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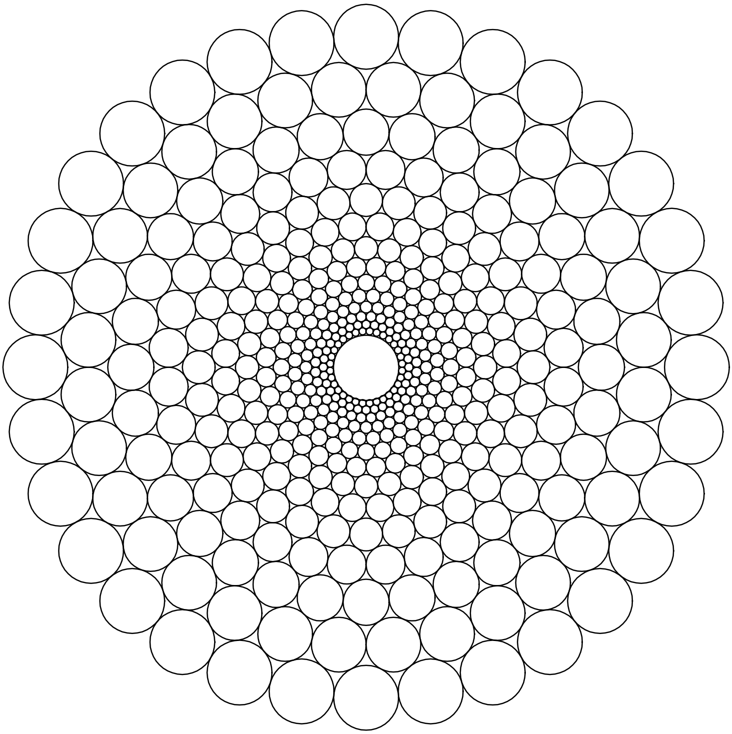
M500 266



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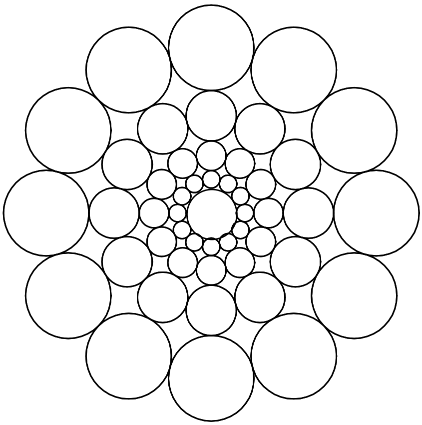
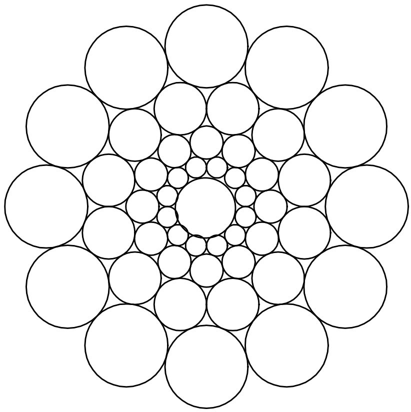
Designing table mats

Bryan Orman

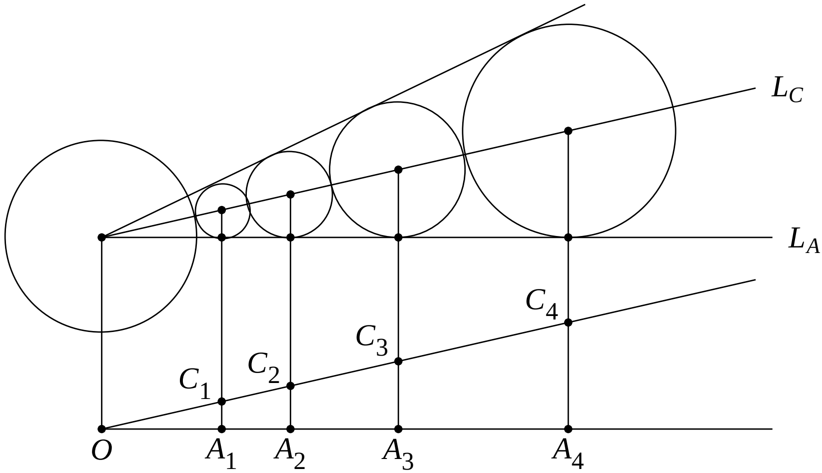
This article will examine the design of a table mat consisting of concentric rings of circular discs. In particular, the table mat will have four rings and the discs in each ring will just touch their neighbouring discs. An additional requirement concerns the central disc. Its radius will be as close as possible to the radius of the discs in the outermost ring.

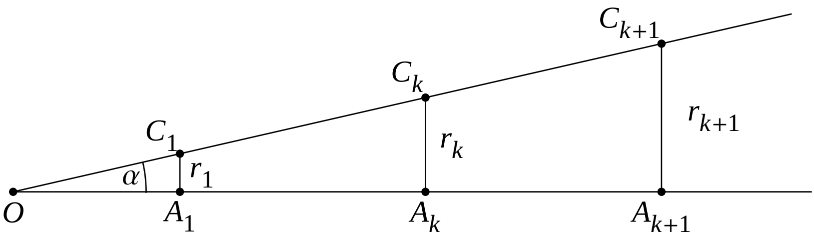
We consider two possible arrangements for such table mats and these are shown below. Note that these do not represent the solution to the problem in that the discs in the outer ring are much larger than the central disc. For each arrangement we will determine the number of discs in each of the four rings in order to meet the requirement concerning the radii of the innermost and outermost discs.





The diagrams on the next page show the specific arrangements for four rings although in the analysis we will consider n rings, for completeness. The centre C_k of the disc D_k, $k = 0, 1, \ldots, n$, lies on the radial line L_C, and the tangent line L_A touches the disc at A_k. The radius of the disc D_k is r_k, $k = 0, 1, \ldots, n$, and it subtends an angle 2α at the centre O of the arrangement.





