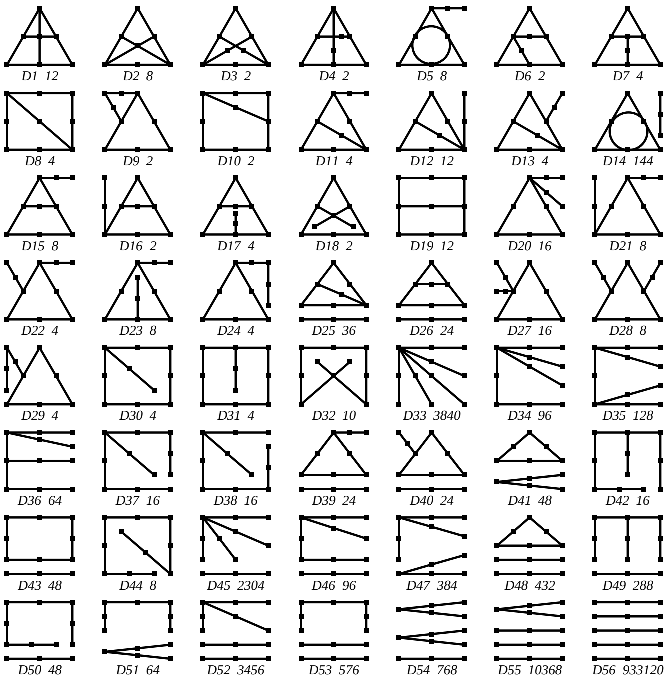
*

ISSN 1350-8539

M500 253



The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The May Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For full details and a booking form see m500.org.uk/may.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. For details see m500.org.uk/winter.htm.

Advice to authors We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

The Gnomon: $\sqrt{2}$ Ancient Greek calculator?

Sebastian Hayes

Ironically, the theorem which is forever associated with the name of Pythagoras undermined the 'numerical paradigm' which he had masterminded and paved the way for the 'geometric paradigm' associated with Plato and which was destined to replace it.

For it soon became clear that only *certain* triangles satisfied the requirement $a^2 = b^2 + c^2$ exactly. Even worse was the realization that, in the simplest possible case, that of an isosceles right-angled triangle, the diagonal was 'incommensurable' with its sides! The now familiar proof of the irrationality of $\sqrt{2}$ (as we would put it) is cited by Aristotle: it is a *reductio ad absurdum* argument showing that, with a, b relatively prime if $a^2 = 2b^2$, or $2 = (a/b)^2$, then b must be both odd and even, which is impossible. We do not know if this argument goes back to the time of Pythagoras and, to me, the neat—rather too neat—logical style points to a much later date. Early societies prefer direct proofs to proofs by contradiction (as I do myself) and, indeed, some people think that the very idea of mathematical proof is a 6^{th} century BC Greek invention. A scholium (commentary) to Euclid Book X states that 'the Pythagoreans were the first to address themselves to the investigation of incommensurability, having discovered it as the result of their investigation of numbers' (T. L. Heath, *A History of Greek Mathematics*). Note that the commentator speaks of 'investigation of numbers' not geometrical investigations. It seems to me much more likely that the Pythagoreans based their conclusion on a property of 'square' and 'rectangular' numbers, namely that a rectangle AD can only be transformed into a square, m^2, if $A = h^2$, $D = l^2$. This excludes at one fell swoop *all* cases where $a^2 = nc^2$ and n is not itself a square number.

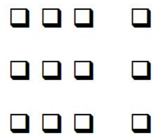
Be that as it may, the Pythagoreans almost certainly had a special method for engendering Pythagorean triples and similar sets of numbers. In a famous passage of *The Republic* Plato discusses 'geometric numbers', making a hard and fast distinction between what he calls the 'irrational diagonal' of a square side 5, and the 'rational diagonal', which is 7, the root, not of $2 \cdot 5^2$ but of $2 \cdot 5^2 - 1$. The passage suggests that the Pythagoreans knew the first terms of the series $a^2 = 2c^2 \pm 1$ which Theon of Smyrna actually provides some centuries later.

Although not himself a mathematician, the giant figure of Plato straddles the history of mathematics like the Colossus of Rhodes. For Plato resolved the conflict between arithmetic and geometry by a separation of the spheres: geometry concerned itself with a transcendental reality of which the physical world was at best a poor imitation while arithmetical number was restricted to the mundane everyday world and essentially 'the affair of craftsmen and traders rather than philosophers'.

