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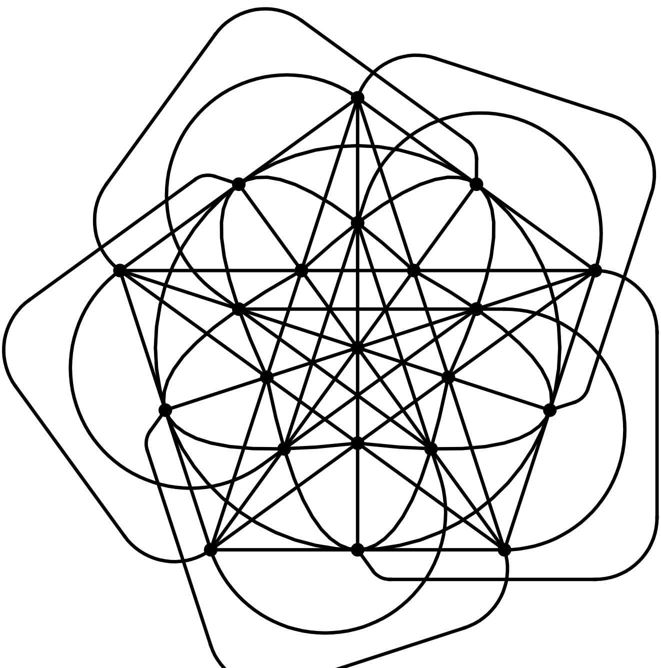
# M500 214



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# Causality and vitalism in mathematics

## Sebastian Hayes

Is there causality in mathematics?

Perhaps we should start by pondering whether causality exists at all. Hume thought not and Wittgenstein dismissed the 'causal connection' between events as a superstition (*Tractatus Logico-Philosophicus*). Certainly no one has ever claimed to see or touch the 'force of causality', and if the supposed 'causal connection' between certain events were self-evident we would not have the difficulties we so manifestly do have in distinguishing bona fide causal pairs from chance associations of events.

For all that, I have never lost any sleep over Hume's attack on causality. *'We know there is causality and there's an end to it.'* as Dr Johnson said about free will. Hume himself, revealingly, admitted that he 'dropped his philosophic scepticism when playing backgammon'.

Belief in causality is undoubtedly a psychological necessity, and thus a biological necessity as well: as a species, we need to believe that we can 'make a difference' and, looking at what we have done to the planet, we've certainly proved that! If modern philosophers have their doubts about the existence of causality, well, so much the worse for them.

Science in the West remained happily married to determinism for three centuries and Claude Bernard actually went so far as to *define* science as the application of causality to the material world. But then, in the course of the twentieth century, physics suddenly got infatuated with indeterminism. Why? The official answer is that this was forced on science by experimental discoveries in the atomic and subatomic domains where 'statistical determinism' rather than 'complete individual determinism' is the norm. Yes, but the phenomenon is far less comprehensive and radical than people think. The individual molecule in a gas is, if you like, allowed freedom of movement—but only because this is very unlikely to affect the overall result. It is like Saddam Hussein giving Iraqis the vote. Also, it is usually only the *order of appearance* of the events that is random, not the events themselves. A good example is the process of photographic development which is a chemical amplification of initial atomic events. It is possible, using very weak exposure, to arrange for the individual photons to arrive one after the other, and if this is done, the photographic image builds up in a way that is completely unpredictable. But the fact remains that all the micro-events have been completely specified in advance (by the object that is being photographed)[1].

In other cases, 'random mutation' for example, the consensus is that the events themselves are basically indeterminate; but this remains an untested and probably untestable hypothesis. Though I am slowly coming round to the possibility, the idea that there can be a completely 'uncaused' event has always struck me as being extremely weird: how then could it have come about?

One suspects that the sudden vogue for indeterminism in physics and elsewhere during the twenties and thirties (strongly resisted by Einstein) was part of the *Zeitgeist*: the senseless slaughter of the Great War and, later, the Wall Street Crash (which no one had predicted) seemed to many people to demonstrate that the world was not fully comprehensible by rational means after all. But the real culprit was logical positivism, a philosophy which has had a crippling effect on the way we think about science and life generally. Whereas common sense always prefers to assume that there is an agent for all changes in the external world — 'Every event has a cause' — logical positivism holds fast to the verification principle instead. Since causality cannot be verified directly, it has no right to exist, therefore it doesn't exist. Stripped of causality, physics becomes an exercise in applied mathematics, while mathematics itself is, according to the moderns, either symbolic logic (Russell) or 'a game played according to fixed rules with meaningless marks on paper' (Hilbert). This effectively puts paid not only to determinism but to objective reality itself, which has become the unwanted ghost in a wholly symbolic machine. Britain's most acclaimed theoretical physicist (Hawking) once admitted disingenuously that he was not really concerned about the underlying truth of a theory but only whether it was 'interesting and fruitful'.

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In the real world by my book events *cause* other events: they do not simply happen to precede them. Coercion is involved, not 'functional covariance'. But when we shift to the logical, sanitized universe we find that all we are left with is 'material implication' $P \Rightarrow Q$.

I don't expect I need to remind readers that the validity of $P \Rightarrow Q$ does not mean we can just invert the terms and conclude that $Q \Rightarrow P$. What is, however, accepted in both logic and mathematics is that the truth of $P \Rightarrow Q$ entails the truth of the contrapositive not $Q \Rightarrow$ not $P$.

'If $\gcd(a, p)) = 1$ and $p$ is prime, $a^{p-1} \equiv 1 \pmod{p}$ for all $a$.' — **True**.

