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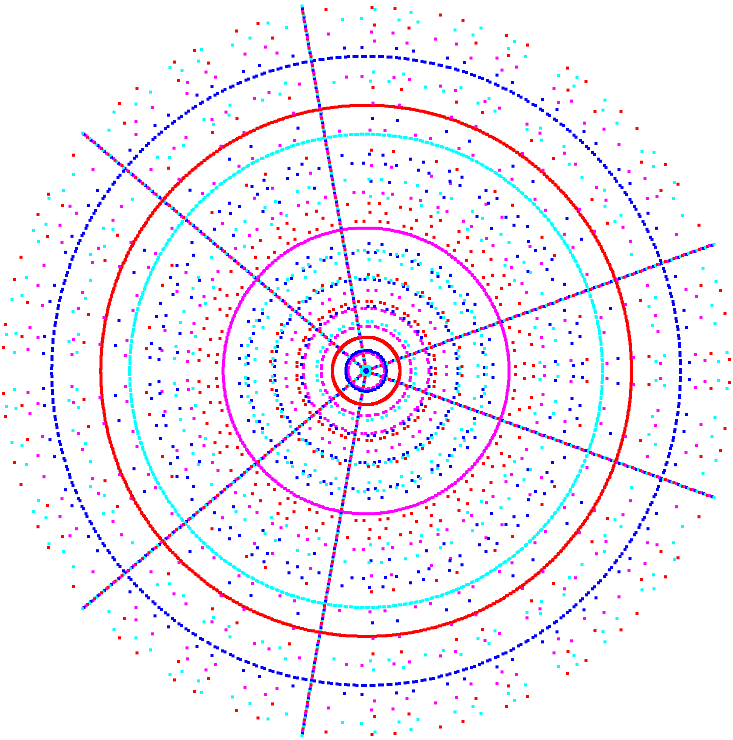

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Solution 261.6 – Determinant equation

Solve

$$\begin{vmatrix} x & 1 & 2 & 3 & 4 & 5 \\ 5 & x & 1 & 2 & 3 & 4 \\ 4 & 5 & x & 1 & 2 & 3 \\ 3 & 4 & 5 & x & 1 & 2 \\ 2 & 3 & 4 & 5 & x & 1 \\ 1 & 2 & 3 & 4 & 5 & x \end{vmatrix} = 0.$$

Stuart Walmsley

Introduction

The determinant to be solved in the problem has a pattern. The symmetry of this pattern determines the solution. To facilitate the solution, the numbers in the determinant are replaced by letters, giving a more general form.

$$|M| = \begin{vmatrix} x & a & b & c & d & e \\ e & x & a & b & c & d \\ d & e & x & a & b & c \\ c & d & e & x & a & b \\ b & c & d & e & x & a \\ a & b & c & d & e & x \end{vmatrix} = 0.$$

The expanded form of the determinant is a polynomial of order 6 in x. Solution yields a product of six factors, each linear in x, and each involving a different root of the polynomial.

The solution may be expressed as a new determinant $|\Lambda|$ in which each solution is an element on the main diagonal and there are zeroes elsewhere. Symbolically, $|M| \to |\Lambda|$. The solution can be found by considering the square matrix corresponding to the determinant. It may be shown that if the product of two square matrices A, B is C, $AB = C$, then the corresponding determinants have the same property, $|A||B| = |C|$.

Suppose M is considered as acting on a linear space: $Mv = w$; that is, M converts a vector v in the space to a vector w. Change the basis of the vectors by a matrix T^{-1}:

$$T^{-1}v = v', \quad v = Tv', \quad T^{-1}w = w', \quad w = Tw';$$

v and v' are the same vector in the two different bases. Substituting and with some simple manipulation, $T^{-1}MTv' = w'$. Since $|T^{-1}||T| = 1$,

$|T^{-1}MT| = |M|$. The problem under consideration is then to find T so that $T^{-1}MT = \Lambda$.

Cyclic matrices and cyclic groups

The matrix M (and the determinant $|M|$) has elements $M_{j,k}$, where j is the row and k the column. The range of j and k is 0, 1, …, 5. The pattern of the matrix can be described:

$$M_{j,k} \;=\; M_{j,j+m} \;=\; M_{0,m} \;=\; M_{0,k-j}, \qquad m = k - j,$$

$j+m$ and $k-j$ to be taken modulo 6. Such a matrix (determinant) is called a *cyclic* matrix (determinant).

The pattern of the matrix is identical to a particular form of the product table for the elements of a group.

Product pq $\quad q =$	x	a	b	c	d	e
$p = q^{-1} = x$	x	a	b	c	d	e
e	e	x	a	b	c	d
d	d	e	x	a	b	c
c	c	d	e	x	a	b
b	b	c	d	e	x	a
a	a	b	c	d	e	x

The array has the form of the product table of a group arranged so that the x, the identity occurs along the main diagonal and that therefore corresponding rows and columns refer to pairs of inverse elements. The group is the cyclic group of order 6. The element a generates a cycle of order 6 and hence the whole group; b is of order 3, c of order 2, d of order 3, e of order 6 and x of order 1.

$$\begin{array}{llllll}
a & aa = b & aaa = c & aaaa = d & aaaaa = e & aaaaaa = x \\
b & bb = d & bbb = x & & & \\
c & cc = x & & & & \\
d & dd = b & ddd = x & & & \\
e & ee = d & eee = c & eeee = b & eeeee = a & eeeeee = x
\end{array}$$

Representations of the group are sought which consist of complex numbers with multiplication as the product rule.

The representative of the generator a must be a number whose sixth power is 1. There are six such numbers

$$e_j \;=\; \exp\left(\frac{2\pi i j}{6}\right), \qquad j = 0,\ 1,\ \ldots,\ 5$$

with the properties

$$\begin{aligned}
e_j^* &= e_{-j}, && j \bmod 6, \\
e_j e_k &= e_{j+k}, && j+k \bmod 6, \\
\textstyle\sum_{j=0}^{5} e_j &= 0, && \\
\textstyle\sum_{k=0}^{5} e_{jk}^* \, e_{kl} &= 6\,\delta_{j,l}, && \delta_{j,l} = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Each e_j generates a representation: a (e_j), b (e_{2j}), $\ldots$, x $(e_{6j} = e_0 = 1)$. Here is the full set.

x	a	b	c	d	e
e_0	e_0	e_0	e_0	e_0	e_0
e_0	e_1	e_2	e_3	e_4	e_5
e_0	e_2	e_4	e_0	e_2	e_4
e_0	e_3	e_0	e_3	e_0	e_3
e_0	e_4	e_2	e_0	e_4	e_2
e_0	e_5	e_4	e_3	e_2	e_1