From the triangle OA_1C_1, and also noting that $OC_1 = r_0 + r_1$, we have $\sin\alpha = \dfrac{r_1}{r_0 + r_1}$, so that $r_1 = \dfrac{r_0 \sin\alpha}{1 - \sin\alpha}$. From the basic quadrilateral $A_k A_{k+1} C_{k+1} C_k$ we have

$$\sin\alpha \;=\; \frac{r_{k+1} - r_k}{r_{k+1} + r_k};$$

so that

$$r_{k+1} \;=\; \frac{1 + \sin\alpha}{1 - \sin\alpha}\, r_k, \qquad k = 1,\ 2,\ \ldots,\ n.$$

Finally, $r_{k+1} = F_{k+1}(\alpha) r_0$, with

$$F_{k+1}(\alpha) \;=\; \sin\alpha\, \frac{(1 + \sin\alpha)^k}{(1 - \sin\alpha)^{k+1}}.$$

For our table mat we require $r_4 = r_0$; that is $F_4(\alpha) = 1$. If we write $t = (1 + \sin\alpha)/(1 - \sin\alpha)$, then this requirement reduces to $t^3(t-1)/2 = 1$, or $t^4 - t^3 = 2$. The Newton–Raphson method gives the solution $t = 1.5437$. As $\sin\alpha = (t-1)/(t+1) = 0.2137$, we have $\alpha = 0.2154 = 12.34°$.

Now, for an actual arrangement, the angle α_N corresponding to N discs is $180°/N$ and so an angle of $12.00°$ corresponds to 15 discs. How close is the radius of an outer disc to r_0?

$$r_4 \;=\; F_4(\alpha_{15})\, r_0 \;=\; 0.9309\, r_0.$$

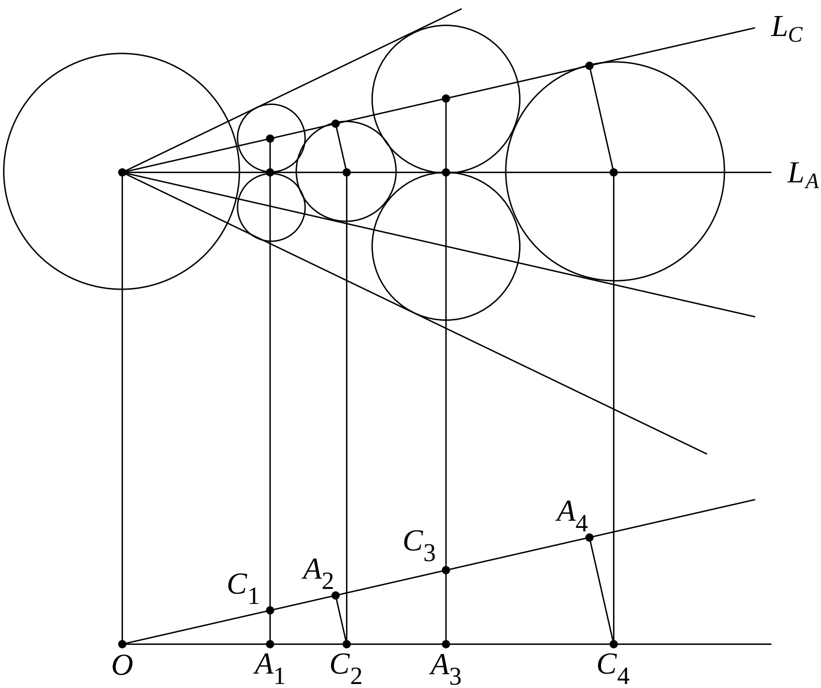
This discrepancy will not be too evident in an actual table mat.

Will $N = 14$ be any better?

$$r_4 \;=\; F_4(\alpha_{14})\, r_0 \;=\; 1.1127\, r_0.$$

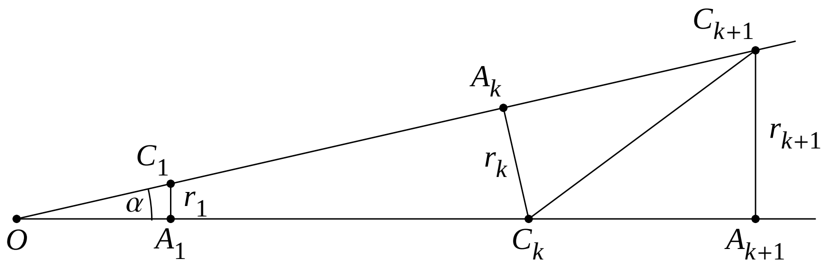
So there is little to choose between these two cases; one is 7 per cent under, the other is 11 per cent over. Note that all intermediate calculations have been performed to six decimal place accuracy and the recorded values given to four decimal places (and angles in degrees to four figures).

We now examine the second arrangement of the discs.



Here the centres of the discs alternate between the lines L_A and L_C, as do the points of contact of the discs with these lines.

There now appear to be two basic quadrilaterals, corresponding to k even and k odd, but they produce the same relationship between r_{k+1} and r_k. Consider the cyclic quadrilateral $C_kA_{k+1}C_{k+1}A_k$ with k even.



Again we have the triangle OA_1C_1 giving

$$r_1 = \frac{1+\sin\alpha}{1-\sin\alpha}\, r_0.$$

Now $OC_k = r_k \operatorname{cosec}\alpha$ and $OA_{k+1} = r_{k+1}\cot\alpha$, and since $C_kC_{k+1} = r_k + r_{k+1}$,

$$C_kA_{k+1} = \sqrt{(r_k + r_{k+1})^2 - r_{k+1}^2} = \sqrt{r_k^2 + 2r_kr_{k+1}}.$$

Using $OA_{k+1} = OC_k + C_kA_{k+1}$, we get

$$r_{k+1}\cot\alpha = r_k \operatorname{cosec}\alpha + \sqrt{r_k^2 + 2r_kr_{k+1}}.$$

This is a quadratic equation in r_{k+1}, namely

$$r_{k+1}^2 - 2r_kr_{k+1}(\sec\alpha + \tan^2\alpha) + r_k^2 = 0,$$

and the solution we require is the larger one since $r_{k+1} > r_k$, so that

$$r_{k+1} = \left(\sec\alpha + \tan^2\alpha + \sqrt{(\sec\alpha + \tan^2)^2 - 1}\right) r_k.$$

Finally, $r_{k+1} = G_{k+1}(\alpha)r_0$, with

$$G_{k+1}(\alpha) = \frac{\sin\alpha}{1-\sin\alpha}\left(\sec\alpha + \tan^2\alpha + \sqrt{(\sec\alpha + \tan^2)^2 - 1}\right)^k.$$

Again we require $r_4 = r_0$, that is $G_4(\alpha) = 1$.