'If $a^{p-1} \equiv 1 \pmod{p}$ for all $a$, then $p$ is prime.' — **False** (because of Carmichael numbers such as 561).

But: 'If it is not the case that $a^{p-1} \equiv 1 \pmod{p}$ for all $a$, then $p$ cannot be prime.' — **True**.

Logically speaking, the contrapositive is equivalent to the original statement because the truth tables are identical. However, if the original statement has a causal basis, this feature disappears when we form the contrapositive. Negating an event is not the same thing as negating an assumption, since something that does not occur can neither cause something else to occur nor positively prevent its occurrence.

> *'I shot my noisy next door neighbour in the head ten minutes ago, so he is now dead.'*

This statement is valid because the underlying causal connection is valid (a shot in the head causes death) whether or not it corresponds to the facts.

> *'If my next door neighbour is currently alive, I cannot have shot him in the head ten minutes ago.'*

is, I suppose, valid reasoning but sounds most peculiar—as if I were a psychopath suffering from recurrent bouts of amnesia. This shows what happens when we empty statements of causal content.

The point is that contrapositives are always a good deal weaker than affirmative statements. One of the reasons why Newtonian physics got off to such a flying start, was because it was formulated positively: *'Every particle attracts every other directly with respect to mass and inversely in proportion to the square of the distance between them.'* Practical people like engineers took to Newtonian mechanics because they could visualize what was going on, *'If rod A makes B go down, then B will make C go up, and C will make D move to the right ....'*

All this has grave consequences for modern mathematics since most modern proofs are indirect (75 percent it has been estimated) and proceed along the lines, *'But if A is not so, then Y, then Z, but Z is nonsense, therefore not-A cannot be true, therefore A.'* Stevenson and Brunel would not have been impressed. Modern mathematics is choc-a-bloc with entities whose only right to exist is that, if they didn't, someone would be contradicting himself somewhere[2]. Compare this with direct proofs, which actually show you how to turn up an example of the thing you are looking for.

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In logic $P \Rightarrow Q$ is *always* valid except when $P$ is true and $Q$ is false. Thus

> *'If all triangles are equilateral, then no square can be inscribed in a circle.'*

and

*'If 8 is a prime number, then G.M. $\leq$ A.M.'*

are both valid (since we have $\mathbf{F} \Rightarrow \mathbf{F}$ and $\mathbf{F} \Rightarrow \mathbf{T}$).

Both these sentences are not even untrue, they are just rubbish because there is no connection between the respective statements.

Examples like the above only go to how show foolish it is to completely ignore meaning when we are setting up a logical system. The rules governing, say, embroidery or bridge are neither here nor there, they are 'meaningless' and none the worse for it. But logic is not embroidery since it can, in principle, have considerable bearing on the decisions we actually make, such as, for example, whether a country is a potential threat to us, and in consequence whether we should go to war or not.

Logic teaches you how not to contradict yourself. But why not contradict yourself if you feel like it? One answer is that this frustrates the main purpose of speech which is to communicate with other people. But there is a second reason which is much more significant. We insist on non-contradiction in logic and mathematics because Nature actually is non-contradictory (at the macroscopic level anyway): it (Nature) obeys a very important principle which I have baptized the **Axiom of Exclusion**, *'An event cannot at one and the same time both occur and not occur at the same spot.'* Without this assumption science would be impossible, for there would be no point in working out, for example, that an eclipse of the sun was going to take place at such and such a locality if it was simultaneously feasible for it not to take place there[3]. Logic is, or should be, the faithful servant of reality rather than the legislator of what is and is not: The Axiom of Exclusion is the justification for, not the consequence of, the logical rule (in bivalent logic) that *'A proposition cannot at one and the same time be true and false.'*

Of course, if the reality we are modelling is inherently fluctuating and ambivalent it is a mistake to make the symbolic system too cut and dried because it will not fit the original. This is basically the reason why literature is able to give a far more convincing picture of actual human behaviour—which seems to be incurably irrational—than biochemistry. The maddening ambiguity and vagueness of language—as the mathematician sees it—become assets if we are dealing with a shifting, inconsistent reality.

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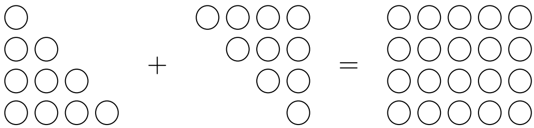
The Buddhist logician Dharmottara considered that

*'There can be no necessary relation other than one based on*

*Identity or Causality.*[4]'

This is admirably concise; so let us apply it to mathematics. If causality has nothing to do with mathematics, which is the usual view, this means that mathematics is entirely based on 'Identity', i.e. it is all one vast tautology.

This seems to be true of mathematical formulae such as those for summing the figurate numbers.



Since the above is perfectly general we can conclude that the sum of the natural numbers commencing with unity can be presented as a rectangle with one side equal to the greatest natural number of the sequence and the other side equal to that amount plus an extra unit. Causality as such does not seem to be involved.

But proofs by rearrangement, though the most convincing of all proofs, are not that common in modern mathematics: one reason why infinite series are such a minefield is that rearrangement can radically alter the nature of the series, the most notorious case being that of $\log 2$.[5]

But what about mathematical induction? Here there is a definite sense of compulsion: if such and such is true for $n$, it must be true for $n + 1$. Certainly, mathematical induction is not mere rearrangement: there is a sequential element which reminds us unmistakeably of a bona fide causal process, steam forcing a piston along the inside of a cylinder whether it wants to go there or not. A large number of functions—all?—can be defined by recursion rather than analytically, and this often seems a much more natural way of doing things. But definition by recursion is very different from analytical definition $n \to f(n)$ because in the recursive process a function is built up piecemeal instead of being there in its entirety from the word go. Philosophically speaking, analytical definition is *being*, definition by recursion *becoming*.