If the general term is represented by e_{jk}, then $e_{jk} = \exp(2\pi ijk/6)$, j, k mod 6. Remembering that $\sum_k e_{jk}^* \, e_{kl} \;=\; 6\,\delta_{j,l}$, a new matrix is defined:

$$T_{j,k} \;=\; \frac{1}{\sqrt{6}}\, e_{jk} \;=\; \frac{1}{\sqrt{6}} \exp\left(\frac{2\pi ijk}{6}\right), \qquad j,\ k \bmod 6.$$

This matrix is unitary and has inverse T^{-1}:

$$T_{j,k}^{-1} \;=\; T_{k,j}^* \;=\; \frac{1}{\sqrt{6}} \exp\left(\frac{-2\pi ijk}{6}\right), \qquad j,\ k \bmod 6.$$

General solution

It will now be shown that the matrix T developed in the previous section provides the solution to the general problem: $T^{-1}MT = \Lambda$, Λ being a diagonal matrix, the non zero elements of which are the solutions of the polynomial problem. In component terms,

$$\sum_q T_{p,q}^{-1} \sum_r M_{q,r} T_{r,s} \;=\; \Lambda_{p,s}.$$

Then

$$\frac{1}{6}\sum_q \exp\left(\frac{-2\pi ipq}{6}\right)\sum_r M_{q,r}\exp\left(\frac{2\pi irs}{6}\right) \;=\; \Lambda_{p,s}.$$

The symmetry of the M matrix,

$$M_{q,r} \;=\; M_{q,q+m} \;=\; M_{0,m} \;=\; M_{0,r-q}, \qquad m = r-q,$$

gives

$$\frac{1}{6}\sum_q \exp\left(\frac{-2\pi ipq}{6}\right)\sum_r M_{0,r-q}\exp\left(\frac{2\pi i(q+m)s}{6}\right) \;=\; \Lambda_{p,s}.$$

Factorize the final factor:

$$\frac{1}{6}\sum_q \exp\left(\frac{-2\pi ipq}{6}\right)\sum_r M_{0,r-q}\exp\left(\frac{2\pi iqs}{6}\right)\exp\left(\frac{2\pi ims}{6}\right) \;=\; \Lambda_{p,s}.$$

Change the sum over r to a sum over m:

$$\frac{1}{6}\sum_q \exp\left(\frac{-2\pi ipq}{6}\right)\sum_m M_{0,m}\exp\left(\frac{2\pi iqs}{6}\right)\exp\left(\frac{2\pi ims}{6}\right) \;=\; \Lambda_{p,s}.$$

The expression may now be factorized into two independent sums

$$\left(\frac{1}{6}\sum_q \exp\left(\frac{-2\pi ipq}{6}\right)\exp\left(\frac{2\pi iqs}{6}\right)\right)\left(\sum_m M_{0,m}\exp\left(\frac{2\pi ims}{6}\right)\right) \;=\; \Lambda_{p,s}.$$

The first factor is $T^{-1}T$, so that

$$\delta_{p,s}\sum_m M_{0,m}\exp\left(\frac{2\pi ims}{6}\right) \;=\; \Lambda_{p,s}$$

and Λ is a diagonal matrix.

It may be noted that the analysis is unchanged if 6 is replaced by any positive integer, so that the result holds for cyclic matrices of any order.

Discussion

The diagonal elements of Λ give the solutions to the determinantal equation which is the problem under direct consideration. To shorten the notation, let

$$e_1 \;=\; \exp\left(\frac{2\pi i}{6}\right) \;=\; \frac{1}{2}+\frac{\sqrt{3}i}{2} \;=\; w.$$

Then

$$e_0 = 1, \quad e_1 = w, \quad e_2 = -w^*, \quad e_3 = -1, \quad e_4 = -w, \quad e_5 = w^*.$$

The solution to the problem then becomes

$$\begin{aligned} x + a + b + c + d + e &= 0, \\ x + wa - w^*b - c - wd + w^*e &= 0, \\ x - w^*a - wb + c - w^*d - we &= 0, \\ x - a + b - c + d - e &= 0, \\ x - wa - w^*b + c - wd - w^*e &= 0, \\ x + w^*a - wb - c - w^*d + we &= 0. \end{aligned}$$

Substitution of the numerical values $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$ leads to

$$\begin{aligned} x &= -15, & x &= 3 + i3\sqrt{3}, & x &= 3 + i\sqrt{3}, \\ x &= 3, & x &= 3 - i\sqrt{3}, & x &= 3 - i3\sqrt{3}. \end{aligned}$$

The full symmetry of the determinant is C_6 and the roots of the polynomial are correspondingly distinct and include complex conjugate pairs. For all the roots to be real, the matrix would have to be symmetric (or if the values a, b, ... include complex numbers, hermitian). In the example here, this would require $a = e$ and $b = d$. The solution to the problem then becomes

$$\begin{aligned} x + 2a + 2b + c &= 0, & x + a - b - c &= 0, & x - a - b + c &= 0, \\ x - 2a + 2b - c &= 0, & x - a - b + c &= 0, & x + a - b - c &= 0. \end{aligned}$$

All the roots are now real and there are two pairs of equal roots. A more detailed analysis would show that the underlying group is the dihedral group of order 12 containing C_6 as a subgroup. The extreme case is $a = b = c = d = e$, when the solutions become:

$$x + 5a = 0, \quad x - a = 0, \quad x - a = 0, \quad x - a = 0, \quad x - a = 0, \quad x - a = 0.$$

The problem now has full permutation symmetry with group S_6. The five repeated roots correspond to the six vertex regular simplex, which is a figure in five dimensions.

$z^n + z^k = 1$

Bryan Orman

In this article the polynomial $p(z) = z^n + z^k - 1$ will be examined for its unimodular roots, that is, the roots that lie on the unit circle $|z| = 1$ in the complex z-plane. The existence of such roots depends on both n and k, and in what follows we assume that $n \geq 2$ and $1 \leq k \leq n-1$ although we start by examining the simple case where the z^k term is omitted.

Consider $z^n = 1$. Evidently *all* the solutions of this equation are unimodular since $|z^n| = |z|^n = 1$, giving $|z| = 1$ for all solutions. Specifically, the required solutions are

$$z_m = \exp\left(\frac{2\pi mi}{n}\right), \qquad m = 0, 1, 2, \ldots, n-1.$$

These are equally spaced around the unit circle at angular positions $2\pi m/n$, with $m = 0$ corresponding to $z_0 = 1$.