There is little point in solving this by the Newton–Raphson method since α_N has to be equal to $180°/N$, as was noted with the first arrangement. It is far easier to examine the two cases $N = 14$ and $N = 13$ since the packing of the discs in this arrangement will be tighter and so the angle will be marginally larger, thereby reducing the number of discs in each ring.

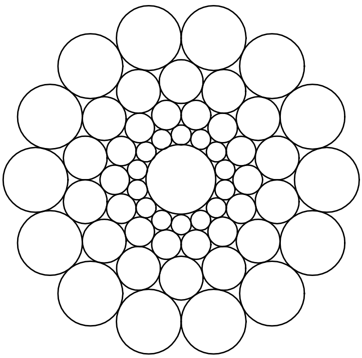
If $N = 14$, then

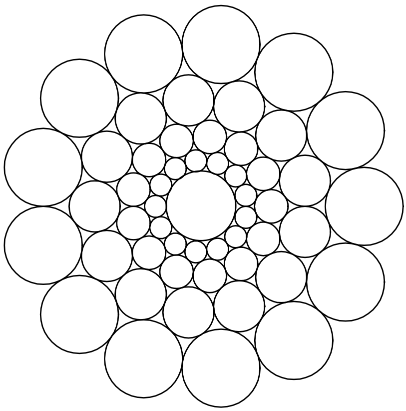
$$\alpha_{14} \;=\; \frac{180°}{14} \;=\; 12.86° \quad \text{and} \quad G_4(\alpha_{14}) \;=\; 0.9276.$$

If $N = 13$, then

$$\alpha_{13} \;=\; \frac{180°}{13} \;=\; 13.85° \quad \text{and} \quad G_4(\alpha_{13}) \;=\; 1.1182.$$

As with the first arrangement, there is little to choose between these two cases; one is 7 per cent under and the other 12 per cent over.





Problem 266.1 – Highly irregular graphs

A graph is *highly irregular* if it is simple (no loops, no multiple edges), connected (all in one piece), and for every vertex v, the neighbours of v have distinct degrees. An example is the 4-vertex path: •—•—•—•. It is easy to see that simple connected graphs of 1 and 2 vertices have the property but the triangle and the 3-vertex path don't.

Show that you cannot have a highly irregular graph of 5 or 7 vertices.

Solution 262.4 – Rational integral

Suppose that a and b are positive integers and that $r > 1$ is a rational number. Show that

$$\int_0^1 (r^b(1-x^{1/a}) + x^{1/a})^{1/b}\, dx$$

is rational.

Tommy Moorhouse

We will use the binomial expansion, repeated here for convenience:

$$(1+z)^\beta \;=\; 1+\beta z+\frac{\beta(\beta-1)}{2!}z^2+\cdots+\frac{\beta(\beta-1)\cdots(\beta-k)}{(k+1)!}z^{k+1}+\cdots.$$

We rewrite the integral as

$$I(a,b) \;=\; r\int_0^1 \left(1-\left(1-\frac{1}{r^b}\right)x^{1/a}\right)^{1/b} dx$$

and substitute $y=(1-1/r^b)^a x$ (which we shorten to $y = Lx$) to get

$$I(a,b) \;=\; \frac{r}{L}\int_0^L (1-y^{1/a})^{1/b} dy.$$

Integration term by term gives

$$L-\frac{L^{1+1/a}}{b(1+1/a)}+\cdots+(-1)^{k+1}\frac{(1-b)(1-2b)\cdots(1-kb)L^{1+(k+1)/a}}{(k+1)!\,b^{k+1}\,(1+(k+1)/a)}+\cdots.$$

The infinite sum is, after some cancellation

$$rF(a,-1/b;a+1;L^{1/a}),$$

that is, r times the hypergeometric function of its arguments. This can be transformed (see Eqn 5.3.9 of [AS]), again after some cancellation of terms in r, to

$$\begin{aligned}&\frac{\Gamma(a+1)\Gamma(1+1/b)}{\Gamma(1)\Gamma(a+1+1/b)}\,\frac{r}{L}\\ &\quad+\frac{\Gamma(a+1)\Gamma(-1-1/b)}{\Gamma(a)\Gamma(-1/b)}\,\frac{1}{r^b-1}\,F\left(1,1-a;2+\frac{1}{b};\frac{-1}{r^b-1}\right),\end{aligned}$$

where we have used $F(\alpha, 0; \beta; z) = 1$. The hypergeometric function with a negative integer as its first or second argument reduces to a polynomial in its last argument. The details can be found in [AS, Eqn 15.4.1] and [WW, Chapter XIV]. The only potentially awkward terms are those like

$$\frac{\Gamma(-1-1/b)}{\Gamma(-1/b)}$$

but these cancel down to rational numbers when the property $\Gamma(1+x) = x\Gamma(x)$ is used repeatedly. This means that the integral $I(a,b)$ is a rational function of r when a and b are positive integers.

References

[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* 4th ed., Cambridge 1927 (reprinted 1992).

[AS] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover 1972.

Solution 256.2 – Three rational numbers

> Numbers $p+2q$, pq^2 and $2pq+q^2$ are rational. Must p and q be rational?

Tony Forbes

I can't remember where it originally came from, but this problem has been bothering me on and off for some time. Then one day I became enlightened.