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For Plato, actual lines and circles were imitations of ideal states of affairs and the imperfect nature of the sublunary world meant there might occasionally be some slight deviations and discrepancies (a tangent drawn in the sand or on papyrus might well touch a circle at more than one 'point' for example). For Newton and Kepler what happened down here was wholly dependent on the prior decisions of a mathematical God, and we still seem to think like this a lot of the time which is why we still talk of the 'laws of Nature'—we do not speak of 'the observed regularities of Nature'. God may not have known, i.e. not bothered to work out in detail, all the particular consequences of his original handful of edicts, but then again He didn't need to. So long as the original laws were basic and far-reaching enough, the world could be left to take care of itself. There is causality of a kind here because there is compulsion: rocks, plants and animals have no choice but to comply with the rules and even man, though he has free will, remains constrained in his physical being. But, according to this paradigm, the causality we find in Nature is not itself 'natural': it has a supernatural origin and purpose.

In the classical (post-Renaissance) world-view there is no real difference between physical and mathematical law, between pure and applied mathematics, so the same schema applies. God determined the axioms and everything else is theorems. But today we no longer believe in an omnipotent intelligent Creator God (most of us anyway) so the 'laws of physics' and 'laws of mathematics' revert to being something rather similar to Platonic Forms, existing out on a limb. This does seem to remove causality from mathematics and physics unless we view the way in which phenomena model themselves on ideal states of affairs—how an actual gas approximates to the behaviour of an ideal gas, for example—as a sort of watered down 'formal causality'. In the Judaeo-Christian world view which was that of Kepler and Newton, everything hinges on actions and decisions made 'in the beginning': someone (God) had sometime in the distant past 'divided the light from the darkness' and distinguished primes from composite numbers. But Platonic Forms and mathematical formulae simply *are*, they do not *do* anything.

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Supposedly, the whole of mathematics can be derived from the half dozen or so axioms of von Neumann or Zermelo Set Theory (see M500 **206** p. 20). But no one ever sat down of a night to see what interesting theorems he or she could derive from them: they are strangely remote like mountains one sees in the distance but which are utterly unrelated to life down here in the plains.

But then most modern mathematics has an insubstantial air: the very way in which we are taught to do our mathematics, to consider the basic entities and procedures, inclines us towards a view of the world where nothing really happens. The old-fashioned viewpoint, still very much in evidence in school textbooks of the pre-war era, goes rather like this. We have a numerical or geometric entity, we do something drastic to it, multiply it, chop it up into bits, rotate it, &c. &c., and then we see what we are left with. The modern way is to 'map' certain values to certain other ones: we make up two sets $(a, b)$ and $(a', b')$ selected according to a rule. Everything exists in a sort of eternal present and we merely move around looking at what's here and comparing it with what's there. The idea of an *unknown* which by dint of intelligent manipulation gets transformed into a *known* is both intuitively clear and exciting: it is like working out the identity of Mr X from circumstantial evidence and witness statements. But the idea of a *variable* is quite different: somehow $x$ has *all* possible values at once (usually an infinite number) each of which incidentally is a constant. Also, in the real world effects always succeed causes which means, mathematically speaking, that the dependent and independent variables are *not* freely invertible—precisely what we are told to assume in calculus. Examples can be multiplied endlessly ....

What nobody seems to have noticed is that the two dominant tendencies in modern mathematics, the axiomatic approach and the analytical, 'functional' presentation (which is essentially Platonic) are pulling mathematics in two completely different directions and may well eventually tear it apart. An axiomatic approach means that deduction is all-important since not only can everything (or nearly everything if we take Gödel into account) be derived from the axioms, but nothing that is not so deducible will crop up (again *pace* Gödel). But deduction involves step by step argument, thus temporal sequence; also, there is a strict hierarchy with certain propositions being much higher up the pecking order, as it were, than others. But the 'functional', analytical treatment is, implicitly at least, atemporal and non-hierarchical. All the properties of $y = f(x)$ are there as soon as we have written down the expression and it is 'our fault' is we don't spot them straightaway. Moreover, all the cross-references between different functions also exist as soon as the functions are properly defined, and in fact prior to their being properly defined (by us). As for some propositions being key ones on which others depend, if everything is already out there nothing 'depends' on anything else, it either is out there or it isn't. There either are odd perfect numbers or there are not. This rather cuts the ground from under the feet of the 'prove-at-all-costs' lobby and, moreover, because computers can usually prove or disprove whether such and such an assertion is true over the domain that concerns us in practice, it ceases to be so important to know if something is 'always' true or not—indeed, some philosopher will shortly come along and tell us that this is a 'metaphysical question' and thus not worth bothering about.

Currently there are only three theories of mathematics left in the running, formalism, logicism and Platonism. Neither of the first two schools of thought can explain the often amazingly good match between mathematics and physical reality, and, while Platonism does explain this, the metaphysical price to pay is a very high one indeed. Even mathematicians who are not afraid to call themselves Platonists (such as Penrose) fight shy of giving any coherent statement of their philosophic position to the general public. Now logicism and formalism do not recognize causal processes at all while Platonism admits only a very watered down sort of causality at best. So this explains the inevitable demise of causality in the scientific world-view.