Adding the z^k term to the left-hand side results in solutions that are either unimodular or inside/outside the unit circle. The existence of unimodular solutions depends on both n and k, so if we assume that $\exp(i\theta)$ is a unimodular solution, where $\theta = \theta(n, k)$, then $p(z) = 0$ becomes

$$e^{in\theta} + e^{ik\theta} = 1.$$

The two unimodular terms on the left-hand side must sum to 1, so they must be complex conjugates. The only possible pair of complex conjugates are $e^{\pm i\pi/3}$. To show this consider $e^{i\alpha} + e^{i\beta} = 1$; then $\cos\alpha + \cos\beta = 1$ and $\sin\alpha + \sin\beta = 0$, giving $\alpha = \pm\pi/3$ and $\beta = \mp\pi/3$. Thus $e^{in\theta} = e^{\pm i\pi/3}$ and $e^{ik\theta} = e^{\mp i\pi/3}$ with general solution

$$n\theta = \pm\frac{\pi}{3} + 2\pi p, \qquad k\theta = \mp\frac{\pi}{3} + 2\pi q,$$

where p and q are integers.

Eliminating θ from these equations gives $n + k = \pm 6(nq - kp)$. This leads to the *necessary* condition on n and k for unimodular solutions of the equation $z^n + z^k = 1$ that $n + k \equiv 0 \pmod 6$. However, this condition is not *sufficient* since, for example, the quartic equation with $n = 4$ and $k = 2$ satisfies this condition but $\zeta = z^2 = (-1 \pm \sqrt{5})/2$, and clearly the roots will not be unimodular. The equation with $n = 9$ and $k = 3$ can be seen to have all non-unimodular roots by putting $\zeta = z^3$.

It is clear from these two examples that we should first reduce the order of the general equation by setting $\zeta = z^\lambda$ so that the equation becomes

$$\zeta^{n/\lambda} + \zeta^{k/\lambda} \;=\; 1.$$

Since we require both powers to be positive integers we set $\lambda = \gcd(n,k)$, $n = n_0\lambda$ and $k = k_0\lambda$, giving the reduced equation of lowest order,

$$\zeta^{n_0} + \zeta^{k_0} \;=\; 1 \qquad \text{with } \gcd(n_0,k_0) = 1.$$

We will now show that if $n_0 + k_0 \equiv 0 \pmod 6$ and $\gcd(n_0,k_0) = 1$, then the *only* unimodular roots of this reduced equation are $e^{\pm i\pi/3}$.

Firstly, it is clear that the above two conditions on n_0 and k_0 imply that neither is divisible by 2 or 3; so both integers are congruent to ± 1 modulo 6. And so $n_0 = 6s \pm 1$ and $k_0 = 6t \mp 1$, where s and t are non-negative integers. A direct calculation shows that if $\zeta = e^{\pm i\pi/3}$, then

$$\zeta^{n_0} + \zeta^{k_0} \;=\; \left(e^{\pm i\pi/3}\right)^{6s\pm 1} + \left(e^{\pm i\pi/3}\right)^{6t\mp 1} \;=\; 1;$$

that is, $e^{\pm i\pi/3}$ are solutions of the reduced equation.

To show that they are the *only* solutions we *assume* that $\zeta = e^{i\theta}$; then we must *show* that $\theta \equiv \pm\pi/3 \pmod{2\pi}$. Now the value of θ is given by $\theta = 2\pi\dfrac{p+q}{n_0+k_0}$ and the values of p and q are found from the following pair of linear Diophantine equations:

$$-6k_0p + 6n_0q \;=\; n_0 + k_0, \qquad 6k_0p - 6n_0q \;=\; n_0 + k_0.$$

We now quote a standard theorem (see D. Burton: *Elementary Number Theory*, page 40): The linear Diophantine equation $ap + bq = c$ has a solution if and only if $d|c$, where $d = \gcd(a,b)$. If p_0, q_0 is any particular solution of this equation, then all other solutions are given by

$$p \;=\; p_0 + \frac{b}{d}\,\mu \quad \text{and} \quad q \;=\; q_0 - \frac{a}{d}\,\mu$$

for varying integers μ. In order to apply this theorem we need to check that $d|c$; that is $\gcd(6k_0, 6n_0)$ divides $n_0 + k_0$. It follows from $\gcd(n_0,k_0) = 1$ that $d = \gcd(6k_0, 6n_0) = 6$, and we have assumed that $n_0 + k_0$ is divisible by 6, so we can apply the theorem.

Inserting $n_0 = 6s \pm 1$ and $k_0 = 6t \mp 1$ into the first of these equations the solution $p_0 = \pm s$ and $q_0 = \pm t$ is found. Inserting them into the second of these equations the solution $p_0 = \pm 5s + 1$ and $q_0 = 5t - 1$ is found. Now

$$p + q = p_0 + q_0 + \frac{(b-a)\mu}{6}.$$

For the particular solution $p_0 = \pm s$ and $q_0 = \pm t$, and from the first equation, $a = -6k_0 = -36t + 6$ and $b = 6n_0 = 36s + 6$; so the value of $p + q$ is $(s + t)(\pm 1 + 6\mu)$. And since $n_0 + k_0 = 6(s + t)$ we have

$$\theta = \frac{2\pi(\pm 1 + 6\mu)}{6}.$$

For the particular solution $p_0 = \pm 5s + 1$ and $q_0 = \pm 5t - 1$, and from the second equation, $a = 6k_0 = 36t - 6$ and $b = -6n_0 = -36s - 6$; so the value of $p + q$ is $(s + t)(\pm 5 - 6\mu)$. In this case

$$\theta = \frac{2\pi(\pm 5 - 6\mu)}{6}.$$

Since μ is any integer these can be taken together and we can write $\theta \equiv \pm\pi/3 \pmod{2\pi}$, as required. Finally, we use $\zeta = z^\lambda$ to complete the analysis and the following theorem can then be used to determine all the unimodular roots of $p(z)$.

The function $p(z) = z^n + z^k - 1$ has exactly 2λ unimodular roots given by

$$z_m = \exp\left(\pm i\left(\frac{\pi}{3\lambda} + \frac{2m\pi}{\lambda}\right)\right), \qquad \text{where } 0 \le m \le \lambda - 1,$$

provided 6 divides $(n + k)/\lambda$. Recall that $\lambda = \gcd(n, k)$.

We end with some consequences of this theorem.

- Since $n + k \equiv 0 \pmod 6$ and $\gcd(n, k) = 1$ imply that neither integer is divisible by 2 or 3, it follows that there are no unimodular roots for $n = 2^\alpha 3^\beta$, where $\alpha, \beta \in \mathbb{N}$.

- If $n \ge 5$ is prime, then there are unimodular roots for which $n + \kappa$ is a multiple of 6, κ being the smallest value of k, and for $k = \kappa + 6m$, $m = 1, 2, \ldots$, with $k \le n - 1$. Since $\lambda = 1$ for n prime there will be exactly one pair of conjugate unimodular roots for each possible k. The polar angles are just $\theta = \pm\pi/3$.