Write $r = p + 2q$, $s = 2pq + q^2$ and solve for p and q by substituting $p = r - 2q$ in the expression for s to give a quadratic in q,

$$s = 2(r-2q)q + q^2, \quad \text{or} \quad 3q^2 - 2rq + s = 0.$$

There are two solutions:

$$q = \frac{1}{3}\left(r + e\sqrt{r^2-3s}\right), \quad p = \frac{1}{3}\left(r - 2e\sqrt{r^2-3s}\right), \qquad e = \pm 1.$$

Compute the other number:

$$t = pq^2 = \frac{rs}{3} - \frac{2\,r^3}{27} - \frac{2e}{27}(r^2-3s)\sqrt{r^2-3s}.$$

Since r, s and t are rational $\sqrt{r^2-3s}$ must be rational. Therefore p and q are rational.

A functional equation

Tommy Moorhouse

It is well known that certain functions satisfy equations that relate their values at different points. A famous example, which gives insight into the behaviour of the function, is the functional equation for the zeta function. Here we will consider a simple family of functional equations and explore whether such equations can be solved in terms of elementary functions.

Consider the equation

$$u(2x) \;=\; f(x)u(x),$$

where we require $u(x)$ and $f(x)$ to be continuous, and we take $u(0) = A$. (Functions u that are not defined at 0 can also be studied, but we will not pursue this here.) We are interested to know whether we can find solutions for any continuous function f. Clearly in the present case (substituting $x = 0$ into the equation) we have $f(0) = 1$. Iterating, we find

$$u(x) \;=\; Af(x/2)f(x/4)\cdots f(x/2^n)\cdots.$$

Given any continuous function f with $f(0) = 1$ we can formally construct this solution. We denote u in the case $A = 1$ by $\Phi[f]$, introducing the idea that Φ is an operator acting on ordinary functions to produce (at least in some cases) other ordinary functions.

The solution, a function defined by an infinite product, may have interesting properties. You could prove that $\Phi[fg] = \Phi[f]\Phi[g]$ where f and g are well behaved in a sense you may wish to explore. It would be interesting to know whether a closed form exists for $\Phi[f]$ for a given f. As an example take $f(x) = e^x$. Then we can construct $\Phi[\exp]$ to find u. You may be able to deduce the solution by inspection in any case. For now we note (and you may want to prove this) that $\Phi[\exp] = \exp$.

A solution looking for a problem Sometimes quite abstract ideas can be put to use in concrete applications. Here we will consider the following problem using an idea based on the above.

Problem Prove that

$$\sum_{k=1}^{\infty} \frac{k}{2^{k+1}} \;=\; 1.$$

Solution An eigenfunction of the operator Φ is a function f such that $\Phi[f] = f$. (We consider here only the case of eigenvalue equal to 1. Can you see why?) From above we have $\Phi[\exp] = \exp$. Now apply Φ again to get

$$\Phi[\Phi[\exp]] \;=\; \Phi[\exp] = \exp\,.$$

We can expand these expressions employing a slight notational compromise:

$$\begin{aligned}
\Phi[\exp](x) &= e^{x/2}e^{x/4}\cdots e^{x/2^n}\cdots\,, \\
e^x &= \Phi[\Phi[\exp]](x) \\
&= \Phi[e^{x/2}]\Phi[e^{x/4}]\cdots \\
&= (e^{x/4}e^{x/8}\cdots)(e^{x/8}e^{x/16}\cdots)(e^{x/16}\cdots)\cdots \\
&= e^{x/4}(e^{x/8})^2(e^{x/16})^3\cdots
\end{aligned}$$

and the result follows. This can clearly be extended to give more complex sums.

More functions Our second example is the equation

$$u(2x) \;=\; \cos(x)u(x).$$

The solution is proportional to $\Phi[\cos](x)$. The graph of $\Phi[\cos](x)$ is symmetric about the origin and looks rather like that of the Bessel function $J_0(x)$. The zeros of $\Phi[\cos](x)$ occur at $x = \pi$, $x = 2\pi$, $x = 2^k\pi$ and so on; and at $x = 3\pi$, $x = 3\cdot 2\pi$, $x = 3\cdot 2^k\pi$ and so on. These (I think) are simple zeros.

It is possible to show that

$$\prod_{k=1}^{M} \cos\left(\frac{x}{2^k}\right) \;=\; \frac{1}{2^{M-1}} \sum_{n=1}^{2^{M-1}} \cos\left(\frac{(2n-1)x}{2^M}\right),$$

and clearly $\Phi[\cos](x)$ is, in some sense, the limit of this sum as $M \to \infty$. It may also be possible to write $\Phi[\cos](x)$ as an infinite product, but this is left to the interested reader to investigate.

Conclusion The operator Φ acts on certain functions to produce other well-defined functions. The reader may feel inspired to explore further.

Problem 266.2 – Snooker without friction

Is it sensible to play snooker on a frictionless table? Assuming it is, devise a strategy for winning a frame in a finite amount of time.

Solution 263.4 – Arctan integral

Show that

$$\int_0^1 \arctan(x^2 - x + 1)\,dx \;=\; \log 2.$$

Richard H. Gould

First integrate by parts, visualising the integrand as $1 \times \arctan(x^2 - x + 1)$. This gives

$$\begin{aligned}
I &= \big[x \arctan(x^2 - x + 1)\big]_0^1 - \int_0^1 \frac{x(2x-1)}{1 + (x^2 - x + 1)^2}\,dx \\
&= \frac{\pi}{4} - \int_0^1 \frac{x(2x-1)}{x^4 - 2x^3 + 3x^2 - 2x + 2}\,dx && (1) \\
&= \frac{\pi}{4} - \int_0^1 \frac{x(2x-1)}{(x^2 - 2x + 2)(x^2 + 1)}\,dx \\
&= \frac{\pi}{4} - \int_0^1 \frac{x}{x^2 - 2x + 2}\,dx + \int_0^1 \frac{x}{x^2 + 1}\,dx && (2) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x - 2 + 2}{x^2 - 2x + 2}\,dx + \frac{1}{2}\int_0^1 \frac{2x}{x^2 + 1}\,dx && (3) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x - 2}{x^2 - 2x + 2}\,dx - \int_0^1 \frac{dx}{1 + (x-1)^2} + \frac{1}{2}\int_0^1 \frac{2x}{x^2 + 1}\,dx && (4) \\
&= \frac{\pi}{4} - \frac{1}{2}\big[\log(x^2 - 2x + 2)\big]_0^1 - \big[\arctan(x-1)\big]_0^1 + \frac{1}{2}\big[\log(x^2 + 1)\big]_0^1 \\
&= \frac{\pi}{4} + \frac{1}{2}\log 2 - \frac{\pi}{4} + \frac{1}{2}\log 2 \\
&= \log 2,
\end{aligned}$$

as required.