But more significantly none of these mathematical schools can explain the surprising *vitality* of mathematics which never ceases to astonish (and sometimes to alarm). Formalism allows for human invention, since that is what in the last resort the whole of mathematics is, but has little to say about how and why inventions come about. I am so far from being a positivist that I see 'vital forces' operating everywhere, not only in the biological and physical domains but also in supposedly abstract areas like pure mathematics and even, in a very rarefied form, in logic. There is perhaps a single unified 'force of necessity' which is (almost) tangible in an arrangement of rods and levers and which, in a good mathematical proof, can be sensed thrusting the tortuous argument on to its triumphant crescendo.

Moreover, this *élan vital* is surely active in mathematics as a whole, ceaselessly pushing it in new and unexpected directions: mathematics, like technology, has a life of its own and individual mathematicians get dragged along whether they want to go in that direction or not. What is absent from the logicist, Platonist and formalist views on mathematics is precisely a recognition of this vital principle. There is just no driving force in Set Theory : it is a steam-engine that has been cleaned up, varnished and put to rest in a science museum. This is why Poincaré, who was a creative mathematician in a sense that Russell and Whitehead were not, dismissed logicism with the crushing retort, 'Logic is sterile but mathematics is the most fertile of mothers.'

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## References and notes

**[1]** See A. P. French and E. F. Taylor, *An Introduction to Quantum Physics*, pp 88–89. The remarkable illustrations show the picture of a girl's face building up from randomly distributed dots.

I have conjectured that there is some sort of a law involved: if all the events are specified in advance, their order of appearance need not be, if not all of the events are specified in advance, there must be strict order.

**[2]** Does anyone, for example, really believe that 'almost all' numbers are transcendental? (I remind readers that a transcendental number is a real number that is not the root of a polynomial equation with integer coefficients.) Apart from $e$ and $\pi$ (and trivial variations) I doubt if anyone reading this could produce one without consulting a dictionary of mathematics. On doing this I find that $10^{-1!} + 10^{-2!} + 10^{-3!} + \ldots$ is also a member of this highly select (but apparently very well attended) club.

**[3]** The trouble with Quantum Mechanics is that it does not verify the Axiom of Exclusion since it permits a physical system to be in incompatible states at the same time. The Many Worlds version of QM *does* verify the Axiom of Exclusion, of course, but there is a heavy price to pay in universes.

[4] F. T. Stcherbatsky, *Buddhist Logic* Vol. 1, p. 259. This is, incidentally, an extremely interesting, readable and, I believe, important book despite its abstruse air. It is more concerned with speculative philosophy than logic as such. The world-view of certain Buddhist thinkers in Northern India during the first few centuries of our era has a distinctly modern feel—they would have been quite happy with Einstein's attempt to describe the physical world in terms of causally related events occurring in a single unified Space–Time field.

This raises the question of why India didn't get there first in terms of the scientific revolution. Needham, in discussing the question with reference to China, concludes that the key notion of natural law was lacking. But this was certainly not lacking in India (the law of *karma*). Maybe these Hinayana Buddhists were too advanced in their conceptions: it was necessary to work through the cruder scientific paradigm of a world made up of 'hard, massy particles' interacting with each other by pushes and pulls before moving on to the vision of evanescent bundles of energy evolving in space–time. Also, of course, there was little motivation to develop science as such: for a Buddhist the physical world was just not important enough to bother about.

[5]
$$\begin{aligned}
\log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} \ldots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\right) = \frac{1}{2}\log 2.
\end{aligned}$$

# Solution 210.2 – Cosecs

Show that

$$\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ \;=\; 6.$$

Are there other interesting identities of the same kind?

## A. J. Moulder

I offer the following proof of the given expression.

$$\begin{aligned}
&\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ \\
&\quad= \frac{1}{\sin 10^\circ} + \frac{1}{\sin(60^\circ - 10^\circ)} - \frac{1}{\sin(60^\circ + 10^\circ)} \\
&\quad= \frac{1}{\sin 10^\circ} + \frac{2}{\sqrt{3}\cos 10^\circ - \sin 10^\circ} - \frac{2}{\sqrt{3}\cos 10^\circ + \sin 10^\circ} \\
&\quad= \frac{1}{\sin 10^\circ} + \frac{4\sin 10^\circ}{3\cos^2 10^\circ - \sin^2 10^\circ} \\
&\quad= \frac{1}{\sin 10^\circ} + \frac{4\sin 10^\circ}{3 - 4\sin^2 10^\circ} \\
&\quad= \frac{3}{3\sin 10^\circ - 4\sin^3 10^\circ} \;=\; \frac{3}{\sin 30^\circ} \;=\; 6.
\end{aligned}$$

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## Patrick Walker

Problem 210.2 in M500 **210** triggered memories. From over 60 years ago I recall

$$\sin 10^\circ + \sin 50^\circ - \sin 70^\circ \;=\; 0$$

and

$$\sin 16^\circ + \sin 20^\circ + \sin 92^\circ \;=\; \sin 52^\circ + \sin 56^\circ.$$

M500 is excellent!

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## Tony Forbes

With a little help from a computer I find that

$$-10^{-100000} \;\le\; \operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ - 6 \;\le\; 10^{-100000}.$$

Hence the answer to the problem must be 6 exactly.

## Problem 214.1 – River crossing

There is a river and a rowing boat which can carry at most two people. A number of married couples are on one bank and they want to cross to the other side of the river. For the usual reason a woman must never be in the presence of a man who is not her husband (on the same bank or in the boat) unless her husband is also present.

(i) Arrange a crossing schedule for one married couple.

(ii) Arrange a crossing schedule for two couples.

(iii) Arrange a crossing schedule for three couples.