By Euclid's time, geometry had so completely swamped arithmetic that most people today, flipping through Euclid's *Elements*, do not realize that he actually devoted three whole books to Number Theory and that a good deal of important number theoretic theorems can be discerned, like extinct insects trapped in amber, embedded in the strictly geometric books as well. Euclid perversely insists on presenting numbers in his (sparse) diagrams as continuous line segments instead of dots or dashes and he scrupulously avoids giving any numerical values whatsoever almost as if he is afraid of contagion.

Certainly, the early Pythagoreans did not employ logical/geometric reasoning *à la* Euclid to deliver particular sets numbers such as Pythagorean triples. So how did they do it? Probably by using 'gnomons'.

A *gnomon* was originally a sort of set-square that could be stood on its edge and used to measure the lengths of shadows—present-day sundials have a 'gnomon' on the top though the shape is more complicated. Thales is supposed to have used a gnomon to estimate the height of the Great Pyramid by employing properties of similar triangles and it was presumably measurements of noonday shadows at places on the same longitude that led the Pythagoreans to conclude that the Earth was a sphere suspended in space.

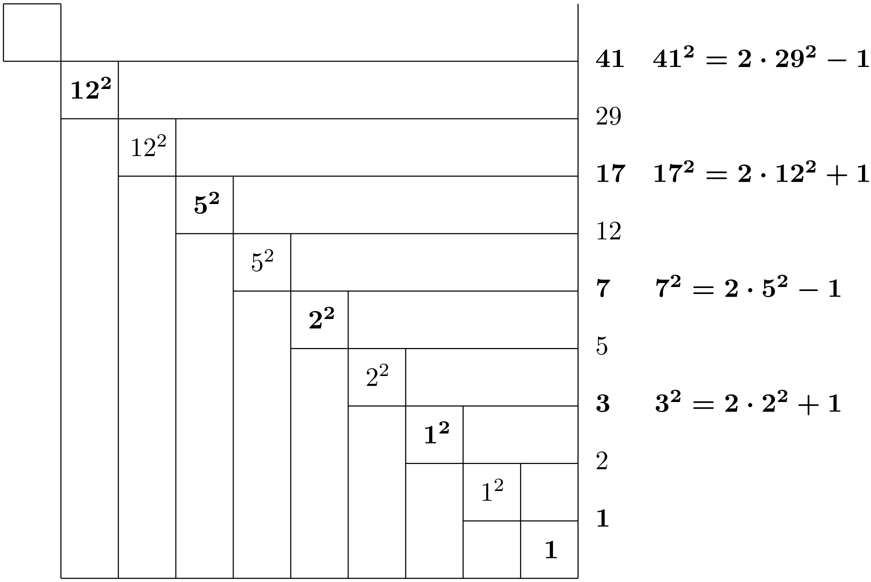


But the gnomon was important in number theory as well since sets of gnomons put together—or drawings of them—exhibit many important arithmetic properties of the integers.

Simply by observation, the early Greek mathematicians deduced that adding on a gnomon, the outer inverted L border, 'preserves the square form' and, more significantly, since the gnomon is always odd, that *The difference of two successive squares is always an odd number.* And someone else had a Eureka moment when he (or she, for the Pythagoreans accepted women into the community as well) realized that *If the gnomon is itself a square, we have a Pythagorean triple* (Note 1).

The gnomon representation, sort of nomograph of the time, can be applied to the problem of the diagonal of a unit square. A cursory examination of squares provides us with two 'near misses' to the $\sqrt{2}$ case, namely

$3^2 = 2 \cdot 2^2 + 1$ and $7^2 = 2 \cdot 5^2 - 1$. Examination of larger squares would have given scribes and hermits reason to suspect that there were any number of cases that came as close as a single unit. So it was all a matter of deriving a procedure that would generate such a sequence and this I decided to do for myself using only the sort of visual or concrete methods of the time though working with representations of gnomons rather than actual ones (because the areas rapidly get too large). I hit on the following diagram.



The principle of formation can be deduced from the first two 'true' examples, $3^2 = 2 \cdot 2^2 + 1$ and $7^2 = 2 \cdot 5^2 - 1$. If we make the 'inner square' of the first example the top left hand corner square of the next gnomon and repeat, we have the next pair of 'doubling squares'. Continuing in this way, it becomes apparent that, if one scales down, one has a method for making the error in $a^2 = 2b^2$ as 'small as we please' to use the 19th century expression (Note 2). Not only does such a mathematical procedure deliver actual numbers that a craftsman could use but, more significantly, it embodies the suggestive idea of progressive approximations to an idealized configuration. The algorithmic process has a thoroughly modern feel and one can imagine the ancient-world equivalent of a PC, a household slave, being 'programmed' to continue such a diagram indefinitely without really understanding what was happening.