Note that in step (1) it was realized that the denominator of the integrand was always positive so could have no real roots. Hence the expression was the product of quadratic terms. Separating alternate terms then made the factorization obvious. Finding the partial fractions at step (2) required the solution of four simultaneous linear equations, which was easily done using Gaussian elimination, and steps (3) and (4) were just a bit of 'conditioning' to avoid making explicit substitutions.

A typical 1960s Further Maths A level question!

John Davidson

Let

$$\begin{aligned}
\arctan\left(x^2-x+1\right) &= y(x) && (1)\\
\Rightarrow\ x^2-x+1 &= \tan y\\
\Rightarrow\ (2x-1)\,\frac{dx}{dy} &= \sec^2 y \;=\; 1+\left(x^2-x+1\right)^2 \;=\; \left(x^2+1\right)\left(x^2-2x+2\right)\\
\Rightarrow\ \frac{dy}{dx} &= \frac{2x-1}{(x^2+1)\,(x^2-2x+2)} \;=\; \frac{1}{(x^2-2x+2)}-\frac{1}{(x^2+1)}\\
\Rightarrow\ y &= \int\left(\frac{1}{(x-1)^2+1}-\frac{1}{(x^2+1)}\right)dx. && (2)
\end{aligned}$$

From (2), and recalling that $\displaystyle\int\frac{du}{(u-b)^2+a^2} = \frac{1}{a}\arctan\left(\frac{u-b}{a}\right)$,

$$y = \arctan(x-1)-\arctan x + C, \qquad (3)$$

where C is an integration constant. Further, from equation (1),

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\arctan\left(x^2-x+1\right) = \frac{\pi}{2}.$$

It follows from equation (3) that $C=\pi/2$; so in (3)

$$\int_0^1 y\,dx = \int_0^1\left(\arctan(x-1)-\arctan x+\frac{\pi}{2}\right)dx. \qquad (4)$$

Consider $\int_0^1 \arctan(x-1)dx$. Under the change of variable $u=x-1$ and recalling the standard integral $\int\arctan u\,du = u\arctan u-\frac{1}{2}\log\left(1+u^2\right)$, it follows that

$$\int_0^1\arctan(x-1)dx = \int_{-1}^0\arctan u\,du = -\frac{\pi}{4}+\frac{1}{2}\log 2. \qquad (5)$$

Also,

$$\int_0^1\arctan x\,dx = \left[x\arctan x-\frac{1}{2}\log\left(1+x^2\right)\right]_0^1 = \frac{\pi}{4}-\frac{1}{2}\log 2. \qquad (6)$$

Substituting equations (5) and (6) in equation (4):

$$\int_0^1 y\,dx = -\frac{\pi}{4}+\frac{1}{2}\log 2-\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)+\frac{\pi}{2} = \log 2. \qquad \text{Q.E.D.}$$

Solution 260.3 – Three dice

> In the game *Crown and Anchor* players bet on numbers 1 – 6 (usually represented by ace, king, queen, jack, crown and anchor) and three dice are thrown. If a player has bet x on n, and n appears i times amongst the three dice, he loses x if $i = 0$ and wins ix if $i > 0$. For an alternative game, we alter the rules. Now players bet on numbers 1 – 5 and, as before, three dice are thrown. But if a 6 shows, everyone loses. Otherwise, a successful player wins double the amount, $2ix$ instead of ix. Analyse the game and hopefully deduce that it is much less unfair.

Mike Lewis

This problem concerns the dice game Crown and Anchor which is notoriously biased in favour of the house. A similar game is played in the USA and is known as Chuck-a-Luck. The game is played with three dice which in Chuck-a-Luck are conventional spot dice. Chuck-a-Luck is analysed in *Scarne on Dice* [*].

The late John Scarne was an expert in gambling fraud and a consultant to various casinos, testified before various Federal Commissions, gave lectures to US Army recruits on Dice and Card Cheats, and was a consultant on the film *The Sting*.

The book *Scarne on Cards* is readily available in Britain but my copy of *Scarne on Dice* is not and my copy had to be ordered from the USA via Amazon. It appeared on my Further Reading list at the end of my Lecture Notes on Probability for an MSc course at the Defence Academy, Shrivenham. *Scarne on Dice* was distributed to US Army recruits to warn them of how they could be cheated.

I will use Scarne's method of analysis of the game; although it lacks some formality it is valid. It should be noted that he was not writing a text book on probability and the intended readership was not maths students but members of the public.

The Basic Game

The game is played with three spot dice and a layout marked with the numbers 1 to 6. Players place their bets on the layout and the payout is as follows.

- Player's number appears on 1 die: even money.
- Player's number appears on 2 dice: 2 to 1.
- Player's number appears on 3 dice: 3 to 1.

In all cases the original stake is returned together with the winnings according to the table above. Assume that £1 is placed on each of the numbers. Three results can occur.

- Three different numbers: bank pays out £3 and takes in £3, bank gain = £0.

- A pair, for example, two 3s and a 4: bank pays out £2 on the pair and £1 on the singleton but takes in a total of £4 from the 1, 2, 5 & 6, bank gain = £1.
- Three of a kind: bank pays out £3 and takes in £5, bank gain = £2.