- It follows that $n = 5$ and $k = 1$ are the smallest possible values of n and k that result in unimodular roots.

An example: $n = 25$. The following satisfy $n + k \equiv 0 \pmod 6$:

$$25 + 5 = 30, \quad 25 + 11 = 36, \quad 25 + 17 = 42, \quad 25 + 23 = 48,$$

and the values of λ are respectively 5, 1, 1, 1.

Since $\lambda = 1$ for $k = 11$, 17, and 23, the unimodular roots have polar angles $\theta = \pm\pi/3$ for these values of k.

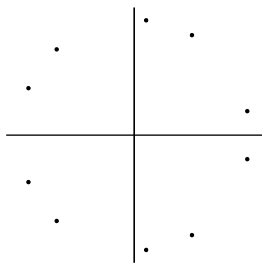
For $k = 5$, we have $\lambda = 5$ and the five pairs of unimodular conjugate roots have polar angles

$$\theta = \pm\frac{\pi}{15}, \ \pm\frac{7\pi}{15}, \ \pm\frac{13\pi}{15}, \ \pm\frac{19\pi}{15}, \ \pm\frac{25\pi}{15};$$

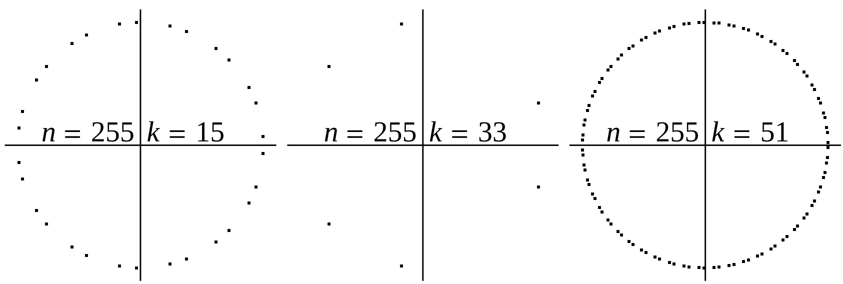
equivalently

$$\theta = \pm\frac{\pi}{15}, \ \pm\frac{5\pi}{15}, \ \pm\frac{7\pi}{15}, \ \pm\frac{11\pi}{15}, \ \pm\frac{13\pi}{15}.$$

These are located around the unit circle at alternating angular separations of $4\pi/15$ and $2\pi/15$, starting with $\pi/15$.



A similar analysis with $n = 255$ and $k = 15$, 33, 51 produces these patterns of solutions.



Integration of polynomial/exponential functions

Ken Greatrix

On two occasions previously, I have shown examples of a technique for the integration of an exponential function with a polynomial component. In the first of these, I claimed that there was a proof by induction. In the second of my examples, I said that a proof would be forthcoming. It is now time to show that such a proof does exist.

I wish to consider the integration of a generalized function, $x^a e^{bx^c}$, which does not submit to standard methods of integration. (In the following, I will be using various indexing numbers (typically j, k). Their value is not necessarily carried from one expression to the next, and only indicates a generalized idea within each expression.)

Expand the exponential term and multiply the series by the polynomial term:

$$\begin{aligned} x^a\, e^{bx^c} &= x^a \left(1 + bx^c + \frac{b^2 x^{2c}}{2!} + \frac{b^3 x^{3c}}{3!} + \cdots + \frac{b^k x^{kc}}{k!} + \ldots\right) \\ &= x^a + bx^{a+c} + \frac{b^2 x^{a+2c}}{2!} + \frac{b^3 x^{a+3c}}{3!} + \cdots + \frac{b^k x^{a+kc}}{k!} + \ldots . \end{aligned}$$

Integrate:

$$\int x^a\, e^{bx^c} dx = \frac{x^{a+1}}{a+1} + \frac{bx^{a+c+1}}{a+c+1} + \frac{b^2 x^{a+2c+1}}{(a+2c+1)2!} + \cdots + \frac{b^k x^{a+kc+1}}{(a+kc+1)k!} + \ldots .$$

Now $x^{(a+1)}/(a+1)$ can be extracted as a common factor. This initial common factor is unique and does not follow the recursive pattern of the common factors in the remainder of this process. Also, since the first term of any exponential expansion is 1 it follows that this is the integral of the polynomial component. With the removal of this common factor, I obtain the first residual series. In the following, for ease of reading, I shall only be showing the process derived from each successive residual series and not the full expression:

$$1 + \frac{(a+1)bx^c}{a+c+1} + \frac{(a+1)b^2 x^{2c}}{(a+2c+1)2!} + \cdots + \frac{(a+1)b^k x^{kc}}{(a+kc+1)k!} + \ldots .$$

This can be separated into a sum of two series – firstly, an exponential:

$$1 + bx^c + \frac{b^2 x^{2c}}{2!} + \frac{b^3 x^{3c}}{3!} + \cdots + \frac{b^k x^{kc}}{k!} + \ldots \;=\; e^{bx^c},$$

and secondly, a remainder series:

$$-\frac{cbx^c}{a+c+1} - \frac{2cb^2x^{2c}}{(a+2c+1)2!} - \cdots - \frac{kcb^kx^{kc}}{(a+kc+1)k!} - \cdots.$$

It will be seen that

$$-\frac{cbx^c}{a+c+1}$$

can now be considered as a common factor in this series. After it is removed by division of the remainder series, and with another cancellation in the factorial term, the next residual series is

$$1 + \frac{(a+c+1)bx^c}{a+2c+1} + \frac{(a+c+1)b^2x^{2c}}{(a+3c+1)2!} + \cdots + \frac{(a+c+1)b^kx^{kc}}{(a+(k+1)c+1)k!} + \cdots.$$

For convenience in writing and for ease of reading I have expressed the final term in k, rather than the now correct value of $k-1$, as my above note explains.