(iv) Can four couples cross the river?

(v) Show that any number of couples can cross if there is an island in the middle of the river.

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## Problem 214.2 – Cuboid in a triangular room

What is the volume of the largest cuboidal box that you can deliver into a triangular room 3 units high with walls of length 5, 5 and $5\sqrt{2}$. There is a door, 2 high by 1 wide, in the centre of one of the $5 \times 3$ walls. If it's any use, you may assume there is a large hole in the floor just outside the door.

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## Problem 214.3 – Consecutive primes

Suppose $p$ and $q$ are consecutive primes. Let

$$\alpha = \sqrt{\frac{p^2 + q^2}{2} - 1}.$$

Show that if $q = p + 2$, then $\alpha$ is an integer. Is the converse true?

Sebastian Martin Ruiz conjectures that it is. If the consecutiveness condition is abandoned, there are many solutions. It might be interesting to investigate the distribution of the values of the smallest $q > p + 2$ which makes $\alpha^2$ a square. If $p$ is large, it seems to be difficult to find a $q$ that is near. For instance, when $p = 1000541$ the smallest (prime) $q$ is 1034177, whereas the next prime after $p$ is 1000547.

A curiosity: the 20000th odd prime is 224743, the 10000th odd prime is 104743, the difference is 120000.

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"If Euclid were alive today, would you consider him to be a remarkable person?"

"Well, yes. He would be over 2000 years old."

# Relations between trig functions of specific values

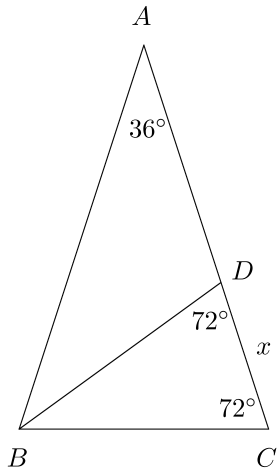
## Richard Boardman

The solution to Problem 210.2 and Patrick Walker's comments [page 10] prompt me to offer the following contribution.

There are many formulae involving sines and cosines of whole numbers of degrees, partly because these numbers involve only a limited set of operations. The sines of 30, 45 and 60 degrees are well known. The value for 18 degrees is equally simple,

$$\sin 18^\circ \;=\; \frac{\sqrt{5}-1}{4}.$$

To see this, draw an isosceles triangle $ABC$ with angle $36^\circ$ at $A$ and $72^\circ$ at $B$ and $C$. Choose $D$ on $AC$ such that $BD$ bisects the angle at $B$ into two angles of $36^\circ$. Triangle $ADB$ has two angles of $36^\circ$ so it is isosceles and $AD = BD$. Triangle $BCD$ has two angles of $72^\circ$ so it is isosceles and $BC = BD$. Scale the figure so that $AD = BD = BC = 1$ and call $DC$ $x$. Then $AC = 1 + x$. Triangles $ABC$ and $BCD$ are similar so that $x/1 = 1/(1+x)$. This gives




$$x \;=\; \frac{\sqrt{5}-1}{2}.$$

But $x = 2\sin 18^\circ$, so that $\sin 18^\circ = (\sqrt{5}-1)/4$.

All of these values involve only rational numbers and the square roots of 2, 3 and 5. Pythagoras shows that cosines and hence tangents only add extra square roots. The formulae for $\sin(a+b)$ and $\cos(a+b)$ show that any sum or difference of these angles will involve only rational numbers and these three square roots. Thus angles which are multiples of 3 degrees have simple sines and cosines. The half angle formulae show that only extra square root operations are added. Thus

$$\sin\frac{45^\circ}{2^n} \;=\; \frac{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}{2},$$

with $n$ nested square root operations. There are similar formulae for $\sin(15^\circ/\,2^n)$ and $\sin(18^\circ/\,2^n)$.

The formulae for $\sin 3\theta$ and $\cos 3\theta$ are cubics, so that $\sin 10^\circ$ and $\cos 10^\circ$ involve nothing worse than cube roots. Again the sums and differences of these angles do not add any fresh complications so that since $180 = 2{\cdot}2{\cdot}3{\cdot}3{\cdot}5$, the sine and cosine of any whole number of degrees must be an expression involving only rationals, square roots and cube roots. Obviously there are many relationships between elements in such a small field.

The symmetric functions of the roots of equations are another fertile source of relationships. Whenever the roots of an equation with simple coefficients are trig functions, then the sums, products and products in pairs of the roots give simple relations. Take the formula for $\tan 3\theta$. Let $t = \tan A$ then

$$\tan 3A \;=\; \frac{t - t^3}{1 - 3t^2}.$$

Rearranging this gives

$$t^3 - 3\tan(3A)t^2 + 3t - \tan 3A \;=\; 0.$$

The roots of this equation are $\tan A$, $\tan(A+120^\circ)$ and $\tan(A+240^\circ)$. Then $3\tan 3A$ is the sum of these roots, $-\tan 3A$ is their product and 3 is the sum of the products in pairs.

Choosing a value for $A$ and reducing all the angles to be in the range 0 to $90^\circ$ can produce some very unlikely looking formulae.

There is a similar situation for angles that have both rational sines and rational cosines. I call these rational angles. These arise from rational solution to the formula $x^2 + y^2 = z^2$ of which there are an infinite number.

Again, the sum and difference formulae show that the sum and difference of any two rational angles is also a rational angle. Apart from integer multiples of $90^\circ$, no rational angle is a whole number of degrees, but taken as a set, they form a most interesting group.

[Acknowledgement. Some of this material is taken from *Trigonometric Delights* by Eli Maor.]