The total number of combinations that can be thrown is 216. Assume that in 216 trials all 216 combinations occur once (equivalent to averaging over an infinite number of trials) and that £1 is placed on each of the numbers. The total staked over the 216 trials is £1296. The number of specific combinations in this series of trials will be:

- three different numbers: 120 combinations, bank gain = £0;
- a pair: 90 combinations, bank gain = £90;
- three of a kind: 6 combinations, bank gain = £12.

Bank profit = £102. Percentage profit = bank advantage = 102/1296 = 7.87%.

Tony Forbes Modified Game

The payouts are modified as follows.

- Player's number appears on 1 die: 2 to 1.
- Player's number appears on 2 dice: 4 to 1.
- Player's number appears on 3 dice: 6 to 1.
- 6 is thrown: all lose.

The number of combinations containing a 6 are as follows.

- On 1 die: 75.
- On 2 dice: 15.
- On 3 dice: 1.

Four results can occur.

- Three different numbers: bank pays out £6 and takes in £2, bank loss = £4.
- A pair, for example, two 3s and a 4: bank pays out £4 on the pair and £2 on the singleton but takes in a total of £3 from the 1, 2 & 5 bank loss = £3.
- Three of a kind: bank pays out £6 and takes in £4, bank loss = £2.
- The throw contains one or more 6s: bank gain = £5.

The number of specific winning combinations in this series of trials will now be as follows.

- Three different numbers: 60 combinations, bank loss = £240.
- A pair: 60 combinations, bank loss = £180.
- Three of a kind: 5 combinations, bank loss = £10.

Total bank loss = £430.

The bank gain due to a 6 being thrown is £5 and in the series of trials the gain would be $91 \times £5 = £455$. The total profit would be £35 giving the bank an advantage of 2.31%. This is comparable with European Roulette, where the advantage is 2.70%.

Grand Hazard

Grand Hazard is an extended version of Chuck-a-Luck played in American casinos. In addition to the bets and pay-offs described for Chuck-a-Luck the following bets can be made.

The following bets on the total thrown pay as follows but lose if a triplet is thrown.

Bet	Probability	Payout	Bank advantage
Total > 10	0.4861	£2	2.78%
Total < 11	0.4861	£2	2.78%
Total is odd	0.4861	£2	2.78%
Total is even	0.4861	£2	2.78%

The remaining bets are as follows.

Bet	Probability	Payout	Bank advantage
4 or 17	0.0139	£60	16.67%
5 or 16	0.0278	£30	16.67%
6 or 15	0.0463	£18	16.67%
7 or 14	0.0694	£12	16.67%
8 or 13	0.0972	£8	22.22%
9 or 12	0.1157	£6	30.56%
10 or 11	0.1250	£6	25.00%
Specific Triplet	0.0046	£180	16.67%
Any Triplet	0.0278	£30	16.67%

The overall advantage to the bank is not less than that for Chuck-a-Luck but the bank profit will depend on betting patterns. Behaviourist psychology suggests that many naïve gamblers will bet on the high pay out combinations. To quote Scarne, ‘Play it smart and remember that the higher the percentage against you, the smaller is your chance of coming out ahead of the game. The banker is bound to get your money in the long run, but if you stick to low percentage bets, he’ll have to work longer for it.’

Reference

[*] J. Scarne, *Scarne on Dice*, 8th revised edition, Wilshire Book Company, North Hollywood, California, 1974.

Solution 245.10 – Every other day

> Find a simple function F, say, that maps a date to either 0 or 1 such that $F(\text{today}) = 1 - F(\text{yesterday})$. This is not just an academic exercise. Such a function will be very useful in those situations where the label on the packet says, 'Take 1 tablet every 2 days'.

Tony Forbes

We assume you know today's date but you just cannot remember whether or not you took a pill yesterday. We may also assume you know the day of the week. However, unless I have overlooked something obvious, I cannot immediately see how that helps. Anyway, the function we seek has the form $F(d, m, y) =$ something mod 2, where d, m and y represent the day, month and year. After some experimentation I came up with this formula:

$$f(d, m) \;=\; d + \lfloor \cos(1.8\, m + 1.3) \rfloor \quad \bmod 2.$$

It assumes there are no leap years, and also that a year has an even number of days. So it will go wrong probably on 1st January and certainly on the day after 29th February. A simple fudge is sufficient to address these two problems properly.

Write $c(y) = \left\lfloor \dfrac{y+99}{100} \right\rfloor$, the century to which year y belongs, and define

$$\begin{aligned} F(d, m, y) \;=\; & d + \lfloor \cos(1.8\, m + 1.3) \rfloor \\ & + \lfloor \arctan(m-1) \rfloor \left(\left\lfloor \cos^2 \frac{\pi y}{4} \right\rfloor + \left\lfloor \cos^2 \frac{\pi y}{100} \right\rfloor + \left\lfloor \cos^2 \frac{\pi y}{400} \right\rfloor \right) \\ & + \frac{y(y-1)(y-3)(y-6)}{8} \\ & + \frac{c(y)(c(y)-1)(c(y)-3)(c(y)-6)}{8} \quad \bmod 2. \end{aligned}$$

Thus $F(1, 1, 1801) = 0$, $F(2, 1, 1801) = 1$ and the $(0, 1)$ pattern continues throughout the range of validity of the Gregorian calendar. The third term obviously takes care of dates from March to December in a leap year. The last two terms account for the cumulative effect of 365-day years. Modulo 2 these adjustments go in eight-year and eight-century cycles of $(0, 0, 1, 0, 1, 1, 0, 1)$.

Q: How can I avoid making mathematical errors?

A: Avoid doing mathematics.