Repeating this process of separation and common factor extraction, we have

$$1 + bx^c + \frac{b^2x^{2c}}{2!} + \cdots + \frac{b^kx^{kc}}{k!} + \ldots \quad = \quad e^{bx^c}$$

and

$$-\frac{cbx^c}{a+2c+1} - \frac{2cb^2x^{2c}}{(a+3c+1)2!} - \cdots - \frac{kcb^kx^{kc}}{(a+(k+1)c+1)k!} - \cdots,$$

where the common factor from the second of these series is now

$$-\frac{cbx^c}{a+2c+1}.$$

After this is divided into the remainder series, and another cancellation of the factorial term, the next residual series becomes

$$1 + \frac{(a+2c+1)bx^c}{a+3c+1} + \frac{(a+2c+1)b^2x^{2c}}{(a+4c+1)2!} + \cdots + \frac{(a+2c+1)b^kx^{kc}}{(a+(k+2)c+1)k!} + \cdots.$$

Having now seen that a pattern is emerging, I can attempt the inductive process. It would appear that at the jth stage in this process, the residual series has the form

$$1 + \frac{(a+jc+1)bx^c}{a+(j+1)c+1} + \frac{(a+jc+1)b^2x^{2c}}{(a+(j+2)c+1)2!} + \cdots + \frac{(a+jc+1)b^kx^{kc}}{(a+(j+k)c+1)k!} + \cdots,$$

which I shall use as a basis for induction. In the above, I have demonstrated that this formula and subsequent process are correct for $j = 0, 1, 2$ (although, to be pedantic, the process derived from the expression with $j = 0$ is not part of the recursive process).

By repeating the above process, this can be shown as a sum consisting of an exponential series

$$1 + bx^c + \frac{b^2x^{2c}}{2!} + \cdots + \frac{b^kx^{kc}}{k!} + \ldots \quad = \quad e^{bx^c}$$

and a remainder series

$$-\frac{cbx^c}{a + (j+1)c + 1} - \frac{2cb^2x^{2c}}{(a + (j+2)c + 1)2!} - \cdots - \frac{kcb^kx^{kc}}{(a + (j+k)c + 1)k!} - \cdots$$

with a common factor of

$$-\frac{cbx^c}{a + (j+1)c + 1}.$$

To complete the inductive step, after removing the common factors, the residual series becomes

$$1 + \frac{(a + (j+1)c + 1)bx^c}{a + (j+2)c + 1} + \cdots + \frac{(a + (j+1)c + 1)b^kx^{kc}}{(a + (j+k+1)c + 1)k!} + \ldots,$$

which is the $(j+1)$th stage, and thus the inductive step is complete.

The general term in the subtraction is

$$\frac{(a + jc + 1)b^kx^{kc}}{(a + (j+k)c + 1)k!} - \frac{b^kx^{kc}}{k!} \cdot \frac{a + (j+k)c + 1}{a + (j+k)c + 1} \quad = \quad -\frac{kcb^kx^{kc}}{(a + (j+k)c + 1)k!}$$

and the general term in the division to remove the common factor is obtained by multiplication of the inverse of the common factor term:

$$\begin{aligned}\frac{kcb^kx^{kc}}{(a + (j+k)c + 1)k!} \cdot \frac{a + (j+1)c + 1}{cbx^c} \quad &= \quad \frac{a + (j+1)c + 1}{a + (j+k)c + 1} \cdot \frac{kb^kx^{kc}}{bx^ck!} \\ &= \quad \frac{a + (j+1)c + 1}{a + (j + (k-1) + 1)c + 1} \cdot \frac{b^{k-1}x^{(k-1)c}}{(k-1)!}.\end{aligned}$$

A negative value divided by another negative value gives a positive result, so I have not shown the negative signs here.

At each stage in this process, a term is 'lost'. Rather than express the general term in $k, (k-1), (k-2)$, etc., I show the 'next' term and thus effectively $k \mapsto k+1$ at each stage in this process. So the general term then becomes

$$\frac{a+(j+1)c+1}{a+(j+k+1)c+1} \cdot \frac{kb^k x^{kc}}{k!}$$

as shown in the inductive step (and also in line with my above note).

Putting all the above components together,

$$\int x^a e^{bx^c} dx = \frac{x^{a+1}}{a+1}\left(e^{bx^c} - \frac{cbx^c}{a+c+1}\left(e^{bx^c} - \frac{cbx^c}{a+2c+1}\left(e^{bx^c} - \dots \right.\right.\right.$$

$$\left.\left.\left. - \frac{cbx^c}{a+kc+1}\left(e^{bx^c} - \dots\right)\right)\right)\right).$$

The common factor extracted from each successive remainder series is such that the following residual series is a sum of the same exponential series and the next remainder series. Conveniently, this is the exponential component of the original integration; but it's probably governed by the nature of an exponential rather than my personal choice (since $\int e^x dx = e^x$; with considerations for constants and function-of-a-function type of expressions).

Since a, b, c are fixed in value and since they and x remain finite, there is a limiting value of k (in the denominator) such that any further terms in the series do not contribute to the required level of accuracy. Not shown here are any constants multiplying the original integration and any possible constants of integration which would be evaluated with given initial conditions.

Looking back to my previous articles; for the standard normal CDF, $a = 0$, $b = -\frac{1}{2}$, $c = 2$ and for the χ^2 CDF (when $\nu = 1$), $a = -\frac{1}{2}$, $b = -\frac{1}{2}$, $c = 1$.

Since I have now proved the process, it's accuracy is also proven. However, as with all recursive processes the limitation of available decimal places in the calculating device means that inaccuracies creep into the calculation. With processes that converge quickly this inaccuracy is lessened. At the time of writing I have no knowledge of the accepted method of evaluation of integrals of this nature, and so I can't make a comparison of relative accuracies. This leaves me with three options: either it's better, it's worse or it actually *is* the accepted method and I've only found something that wasn't even lost.

Hindsight is a wonderful thing. I now realise that had I done this first, I wouldn't have needed to produce two example iterations. But then again, it's always useful to have worked examples to enhance an explanation.

Initially I shied away from this proof because I didn't think it would be straightforward enough and had doubts that my knowledge of maths was sufficient. Thus I demonstrate some of the many dangers of making assumptions. The first was assuming that a proof existed without seeking to find one. Secondly I doubted my capability – before investigating the possibility of a proof. Obviously it's a case of *'try before you shy!'*

Solution 213.2 – e

Define a sequence $\epsilon_2, \epsilon_4, \epsilon_6, \ldots$ by

$$\epsilon_k \;=\; k + \frac{1+\epsilon_{k+2}}{1+2\epsilon_{k+2}}, \qquad \epsilon_\infty \;=\; 0.$$

Show that $e = 2 + \epsilon_2/(1+\epsilon_2)$.

Steve Moon

First we perform a few iterations on the expansion $2 + \epsilon_2/(1+\epsilon_2)$:

$$t_1 \;=\; 2 + \frac{\epsilon_2}{1+\epsilon_2} \;=\; \frac{2+3\epsilon_2}{1+\epsilon_2}, \qquad t_2 \;=\; \frac{11+19\epsilon_4}{4+7\epsilon_4},$$

$$t_3 \;=\; \frac{106+193\epsilon_6}{39+71\epsilon_6}, \qquad t_4 \;=\; \frac{1457+2721\epsilon_8}{536+1001\epsilon_8}$$

and so on. We are given that $\epsilon_0 = \infty$; so we deduce $\lim_{k\to\infty} \epsilon_k = 0$ and hence the sequence $t_1, t_2, \ldots$ converges as $k \to \infty$. If we now substitute $\epsilon_{k+2} = 0$ in each t_k we obtain a sequence

$$T \;=\; \left(\frac{2}{1}, \;\frac{11}{4}, \;\frac{106}{39}, \;\frac{1457}{536}, \;\ldots\right)$$

and we can say this converges to some number as $k \to \infty$.