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'It can be of no practical use to know that $\pi$ is irrational, but if we can know, it surely would be intolerable not to know.' —**E. C. Titchmarsh**.

[Titchmarsh (Edward Charles, 1899–1963) seems to have been one of those mathematicians who would never use more than was absolutely necessary for an argument. For instance, in his proof of Lemma 6.12 in *The Theory of the Riemann Zeta-Function* there is this interesting inequality:

$$\frac{1}{2} - \frac{3}{8} - \frac{1}{21} \;\geq\; \frac{1}{14}.$$

—ADF]

# Trigonometric identities

## Tony Forbes

The two identities quoted by Patrick Walker on page 10, namely

$$\sin 10^\circ + \sin 50^\circ - \sin 70^\circ \;=\; 0 \tag{1}$$

and

$$\sin 16^\circ + \sin 20^\circ + \sin 92^\circ - \sin 52^\circ - \sin 56^\circ \;=\; 0, \tag{2}$$

actually have very simple interpretations when the trigonometric functions are converted to their complex exponential forms.

Write $\phi = e^{\pi i/180}$. Then, observing that $\phi^{180} = -1$, the usual formula for the sine function becomes

$$\sin x^\circ \;=\; \frac{\phi^x + \phi^{180-x}}{2i}.$$

So for the first identity we only have to show that

$$\phi^{10} + \phi^{170} + \phi^{50} + \phi^{130} - \phi^{70} - \phi^{110} \;=\; 0.$$

Rearranging the left-hand side, we have

$$\begin{aligned}
&\phi^{10}(1 + \phi^{40} - \phi^{60} - \phi^{100} + \phi^{120} + \phi^{160}) \\
&\qquad = \; \phi^{10}(1 + \phi^{40})(1 - \phi^{60} + \phi^{120}) \\
&\qquad = \; \phi^{10}(1 + \phi^{40})(1 + \phi^{120} + \phi^{240}).
\end{aligned}$$

But 1, $\phi^{120}$ and $\phi^{240}$ are the three cube roots of 1. Hence they sum to zero, as required.

Similarly, the other identity, (2), is equivalent to

$$\phi^{16} + \phi^{164} + \phi^{20} + \phi^{160} + \phi^{92} + \phi^{88} - \phi^{52} - \phi^{128} - \phi^{56} - \phi^{124} \;=\; 0.$$

This time the left-hand side factorizes as

$$\begin{aligned}
&\phi^{16}(1 + \phi^{4})(1 - \phi^{36} + \phi^{72} - \phi^{108} + \phi^{144}) \\
&\qquad = \; \phi^{16}(1 + \phi^{4})(1 + \phi^{72} + \phi^{144} + \phi^{216} + \phi^{288})
\end{aligned}$$

and the last factor is zero because it is the sum of the five 5th roots of 1.

We can also use this kind of argument to deal with Problem 210.2 – Cosecs, solved on page 10 in this issue. Here we are to prove that

$$\operatorname{cosec} 10^\circ + \operatorname{cosec} 50^\circ - \operatorname{cosec} 70^\circ - 6 \;=\; 0. \tag{3}$$

Regarding the similarity between (1) and (3), I cannot offer any obvious explanation as to how the changing of sin to cosec induces the appearance of the extra term, $-6$.

Putting $\psi = e^{\pi i/18}$ and using the equalities $\psi^9 = i$ and $\psi^{18} = -1$, we transform the left-hand side of (3) into

$$\frac{2\psi^9}{\psi + \psi^{17}} + \frac{2\psi^9}{\psi^5 + \psi^{13}} - \frac{2\psi^9}{\psi^7 + \psi^{11}} - 6. \tag{4}$$

Rearranging (4) as a ratio of two polynomials in $\psi$, and using only the property $\psi^{18} = -1$, we obtain after a considerable amount of work:

$$\frac{-4\,\psi\,(1+\psi^4)\,(1-\psi^6+\psi^{12})}{\psi - \psi^3 + \psi^5 - \psi^7 - \psi^{11} + \psi^{13} - \psi^{15} + \psi^{17}}. \tag{5}$$

Now we can see immediately that the numerator is zero because one of its factors is the sum of the three cube roots of 1, namely 1, $\psi^{12}$ and $\psi^{24} = -\psi^6$.

Also on page 10 I offered an alternative proof of (3). I simply used a computer to confirm that the left-hand side of (3) is zero to 100000 decimal places and asserted that this was sufficient 'proof'.

It's not as daft as it sounds. As we have seen, if you have a trigonometric identity such as (3) involving integer multiples of tens of degrees, then buried under the surface is a quantity such as (5) involving a 36th root of 1, the algebraic integer $\psi$. Thus (5) is the ratio of two algebraic integers in $\mathbb{Q}(\psi)$, the field obtained by adjoining the complex number $\psi$ to the field of rationals. So there is a severe restriction on the kinds of values (5) can take. In particular, it is impossible for (5) with its simple numerator and 'small' algebraic integer denominator to have a value that is very close to zero other than zero precisely. In fact, the denominator of (5) has absolute value 1.

In general, any reasonably simple trigonometric identity involving integer coefficients and integer degrees ought to be provable by a sufficiently accurate numerical computation. Unfortunately I don't really know what 'sufficiently accurate' means. In the example I have given, I know that 100000 decimal places are more than sufficient, but I have no clear estimate of the minimum accuracy needed.

# Philosophical implications of the discovery of the natural algebras

## Dennis Morris

The central dichotomy of philosophy is the idealist world-view and the materialist world-view. The idealist view is that the mind, or a mind (god), exists and that all material things exist only as a thought within that mind. All religions are of this view. The materialist view is that material things exist and that all thoughts are the product of the material universe—humanity produces god not god produces humanity.