Solution 264.3 – Determinant

An $n \times n$ matrix has a in each entry on the diagonal and b everywhere else. What is its determinant? For example, when $a = 6$ and $b = 5$ we get

$$\det \begin{bmatrix} 6 & 5 & 5 & 5 & 5 & 5 \\ 5 & 6 & 5 & 5 & 5 & 5 \\ 5 & 5 & 6 & 5 & 5 & 5 \\ 5 & 5 & 5 & 6 & 5 & 5 \\ 5 & 5 & 5 & 5 & 6 & 5 \\ 5 & 5 & 5 & 5 & 5 & 6 \end{bmatrix} = 31.$$

Dave Wild

The value of a determinant remains unchanged if a multiple of one row is added to another row, or a multiple of one column is added to another column. If we subtract the last row from each of the other rows then most of the elements in the matrix become zero. If we then add each of the other columns in turn to the last column, then the only non-zero elements left occur on the diagonal and in the last row. These transformations are shown below where $c = a - b$.

$$\det \begin{bmatrix} a & b & b & \dots & b & b & b \\ b & a & b & \dots & b & b & b \\ b & b & a & \dots & b & b & b \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a & b & b \\ b & b & b & \dots & b & a & b \\ b & b & b & \dots & b & b & a \end{bmatrix} = \det \begin{bmatrix} c & 0 & 0 & \dots & 0 & 0 & -c \\ 0 & c & 0 & \dots & 0 & 0 & -c \\ 0 & 0 & c & \dots & 0 & 0 & -c \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c & 0 & -c \\ 0 & 0 & 0 & \dots & 0 & c & -c \\ b & b & b & \dots & b & b & a \end{bmatrix}$$

$$= \det \begin{bmatrix} c & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & c & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & c & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & c & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & c & 0 \\ b & b & b & \dots & b & b & a + (n-1)b \end{bmatrix}.$$

So the determinant equals $(a - b)^n + nb(a - b)^{n-1}$. The example given in the question has $n = a = 6$ and $b = 5$; so the determinant is $1 + 6 \cdot 5 = 31$, as stated.

Solution 210.1 – Determinant

Compute

$$\left|\begin{matrix} 4 & a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 \\ a+b+c+d & a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 \\ a^2+b^2+c^2+d^2 & a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 \\ a^3+b^3+c^3+d^3 & a^4+b^4+c^4+d^4 & a^5+b^5+c^5+d^5 & a^6+b^6+c^6+d^6 \end{matrix}\right|.$$

Dave Wild

This is an alternative solution to the one given in M500 **251**. The given matrix can be written as AA^{T}, where

$$A = \left|\begin{matrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{matrix}\right|.$$

Then $\det A$ is the Vandermonde determinant and equals $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$. Therefore the determinant of the given matrix equals $\det(AA^{\mathrm{T}}) = \det(A)\det(A^{\mathrm{T}}) = \det(A)^2$.

The supplementary problem suggested by Tony Forbes [show that

$$\left|\begin{matrix} a^t+b^t+c^t & a^{t+1}+b^{t+1}+c^{t+1} & a^{t+2}+b^{t+2}+c^{t+2} \\ a^{t+1}+b^{t+1}+c^{t+1} & a^{t+2}+b^{t+2}+c^{t+2} & a^{t+3}+b^{t+3}+c^{t+3} \\ a^{t+2}+b^{t+2}+c^{t+2} & a^{t+3}+b^{t+3}+c^{t+3} & a^{t+4}+b^{t+4}+c^{t+4} \end{matrix}\right|$$

$$= (abc)^t(a-b)^2(a-c)^2(b-c)^2]$$

may be tackled in the same manner by setting

$$A = \left|\begin{matrix} a^{t/2} & b^{t/2} & c^{t/2} \\ a^{1+t/2} & b^{1+t/2} & c^{1+t/2} \\ a^{2+t/2} & b^{2+t/2} & c^{2+t/2} \end{matrix}\right|.$$

Problem 266.3 – Equilateral triangle

There is an equilateral triangle. Point P is at distance a from one vertex and b from another vertex. What is the largest possible distance P can be from the third vertex?

Thanks to Dick Boardman for suggesting this problem and for referring me (TF) to 'Nick's mathematical Puzzles' (http://www.qbyte.org/puzzles/), number 139, which is essentially the special case $a = 3$, $b = 4$.

Problem 266.4 – Determinants

Tony Forbes

Let a_k be the coefficient of x^k in the Taylor expansion of $(1+x)\sqrt{1+4x}$:

$$\begin{aligned}(x+1)\sqrt{1+4x} &= \sum_{k=0}^{\infty} a_k x^k \\ &= 1 + 3x + 2x^3 - 6x^4 + 18x^5 - 56x^6 + 180x^7 - 594x^8 + \dots.\end{aligned}$$

Show that if n is a multiple of 3 and $m = \lfloor n/2 \rfloor$, then

$$\det \begin{bmatrix} a_{3+2m-n} & a_{4+2m-n} & \dots & a_{m+1} \\ a_{4+2m-n} & a_{5+2m-n} & \dots & a_{m+2} \\ \dots & \dots & \dots & \dots \\ a_{m+1} & a_{m+2} & \dots & a_{n-1} \end{bmatrix} = 0.$$

Observe that the matrix is symmetric with $n-m-1$ rows and columns, the top-left entry is a_2 for odd n, a_3 for even n, and you get row $r+1$ from row r simply by adding 1 to the subscripts. For example, when $n=3$ the expression reduces to $\det[a_2] = a_2 = 0$ whereas for $n=6$ and $n=9$ we have

$$\det \begin{bmatrix} a_3 & a_4 \\ a_4 & a_5 \end{bmatrix} = \det \begin{bmatrix} 2 & -6 \\ -6 & 18 \end{bmatrix} = 0,$$

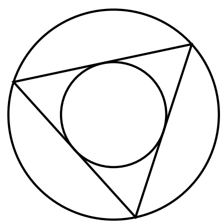
$$\det \begin{bmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} = \det \begin{bmatrix} 0 & 2 & -6 & 18 \\ 2 & -6 & 18 & -56 \\ -6 & 18 & -56 & 180 \\ 18 & -56 & 180 & -594 \end{bmatrix} = 0,$$

and by developing the series a little further you can verify that

$$\det \begin{bmatrix} 2 & -6 & 18 & -56 & 180 \\ -6 & 18 & -56 & 180 & -594 \\ 18 & -56 & 180 & -594 & 2002 \\ -56 & 180 & -594 & 2002 & -6864 \\ 180 & -594 & 2002 & -6864 & 23868 \end{bmatrix} = 0,$$

corresponding to $n = 12$.