Now consider the continued fraction expansion of e, which is

$$e \;=\; [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots] \;=\; 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}. \quad (1)$$

By evaluating the convergents we obtain the sequence

$$\frac{2}{1},\ \frac{3}{1},\ \frac{8}{3},\ \frac{11}{4},\ \frac{19}{7},\ \frac{87}{32},\ \frac{106}{39},\ \frac{193}{71},\ \frac{1264}{465},\ \frac{1457}{536},\ \frac{2721}{1001},\ \ldots,$$

which can readily be checked. If the nth convergent is p_n/q_n and a_n is the nth term in the continued fraction expansion, then

$$\begin{array}{rclrclrclrcl} p_1 &=& a_1, & q_1 &=& 1, & p_2 &=& a_1a_2+1, & q_2 &=& a_2, \\ p_n &=& a_np_{n-1}+p_{n-2}, & & & & q_n &=& a_nq_{n-1}+q_{n-2}, & & & n \geq 3. \end{array} \qquad (2)$$

It is a standard result that the alternate 'odd' convergents 2/1, 8/3, 19/7, ... form a subsequence which converges to e from below and the 'even' terms converge to e from above.

Now the terms in T comprise two alternating subsequences, (a) 2/1, 106/39, ..., where we see the first and fourth members of the subsequence that converges to e from below, and (b) 11/4, 1457/536, ..., where we see the second and fifth members of the subsequence that converges to e from above. Hence T comprises a subsequence that converges to e from below, and a subsequence that converges to e from above. We deduce that $2+\epsilon_1/(1+\epsilon_2) = e$, as required, and the convergence is rapid compared with (1) as we calculate every 3rd term only.

Also notice that in t_1, t_2, t_3 and t_4, the coefficients of ϵ_k form the fractions 3/1, 19/7, 193/71 and 2721/1001, which are also convergents of e. If we rewrite the problem defining $\epsilon_\infty = \infty$, then, for example, we can rewrite t_4 as

$$t_4 \;=\; \frac{2721\epsilon_8 + 1457}{1001\epsilon_8 + 536} \;=\; \frac{2721 + 1457/\epsilon_8}{1001 + 536/\epsilon_8}$$

and if we set '$\epsilon_8 = \infty$', $t_4 = 2701/1001$. Hence we can use converging fractions formed by the coefficients of ϵ_k to obtain e, in the limit, as before, by following the analysis through. These fractions again form two subsequences converging to the same limit, e.

Tony Forbes

'Where does (1) come from?', I have often and repeatedly asked myself without—until the above article arrived—ever making any more than a trivial amount of effort to find out. At last I am enlightened. After a brief search on the Web I discovered a clever proof due to Henry Cohn.

Suppose we are given the continued fraction in (1). Recalling the definitions of a_n, p_n and q_n in (2), we have

$$\begin{aligned} p_{3n} &= 2np_{3n-1} + p_{3n-2}, & q_{3n} &= 2np_{3n-1} + p_{3n-2}, \\ p_{3n+1} &= p_{3n} + p_{3n-1}, & q_{3n+1} &= p_{3n} + p_{3n-1}, \\ p_{3n+2} &= p_{3n+1} + p_{3n}, & q_{3n+2} &= p_{3n+1} + p_{3n}, \end{aligned} \tag{3}$$

$n = 1, 2, \ldots$. Now define three integrals:

$$A_n = \int_0^1 \frac{x^n(x-1)^n}{n!}\, e^x dx, \qquad B_n = \int_0^1 \frac{x^{n+1}(x-1)^n}{n!}\, e^x dx,$$

$$C_n = \int_0^1 \frac{x^n(x-1)^{n+1}}{n!}\, e^x dx.$$

Then

$$A_1 = e - 3 = q_2 e - p_2, \quad B_1 = 8 - 3e = p_3 - q_3 e, \quad C_1 = 11 - 4e = p_4 - q_4 e,$$

$$A_n = -B_{n-1} - C_{n-1}, \quad B_n = -2nA_n + C_{n-1}, \quad C_n = B_n - A_n, \quad n \geq 2.$$

The expressions for A_1, B_1 and C_1 follow by elementary calculus, and the equality for C_n is straightforward. To prove the equalities for A_n and B_n, differentiate $x^n(x-1)^{n+\alpha}e^x$ to get

$$nx^{n-1}(x-1)^{n+\alpha}e^x + (n+\alpha)x^n(x-1)^{n+\alpha-1}e^x + x^n(x-1)^{n+\alpha}e^x. \tag{4}$$

Putting $\alpha = 0$ in (4) and integrating from 0 to 1 gives

$$n!\big(C_{n-1} + B_{n-1} + A_n\big) = \big[x^n(x-1)^n e^x\big]_0^1 = 0,$$

which implies $A_n = -B_{n-1} - C_{n-1}$. Similarly but with $\alpha = 1$ we obtain

$$n!\big(nA_n - C_{n-1} + (n+1)A_n + B_n - A_n\big) = \big[x^n(x-1)^{n+1}e^x\big]_0^1 = 0,$$

which simplifies to $B_n = -2nA_n + C_{n-1}$.

Therefore, by induction and making use of (3), we see that A_n, B_n and C_n can be evaluated in terms of p_k and q_k:

$$A_n = q_{3n-1}e - p_{3n-1}, \quad B_n = p_{3n} - q_{3n}e, \quad C_n = p_{3n+1} - q_{3n+1}e, \quad n \geq 1.$$

But A_n, B_n and C_n tend to 0 as $n \to \infty$. So $q_n e - p_n$ tends to 0 and hence p_n/q_n tends to e as $n \to \infty$ thus confirming that the continued fraction (1) really is e.

Problem 263.1 – 100 people and 100 boxes

One hundred persons numbered 1–100 play a game involving 100 boxes also numbered 1–100. Box i contains a number $\phi(i)$, where ϕ is some random permutation of $(1, 2, \ldots, 100)$. The object of the game is for each person to guess which box contains his/her number. To help achieve this, he/she is allowed to select 50 boxes and view their contents—so a pure guess is required only when the number is not thereby revealed. Although they may discuss a strategy beforehand, players must not communicate with each other whilst the game is in progress.