Axiomatic mathematics is a structure based on axioms that come out of the mind of mathematicians rather than are discovered in the muck of reality. A mathematical statement is true if it can be deduced from the axioms. In contrast, physical laws are rough hewn from reality, and their truth is tested against reality. Axiomatic mathematics is an idealist world-view; physics is a materialist world-view.

In many ways, mathematics is the greatest of religions. It boasts certainty of truth. It does not require its constructs to exist in the material universe.[1] All that is required is to have faith in the axioms. Great priests can change the axioms or add another one, but not even the greatest of priests may question the axiomatic structure.[2]

The natural algebras are non-axiomatic. They are discovered rather than invented. The truths of the natural algebras are tested against reality. The natural algebras begin by finding real numbers. They must exist because if they did not there would be zero of them, and zero is a number. That real numbers exist is sufficient to cause addition and multiplication (repeated addition) to exist. This multiplication of real numbers is discovered to be distributive, and, from this, it follows that linear transformations exist:

$$
\begin{aligned}
(20+3)(30+7) &= 600+140+90+21, \\
\begin{bmatrix} 20 & 3 \\ 3 & 20 \end{bmatrix}\begin{bmatrix} 30 & 7 \\ 7 & 30 \end{bmatrix} &= \begin{bmatrix} 600+21 & 140+90 \\ 90+140 & 600+21 \end{bmatrix}.
\end{aligned}
$$

Within the set of all linear transformations are discovered subsets that are closed under multiplication etc. These subsets are the natural algebras. Within the natural algebras are discovered the natural spaces. The two 2-

[1] Anyone who has studied metric spaces will know this!

[2] Well, perhaps Gödel bruised it a little.

dimensional spaces together form the apparently 4-dimensional space-time of the real universe. It might be that the invariants within the higher-dimensional algebras form the conservation laws of the atomic particles.

Prior to the discovery of the natural algebras, mathematicians invented mathematics. Algebras were invented and inner products were invented to put upon those algebras. The natural algebras are discovered rather than invented. The inner product upon a natural algebra is the one it is found with, and it will allow no other.

Mathematics that is invented is idealist mathematics. Mathematics that is discovered is materialist mathematics. The discovery of the natural algebras is the slaying of the last and the greatest of all religions in that it turns mathematics into a materialist science rather than an idealist construct. Sitting atop the natural algebras, invented mathematics appears very much to be the work of alchemists.

---

## Problem 214.4 – 103653

(i) Prove that

$$
\begin{aligned}
&01020304050607080910111213121110090807060504030201 \\
&\times\ 103653 \\
=\ &105757575757575757575757575736424242424242424242424242424253.
\end{aligned}
$$

(ii) Remove the 1213 from the middle of the first number and remove a 57 and a 42 from the third number.

(iii) Continue in the same way, removing four digits from the middle of the first number and 57 and 42 from the third number, until you get to

$$01 \times 103653 \ = \ 103653.$$

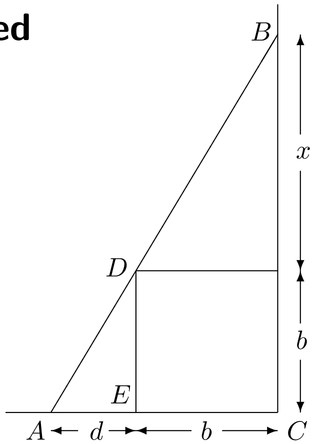
(iv) Be amazed by what you see.

(v) Now go the other way. Extend the first number by the appropriate four digits and the third number by 57 and 42. Show that this works all the way up to

$$0102 \ \ldots \ 9798999897 \ \ldots \ 0201.$$

(vi) Are there any other examples?

# Solution 209.4 – Ladder, revisited




A ladder of length 1 stands against a vertical wall just touching a shed of height and width $b$. Find $d$, the distance of the ladder bottom from the shed.

## Nick Hobson

The fact that the situation is symmetric in $d$ and $x$ suggests a slightly different approach from that presented by Steve Moon in M500 **212**. By similar triangles, we have

$$d\,x \;=\; b^2. \tag{1}$$

By Pythagoras,

$$(d+b)^2 + (x+b)^2 \;=\; 1.$$

So

$$d^2 + 2\,b\,d + x^2 + 2\,b\,x + 2\,b^2 \;=\; 1. \tag{2}$$

Then,

$$(d+x)^2 \;=\; d^2 + 2\,d\,x + x^2 = d^2 + x^2 + 2\,b^2,$$

from (1). Hence (2) may be rewritten as

$$(d+x)^2 + 2\,b\,(d+x) - 1 \;=\; 0.$$

Rejecting the negative root, we obtain

$$d + x \;=\; \sqrt{b^2+1} - b.$$

Having found $d\,x$ and $d+x$ in terms of $b$, we use Viète's relations to write down the quadratic of which $d$ and $x$ are the roots:

$$z^2 - (\sqrt{b^2+1} - b)\,z + b^2 \;=\; 0,$$

which has the same solutions as those given by Steve Moon. This problem is similar to one given on my puzzle site: http://www.qbyte.org/puzzles/p076s.html#ladder.

# Problem 214.5 – 1000000 tarts

There are one million tarts; all weigh the same except for 100, which are too light. There is also a weighing machine that will indicate whether or not a batch of tarts has the correct weight. Devise a testing strategy to identify the 100 defective tarts with a small maximum number of weighings.