This result is not unrelated to the fact that you can draw an equilateral triangle whose vertices lie on a circle of radius 2 and whose sides are tangent to a circle of radius 1. But the point of this exercise is to see if there is a direct proof that does not involve geometry.



"I'm even scoring in binary!"

Ken Greatrix

At the archery club recently it was our annual indoor competition, consisting of sixty shots, three at a time on a target with ten rings.

In golf it's known as 'the yips', an involuntary movement causing a bad shot. We have a similar thing in archery. One of my fellow competitors who is also a computer engineer had a bad case of this so that when it came to scoring he called out, "Ten, ten, one."

Absolutely disgusted with himself he continues: "I've worked with binary, octal and hexadecimal and now I'm even scoring in binary!"

Realizing that he had written '10101' on his score-card, I said, "And it makes the same score."

I then wondered if any other scores fitted this pattern, but I was only able to bring two more examples to mind. If we ignore the conventions of calling scores in descending order and of recording 'M' for a miss then ten, ten, zero and zero, zero, one also have total scores that look the same when read as binary numbers.

Are there any other scores that have this property with three shots on a ten-ring target? What about any number of shots on a target with any number of rings?

PS. If anyone is really interested – I won the bronze medal.

Life expectancy – the unprofessional calculation

Peter L. Griffiths

The whole life expectancy of 80 years should be compared with the lowest age for certainty of death say 125 years to give a fraction of 80/125 equalling 0.64, whose shortfall from 1 is 0.36, which is the fraction applied to the actual present age to be added to the 80 years to give the part life expectancy at the actual present age for a number of years higher than 80 but less than 125 ($80/0.64 = 125$).

Age		Part life expectancy
0	$\times\ 0.36 + 80$	80 (whole life)
20	$\times\ 0.36 + 80$	87
40	$\times\ 0.36 + 80$	94
55	$\times\ 0.36 + 80$	100
70	$\times\ 0.36 + 80$	105
80	$\times\ 0.36 + 80$	108
100	$\times\ 0.36 + 80$	116
120	$\times\ 0.36 + 80$	123

Solution 262.1 – Binomial ratio

Let r and s be positive integers and suppose m is an integer other than -2, -1, 0, 1. Show that $\dfrac{m^s-1}{m^r-1}$ and $\dfrac{s}{r}$ are either both integers or both non-integers. Or find a counter-example.

Ledger White

Let $x = m^r - 1$ then:

$$\frac{m^s-1}{m^r-1} = \frac{(x+1)(m^{s-r})-1}{x} = \frac{xm^{s-r}+m^{s-r}-1}{x}$$

$$= m^{s-r} + \frac{m^{s-r}-1}{x} = m^{s-r} + \frac{m^{s-r}-1}{m^r-1}.$$

The second term is the same format as the original, and therefore it may be similarly expanded too, and further expansion can be continued with $n = 2,\ 3,\ \ldots$

$$\ldots + m^{s-nr} + \frac{m^{s-nr}-1}{m^r-1} \ \ldots$$

$\ldots$ until $s - nr \le r$. Thus, the original expression can be written as a sum of a [possibly empty] series of integers m^{s-nr}, $n = 1,\ 2,\ 3,\ \ldots$ plus a final term. If, in the final term, $s - nr = r$ then the final term is 1 and both $(m^s-1)/(m^r-1)$ and s/r are integers. Otherwise [recalling that $m \notin \{-2,-1,0,1\}$] the final term is a non-zero fraction in $(-1,1)$ and they are both non-integers.

The reverse is true, begin with $(s-nr) \le r$ and build the series from right to left. Wherever you stop both divisions are integers or they are both non-integers. Thus

$$\frac{9^{28}-1}{9^7-1} = 9^{21}+9^{14}+9^7+1$$

and

$$\frac{9^{27}-1}{9^7-1} = 9^{20}+9^{13}+9^6+\frac{9^6-1}{9^7-1}.$$

Given the simplicity of this argument I wonder if the answer to the original problem really is (i) [obviously true]. If not, it is certainly very close!

TF writes. If, like me, you are having difficulty with the meaning of the word ‘simplicity’, you might like to have a go at the problem on the next page.

Problem 266.5 – Binomial ratio revisited

Suppose r and s are positive integers such that r does not divide s. Let $n = \lfloor s/r \rfloor$. Show that for integer m, $(m^{s-nr} - 1)/(m^r - 1)$ is not an integer except possibly when $m \in \{-2, -1, 0, 1\}$.

M500 Winter Weekend 2016

The **thirty-fifth M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 8$^{\text{th}}$ – Sunday 10$^{\text{th}}$ January 2016.

Cost: £205 to M500 members, £210 to non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. You can obtain a booking form either from the M500 web site,

http://www.m500.org.uk,

or by emailing the Winter Weekend Organizer at winter@m500.org.uk.

The Winter Weekend provides you with an opportunity to do some non-module-based, recreational maths with a friendly group of like-minded people. The relaxed and social approach delivers maths for fun. And as well as a complete programme of mathematical entertainments, on Saturday we will be running a pub quiz with Valuable Prizes.

M500 Mathematics Revision Weekend 2016

The M500 Revision Weekend 2016 will be held at

Yarnfield Park Training and Conference Centre,

Yarnfield

Staffordshire ST15 0NL

from Friday 13th to Sunday 15th May 2016.

We expect to offer tutorials for most undergraduate and postgraduate mathematics Open University modules, subject to the availability of tutors and sufficient applications. Application forms will be sent via email to all members who included an email address with their membership application or renewal form, and are included with this magazine mailing for those who did not.

Contact the Revision Weekend Organizer, **Judith Furner**, at email address weekend@m500.org.uk if you have any queries about this event.

04.00

Front cover One of Bryan's table mats. See page 1. Observe that the discs in the outer ring have very nearly the same radius as the central circle. In this example $G(14, \pi/32) \approx 0.994689$.