If every guess is correct, it is good; otherwise not. Show that the probability of a good outcome can be as high as 0.33. A surprising result considering that the probability of an individual guessing correctly is only 0.51.

For example, they might agree that player p will inspect boxes with opposite parity to p and guess box p if his/her number is not revealed. But the chance of success with this strategy is almost zero. Hint: See Problem 263.5.

Problem 263.2 – Sequences

Show that the number of binary $\{0, 1\}$ sequences of length n that do not contain two consecutive 1s is a Fibonacci number. For example, with $n = 4$ we have

$$\{0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010\},$$

eight sequences, and indeed 8 is the 4th or 5th or 6th Fibonacci number, depending on where you are supposed to start counting them.

Is there an equally familiar characterization of decimal sequences that avoid consecutive nines? The numbers for $n = 1, 2, \ldots, 6$ are (TF thinks) 10, 99, 981, 9720, 96309, 954261.

Problem 263.3 – Binomial coefficient gcd

For which positive integers n is it true that

$$\gcd\left(\binom{n}{a}, \binom{n}{b}, \binom{n}{c}\right) \;>\; 1 \qquad \text{for all } a, b, c \;\in \{1, \ldots, n-1\}?$$

Problem 263.4 – Arctan integral

Show that

$$\int_0^1 \arctan(x^2 - x + 1)\, dx \;=\; \log 2.$$

Solution 188.4 – Sixteen tarts

> There are 16 indistinguishable jam tarts. Some jam has been removed from one and put back into another so that fourteen weigh the same, one weighs a bit more and one a bit less. Devise a scheme to find the light and heavy tarts in five weighings.
>
> By a *weighing* we mean the process of selecting two sets of tarts, A and B, and determining whether A is lighter than B, A weighs the same as B, or A is heavier than B.

Tony Forbes

As I suggested in M500 **188**, the problem is interesting because the number of light–heavy possibilities, $16 \times 15 = 240$, is only slightly less than the number of observations provided by five weighings, $3^5 = 243$. Dick Boardman presented a similar problem involving nine tarts and four weighings [M500 **182**], and in my solution it was possible to specify a fixed set of four weighing instructions [M500 **184**]. For 16 tarts, however, all my attempts to provide a single set of five instructions have failed. The 16-tarts problem is solvable—but not quite as elegantly as I had hoped.

Label the tarts 0, 1, ..., 9, A, B, C, D, E, F. For a slightly more difficult problem, we assume only that *at most* one pair of tarts has been interfered with. The first weighing will be

W1: weigh {0, 1, 2, 3} against {4, 5, 6, 7}.

If the result of W1 is balanced, do

W2: weigh {0, 1, 4, 5, 8} against {6, 7, 9, A, B},
W3: weigh {2, 5, 6, A} against {3, 7, B, D},
W4: weigh {4, B, C, E} against {1, 6, 7, D}.

Otherwise do

W2: weigh {0, 1, 2, 4, 5, 6, 8} against {3, 7, 9, A, B, C, D},
W3: weigh {5, B, C, D} against {7, 8, E, F},
W4: weigh {0, 4, 9, B, E} against {1, 6, A, C, F}.

From the data provided by W1–W4 we can reduce the light–heavy possibilities to at most three. For convenience, they have been calculated for every combination and the details are presented in the table on the next page. In the first column we indicate the results of W1–W4 by letters L, H, B, according to the relative weight of the tarts on the *left*; L light, H heavy, B balanced. The second column gives the corresponding two or three possible light–heavy combinations.

A carefully chosen fifth weighing, W5, will correctly identify the light and heavy tarts. Suppose there are three possibilities,

$$\{a \text{ light } x \text{ heavy, } b \text{ light } y \text{ heavy, } c \text{ light } z \text{ heavy}\}.$$

If a, b and c are distinct, weigh a against b; otherwise if x, y and z are distinct, weigh x against y. Without loss of generality the only other possible triple is

$$\{a \text{ light } x \text{ heavy, } a \text{ light } y \text{ heavy, } b \text{ light } x \text{ heavy}\},$$

in which case weigh a and x against two unaltered tarts. If there are only two l–h entries in the table, the final identification is even easier. Observe that BBBB includes the possibility that all of the tarts weigh the same.

	l h	l h	l h
BLLL	4 7	5 7	8 D
BLLB	0 3	C B	E B
BLLH	1 3	8 B	F B
BLBL	5 6	C 9	E 9
BLBB	8 9	8 F	F 9
BLBH	8 C	8 E	D B
BLHL	4 6	C A	E A
BLHB	0 2	8 A	F A
BLHH	1 2	D 9	D A
BBLL	C D	E D	F D
BBLB	2 3	6 7	A 9
BBLH	5 4	9 B	A B
BBBL	0 1	C F	E F
BBBB	C E	E C	- -
BBBH	1 0	F C	F E
BBHL	4 5	B 9	B A
BBHB	3 2	7 6	9 A
BBHH	D C	D E	D F
BHLL	2 1	9 D	A D
BHLB	2 0	A 8	A F
BHLH	6 4	A C	A E
BHBL	B D	C 8	E 8
BHBB	9 8	9 F	F 8
BHBH	6 5	9 C	9 E
BHHL	3 1	B 8	B F
BHHB	3 0	B C	B E
BHHH	7 4	7 5	D 8

	l h	l h	l h
LLLL	0 7	0 F	2 F
LLLB	0 E	1 F	2 7
LLLH	1 7	1 E	2 E
LLBL	0 A	2 A	E 7
LLBB	0 9	1 A	8 7
LLBH	1 9	2 9	F 7
LLHL	0 C	0 D	2 C
LLHB	0 B	1 C	2 D
LLHH	1 B	1 D	2 B
LBLL	0 8	9 7	B 7
LBLB	2 8	3 7	D 7
LBLH	1 8	A 7	C 7
LBBL	0 6	2 6	3 A
LBBB	0 4	1 6	
LBBH	1 4	2 4	3 9
LBHL	0 5	3 C	8 6
LBHB	2 5	3 D	8 5
LBHH	1 5	3 B	8 4
LHLL	3 F	B 6	D 6
LHLB	3 8	B 4	C 6
LHLH	3 E	C 4	D 4
LHBL	3 6	9 6	B 5
LHBB	9 4	A 6	D 5
LHBH	3 4	A 4	C 5
LHHL	9 5	E 5	E 6
LHHB	3 5	E 4	F 6
LHHH	A 5	F 4	F 5

	l h	l h	l h
HLLL	4 F	5 A	5 F
HLLB	4 E	5 3	6 F
HLLH	5 9	5 E	6 E
HLBL	4 3	4 A	5 C
HLBB	4 9	5 D	6 A
HLBH	5 B	6 3	6 9
HLHL	4 C	4 D	E 3
HLHB	4 B	6 C	8 3
HLHH	6 B	6 D	F 3
HBLL	4 8	5 1	B 3
HBLB	5 2	5 8	D 3
HBLH	5 0	6 8	C 3
HBBL	4 1	4 2	9 3
HBBB	4 0	6 1	
HBBH	6 0	6 2	A 3
HBHL	7 A	7 C	8 1
HBHB	7 3	7 D	8 2
HBHH	7 9	7 B	8 0
HHLL	B 1	B 2	D 1
HHLB	B 0	C 1	D 2
HHLH	C 0	C 2	D 0
HHBL	7 F	9 1	9 2
HHBB	7 8	9 0	A 1
HHBH	7 E	A 0	A 2
HHHL	7 1	E 1	E 2
HHHB	7 2	E 0	F 1
HHHH	7 0	F 0	F 2