---

We do not demand an optimal strategy—too difficult. But we are interested in schemes which are merely very good. Here is a not bad example.

(i) Divide the tarts into 1000 batches of 1000 and test each batch. Assume the worst possible scenario—that the 1000th batch is bad and hence that 1000 tests are actually required. Again assuming the worst possible case, we have narrowed down the population to be screened from 1000000 to at most 100000 tarts.

(ii) Divide the remaining tarts into 1000 batches of 100, reducing the number of suspect cases to at most 10000.

(iii) Divide them into 1000 batches of 10 to leave at most 1000.

(iv) Test the remaining tarts individually.

Hence this strategy can identify the 100 bad tarts in at most 3990 tests. We save 10 tests in stage (iv) because there are either at most 99 batches of 10 to test, or exactly 100 batches each containing exactly one bad tart. And with a refinement of this argument as well as similar arguments for stages (i)–(iii) you can reduce the figure even further.

Unlike those other tart problems that have appeared from time to time in M500, this one might possibly have practical application outside the catering industry. Of course, there is nothing special about 1000000, and knowledge of the *exact* number of defective tarts is, to say the least, a little contrived. But imagine a real-life situation where one must perform costly tests for a rare defect in a large population. If samples can be combined (as with blood taken from humans, for instance), then something along the lines of the strategy outlined above would achieve a substantial saving.

---

# Russell's attic

## Eddie Kent

Russell's attic is a room containing countably many pairs of shoes and countably many pairs of socks. It is easy to see that there are countably many shoes, for instance by matching the left shoes to the odd numbers and the right shoes to the even numbers. But can you say how many socks there are?

# Letters to the Editor

## Vector algebra

Dear Tony,

I feel that Dennis Morris is overstating his case on page 1 of M500 **212**. The word 'algebra' comes from the arabic and has been used for hundreds of years to describe systems in which letters symbolized objects and were manipulated according to rules. The very limited meaning he gives to the word is comparatively recent and never widely used.

People who criticize physicists who use inelegant mathematics 'because it works' should remember that Newton used calculus but could never provide a rigorous proof, and that now, when we are aware of the quantization of matter and energy, the calculus is used where limiting processes could never actually be applied. It still works!

**Dick Boardman**

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## Riddles

That's an arresting pattern on the cover of No. **211**. Also, my eye was caught by the 'Sentences' notes on page 21. My dog-eared copy of *Riddles in Mathematics* (Eugene P. Northrop, Pelican 1960) gives a logic paradox in this connection. Consider, 'the least integer not namable in fewer than nineteen syllables'. This has eighteen syllables. Finally, you might like to ponder the following.

The Goldbach conjecture can be stated:

*Every even integer* $> 6$ *can be expressed as* $p + q$, *where* $p, q$ *are odd primes,* $p < q$. (A)

It can also be stated:

*Every even integer* $> 6$ *can be expressed as* $(r + s)/2$, *where* $r, s$ *are odd primes,* $r < s$ (midway between two primes). (B)

For any even integer $2x$ that satisfies (B), (B) implies (A) is true for $x$. What about the converse? If the converse is true, the Goldbach conjecture is true for $2x$ if true for $x$, etc. Possibly too good to be true.

Regards,

**Hugh McIntyre**

# Six-region sudoku

## Tony Forbes

A different sudoku-like puzzle. There are six types of region, each of which must contain the symbols $\{1, \ldots, 9\}$: rows, columns, boxes, split rows, split columns and split boxes. As usual, the solution is unique.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | | | | | | | | |
| | | | 2 | | | | | |
| | | | 6 | | | 5 | | |
| | | 9 | | | | 8 | | |
| | | 7 | | | 4 | | | |
| | | | | | 3 | | | |
| | | | | | | | | |
| | | | | | | | | |

A split row consists of three rows of three spaced three apart in one of the three column blocks, like the cells marked 'r' and 'x' in the upper array on the right. A split column consists of three columns of three spaced three apart in one of the three row blocks ('c' and 'x'). A split box is a $3 \times 3$ square array of cells spaced three apart in both directions, ('b').

The number of starter-digits, eight, is the smallest possible, and there is a mathematically nice characterization. If we label the rows and columns with base-3 numbers, 00, 01, 02, 10, 11, 12, 20, 21, 22, then a cell has four coordinates, *abcd*, where *ab* is the row and *cd* is the column. A region is precisely the set of cells obtained by fixing two coordinates. [Thanks to Peter Cameron of QMUL for the idea.]

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | c | | | c | | | c |
| r | r | x | | | c | | | c |
| b | | c | b | | c | b | | c |
| | | | | | | | | |
| r | r | r | | | | | | |
| b | | | b | | | b | | |
| | | | | | | | | |
| r | r | r | | | | | | |
| b | | | b | | | b | | |

HINT. Suppose the top-left cell is 1. It is clear that the cells marked 'x' in the lower array on the right cannot contain 1. Not so obvious is that the cells marked 'z' cannot contain 1. To see this, observe that one of the cells marked 'a' must be a 1. Similarly 'b', 'c' and 'd'. Four boxes, four cells, four ones. Hence the corresponding cells in the other five boxes must be 1-free.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | x | x | x | x | x | x | x | x |
| x | x | x | x | a | a | x | b | b |
| x | x | x | x | a | a | x | b | b |
| x | x | x | x | | | x | | |
| x | c | c | | z | z | | z | z |
| x | c | c | | z | z | | z | z |
| x | x | x | x | | | x | | |
| x | d | d | | z | z | | z | z |
| x | d | d | | z | z | | z | z |

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