 + 9 + 11 + 13). \end{aligned}$$

In the above there are five terms, and $q = 5$. We may conjecture that there will always be q terms, which is easily proved since

$$\begin{aligned} &(r - q + 1) + (r - q + 3) + \cdots + (r + q - 1) \\ =\;& (r - q + 1) + (r - q + 3) + \cdots + (r - q + (2q - 1)) \\ =\;& q(r - q) + (1 + 3 + \cdots + (2q. - 1)) \;=\; q(r - q) + q^2 \;=\; rq. \end{aligned}$$

If N is a perfect square, $N = r^2$ and so $r = q = (2r/2)^2 - 0^2$ (allowing zero). The formula actually still works since $(r - q + 1) = 1$ if $r = q$. Moreover, if we allow negative odd integers, we do not need to stipulate that $r \geq q$. Thus if $r = 5$ and $q = 9$, we obtain

$$N = (-3) + (-1) + (1) + \cdots + (13) \quad (\text{9 terms}).$$

In how many ways can an integer be expressed in this fashion?

I found it necessary to tackle this in piecemeal fashion.

Suppose N is p^m where m is an odd number. If we keep to positive integers for the time being, we find that N_G, or the number of ways an integer can be expressed as a sum of consecutive odd integers is $(m+1)/2$ for m odd and $(m+2)/2$ for m even.

If $p = 2$, we lose a pair of factors and the number of ways for 2^m reduces to $(m-1)/2$ for m odd and $m/2$ for m even.

If N is not a power, it has a (unique) expression as prime numbers raised to powers i.e. $N = p^a q^b r^c \ldots$, where p, q, r, $\ldots$ are primes.

If N is odd, N_G, as far as I can make out, is

$$\frac{a+1}{2} \frac{b+1}{2} \frac{c+1}{2} \cdots$$

if a, b, c, $\ldots$ are odd, and we make the corresponding changes to $a/2$, $b/2$, $c/2$, $\ldots$ if a, b, c, $\ldots$ are even. If we have $N = 2^a q^b r^c \ldots$, we lose one set of solutions, so $N_G = (a-1)/2$ or $a/2 \ldots$. I have no doubt that someone versed in combinations and permutations will provide a more elegant general formula without too much effort.

This question leads on to other, much tougher questions in Number Theory, notably whether Goldbach's Conjecture is right—Goldbach suggested that every even number greater than 4 is the sum of two odd primes.

Note: I refer to the series

$$1 + 3 + 5 + \cdots + (2n - 1)$$

as Galileo's series because of his epoch-making observation that 'the distances traversed during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity.'

'Religion affects us all; it shapes our world and those around us ...'

—R4 trailer (spotted by JRH).

Euler relation
David L. Brown

How's this for an 'Euler Boiler'? Look what happens if we find $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ by two methods, then equate the results. If $i = \sqrt{-1}$ then $1 + x^2 = (1 - ix)(1 + ix)$. Put

$$
\begin{aligned}
\frac{1}{1+x^2} &= \frac{A}{1-ix} + \frac{B}{1+ix} \\
&= \frac{A(1+ix) + B(1-ix)}{(1-ix)(1+ix)} \\
&\rightarrow\ A(1+ix) + B(1-ix) = 1.
\end{aligned}
$$

Therefore $A + B = 1$ and $A - B = 0 \rightarrow A = B = 1/2$. Hence

$$
\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \frac{1}{2}\int_0^1 \left\{ \frac{1}{1-ix} + \frac{1}{1+ix} \right\} dx \\
&= \frac{1}{2i}\Big[-\log(1-ix) + \log(1+ix) \Big]_0^1 \\
&= \frac{1}{2i}\left[\log\frac{1+ix}{1-ix} \right]_0^1 = \frac{1}{2i}\log\frac{1+i}{1-i} \\
&= \frac{1}{2i}\log\frac{(1+i)^2}{(1-i)(1+i)} = \frac{1}{2i}\log\frac{1+2i+i^2}{1-i^2} = \frac{1}{2i}\log i.
\end{aligned}
$$

But we know that

$$
\int_0^1 \frac{dx}{1+x^2} = \Big[\arctan x \Big]_0^1 = \frac{\pi}{4}.
$$

Therefore

$$
\frac{1}{2i}\log i = \frac{\pi}{4} \rightarrow \log i = \frac{\pi i}{2};
$$

i.e. $e^{\pi i/2} = i$. Squaring both sides we get $e^{\pi i} = -1$; i.e. $e^{\pi i} + 1 = 0$. Has this strange Euler relationship been previously obtained via the Calculus? The relationship can be expanded:

Since $e^{\pi i} = -1$, squaring both sides gives $e^{2\pi i} = 1$. But $1^n = 1$, where n is any real integer (positive or negative). Therefore $e^{2\pi n i} = 1^n = 1$, i.e. $e^{2\pi n i} = 1$ for all n. Hence the relationship is even stranger! Does this mean it is cyclic? Does anybody understand what it means?

Prime numbers and groups

Grant Curry

If the set of all prime numbers is a group then there must exist a closed binary operation between all pairs of primes. If simple operations such as $+$, $-$, $\times$, $/$ are considered then it is apparent these are not the binary operation needed.

Let P be the set of prime numbers: $7, 13 \in P$; however, $13 - 7 = 6 \notin P$ and $13 + 7 = 20 \notin P$. But by definition, $13 \times 7 \notin P$ and $13/7 \notin P$.

The set P is not a group under modulo multiplication because the Identity, $1 \notin P$. Similarly for modulo addition, $0 \notin P$.

A natural question to ask is there any binary operation for P to be a group? What we are looking for is a function of three variables. It can be shown that a polynomial of two variables cannot represent prime numbers and this can be extended to a polynomial of three variables as follows.

Let p, q, $r \in P$ and let

$$p = a_n q^n + a_{n-1} q^{n-1} + \cdots + aq + a_0 + b_m r^m + b_{m-1} r^{m-1} + \cdots + br + b_0. \quad (1)$$

As there are infinitely many primes, when q and r run through the primes they will simultaneously reach a_0 and b_0 and hence the right hand side of (1) can be factorized in which case p is not prime. (If a_0 or b_0 are composite then the RHS can still be factorized because of the fundamental theorem of arithmetic). Hence there does not exist a polynomial of three variables which generates primes. This could be extended to a function of n variables.

Now Taylor's Theorem states that if a function f is $(n+1)$-times differentiable on an open interval containing the points a and x then $f(x)$ can be expressed as a polynomial with $f(a)$ as a constant. Hence it follows that if a function has at least a first derivative then it can be represented by a polynomial and from the above argument cannot represent only primes. It also follows that any number of combinations of differentiable functions cannot represent primes and therefore no differentiable function can be a binary operation for P to be a group.

Apology

We wish to apologize to Ralph Hancock for our treatment of his Latin poem *De pharo eddystoniensi* [M500 **164** 14–15]. We wish to make it clear that the idiotic English translation and facetious comments at the end were in fact due to one of us (EK) and indeed should have been omitted, for Ralph was aware of EK's 'contribution' and had specifically asked ADF not to include it.

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