To complete the solution we really ought to at least provide some indication of how one might go about verifying that the table is correct. The obvious direct way would be to work through every entry using human ingenuity to deduce the l–h combinations from the weighing results. Let us see how you could do this.

Consider LHLH, for instance. Remembering that the second set for W2–W4 applies, the results of the weighings are

$$\begin{array}{lllcl}
\text{W1:} & \{0, 1, 2, 3\} & < & \{4, 5, 6, 7\}, \\
\text{W2:} & \{0, 1, 2, 4, 5, 6, 8\} & > & \{3, 7, 9, \text{A}, \text{B}, \text{C}, \text{D}\}, \\
\text{W3:} & \{5, \text{B}, \text{C}, \text{D}\} & < & \{7, 8, \text{E}, \text{F}\}, \\
\text{W4:} & \{0, 4, 9, \text{B}, \text{E}\} & > & \{1, 6, \text{A}, \text{C}, \text{F}\}.
\end{array}$$

From W1, tarts 0, 1, 2, 3 cannot be heavy and 4, 5, 6, 7 cannot be light. So, using this information together with similar deductions from W2, W3, W4, we see that

(i) only 3, A, C, D can be light,

(ii) only 4, 8, E can be heavy,

(iii) 0, 1, 2, 5, 6, 7, 9, B, F must have the correct weight.

By cancelling known good tarts we reduce the results to

$$\begin{array}{lllcl}
\text{W1:} & \{3\} & < & \{4\}, \\
\text{W2:} & \{4, 5, 6, 8\} & > & \{3, \text{A}, \text{C}, \text{D}\}, \\
\text{W3:} & \{\text{C}, \text{D}\} & < & \{8, \text{E}\}, \\
\text{W4:} & \{4, \text{E}\} & > & \{\text{A}, \text{C}\}.
\end{array}$$

It is now clear from W1 that 4 is heavy or 3 is light (or both). Assume 4 is heavy. But 4, 3 and A are not weighed in W3 and therefore 3 and A cannot be light. On the other hand, assume 3 is light, Now observe that W3 omits 3 and 4, and W4 omits 3 and 8, together implying that 4 and 8 cannot be heavy. So by (i) and (ii), LHLH $\to$ {3 light E heavy, C light 4 heavy, D light 4 heavy}.

All you have to do is repeat a similar tedious analysis for the other 80 entries. Fortunately there is a better alternative. Simply record the results of the four weighings for each of the 240 light–heavy combinations to create a function that never maps more than three to one. For example, you should find that *only* {3 light E heavy, C light 4 heavy, D light 4 heavy} $\to$ LHLH. This is in fact exactly how the table was generated—and since it was done by a computer there is zero probability of human error.

Problem 263.5 – Cycles

Of the $n!$ permutations of $\{1, 2, \ldots, n\}$, how many consist entirely of cycles of length at most m? For example, when $n = 4$ and $m = 2$ there are 10 permutations with the stated property, namely $()$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,4)$, $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$.

Verbal arithmetic

A cryptarithmic puzzle with the added feature that it consists entirely of true arithmetical statements. This one is due to Wei-Hwa Huang, who refers to these things as *verbal arithmetic*.

$$\begin{aligned} \text{ONE} + \text{ONE} &= \text{TWO} \\ \text{TWELVE} + \text{TEN} &= \text{TWENTY} + \text{TWO} \end{aligned}$$

Find the one-to-one function that maps letters to digits $\{0, 1, \ldots, 9\}$.

Can readers invent further examples? We would be particularly interested if you can create a similar puzzle where all ten digits are represented.

The Cotes formula has two square roots

Peter L. Griffiths

The first square root of $\cos u + i \sin u = \exp(iu)$ is $\cos(u/2) + i \sin(u/2) = \exp(iu/2)$. For the second and subsequent roots, the number of 360°s to be added is one less than the number of roots required which is the divisor of u. The second square root of $\cos u + i \sin u = \exp(iu)$ is $\cos((u+360^\circ)/2) + i \sin((u+360^\circ)/2) = \exp(iu/2)\exp(i180^\circ)$, whose square is $\exp(i(u + 360^\circ)) = \exp(iu)$.

But this second square root, $\exp(iu/2)\exp(i180^\circ)$ is not equal to the first square root $\exp(iu/2)$ because the second square root contains the factor of $\exp(i180^\circ) = -1$, shown as follows: $\cos 180^\circ + i \sin 180^\circ = \exp(i180^\circ) = -1$. This confirms that -1 reflects the difference between the two different roots. Sometimes it is the obvious which is difficult to explain. All the roots of the Cotes formulae can be demonstrated in the Cotes format.

Jack is looking at Anne, but Anne is looking at George. Jack is married, but George is not. Is a married person looking at an unmarried person?

(A) Yes; (B) No; (C) Cannot be determined.

Sent by Eddie Kent

04.00

Problem 263.6 – Simplification

Tony Forbes

Prove that this monstrous expression

$$\begin{aligned}&\frac{1}{1+r+3r^2-5r^3}\bigg(-1-r^2-2r^4+r^3\Big(4+\sqrt{(2-r)r}\Big)\\
&\quad+\sqrt{(r-2)r^3\Big(-1-2r^2+3r^4-4r\sqrt{(2-r)r}-4r^3\big(2+\sqrt{(2-r)r}\big)\Big)}\\
&\quad-r\bigg(\sqrt{(2-r)r}\\
&\qquad+\sqrt{r^2+2r^4-3r^6+4r^3\sqrt{(2-r)r}+4r^5\big(2+\sqrt{(2-r)r}\big)}\bigg)\bigg)\end{aligned}$$

is equal to $r-1$ (arguing by continuity at the roots of $1+r+3r^2-5r^3$).

Front cover Plot of kz, where $z^{3003}+z^k=|z|=1$; see page 6.