The complete precedence of geometry over arithmetic which seems to have become *de rigueur* by the Alexandrian era must have seriously inhibited mathematicians from investigating further not only irrationals but series generally. For the 'problem' had been solved once and for all: hence no point in bothering about progressively more accurate 'solutions'. And as a matter of fact, we have extremely few good Greek estimates of 'irrational numbers' (though practising engineers must have had their own procedures and tables).

All this had immense repercussions and may well have been an important factor in holding back Greece, or rather Alexandria, from entering the industrial era directly (Note 3) and thus dispensing with a troubled interlude of nearly one thousand six hundred years! Hero of Alexandria was well acquainted with the motive power of steam and he designed temple doors which opened automatically not unlike ours (but powered by compressed air). However, what a nascent industrial society needs from mathematics is not so much recondite investigations into conic sections but ways of generating increasingly precise values for certain typical physical processes, especially those used in manufacturing. In other words it needs numerical series. Later Greek higher mathematics was singularly ill-equipped to provide such a back-up since it concerned itself exclusively with timeless, geometric realities. With the arrival of heat machines and complicated machines generally, the concept of 'steady state' or dynamical equilibrium entirely replaces the unreal notion of static equilibrium which reigned supreme in Greek science. Interestingly, at the very moment when the age of steam was taking off, Fourier came up with a completely new mathematical device that not only produced progressively more accurate approximations to *any* 'normal' function but could even cope with (a countable number of) discontinuities. Newton and the Enlightenment mathematicians would probably have disapproved of Fourier series and Laplace transforms—and most undergraduates start off by disliking them—but these messy mathematical procedures were precisely what was needed to model the complex and perpetually changing industrial processes.

We inherit not only our democratic institutions and nascent science from the Greeks but also our priorities and scale of values. Algebra, a Renaissance invention, removed mathematics even further from the physical world than geometry and modern pure mathematicians are, almost to a man (or woman) convinced Platonists in the sense that they believe mathematics deals in truths that cannot be empirically falsified in the way scientific theories can be. Plato would have wholeheartedly approved.

Curiously though, we are currently in the midst of a resurgence of the Babylonian/Pythagorean numerical tradition since our society has witnessed the decisive victory of the digital computer over the analogue and one notes a growing trend in the sciences to solve equations numerically rather than analytically.

Would mathematics have been the poorer if we had followed the numerical option from the beginning? On one level, clearly yes. But the numerical approach would have brought about a much better understanding of so-called 'infinite series' much earlier and likewise promoted the development of a science of motion which the Greeks completely lacked. Recursive procedures would have taken precedence over geometrical and analytical ones. This is a radically different way not only of doing mathematics but of envisaging reality and seems much closer to the way Nature actually works which is by progressively approximating to a desired evolutionary goal rather than by passively obeying changeless laws that cannot be improved upon. Whether present day mathematicians like it or not, the switch from the geometrical/analytical to the numerical/recursive is coming in fast anyway with the increasing use of numerical approximations, computer simulations, genetic algorithms, cellular automata and the like.

Note 1. The 'simple' gnomon $(2n+1) = m^2$ only generates triples of the form

$$\left(\frac{m^2+1}{2}\right)^2 \; = \; m^2 + \left(\frac{m^2-1}{2}\right)^2$$

but if we allow the gnomon to have more than one layer as it were, we can generate other sets. For example, a gnomon with three layers gives the equation $2(3r) + 3^2 = m^2$; so all we need is an odd square larger than 9 and divisible by 3^2.

Note 2. The $\sqrt{3}$ and $\sqrt{5}$ cases can be dealt with in a similar way but beyond this we need algebra—or I do.

Note 3. The usual explanation is that the abundance of cheap labour (because of slavery) made it hardly worth the trouble building complicated machines. This was certainly a major factor but we may also suppose that, in a Spenglerian sense, Greek civilization was by this time exhausted by its mammoth intellectual and artistic achievements and ready to cede first place to the more efficiently organized but unmathematical Romans.

Pick's theorem

Tony Forbes

At one of the weekly Mathematics Study Group meetings at the London South Bank University, myweb.lsbu.ac.uk/∼whittyr/MathsStudyGroup/, I was pleased to be listening to a talk by Robin Whitty on the LLL algorithm, the subject of a forthcoming *Theorem of the Day*, www.theoremoftheday.org/. But the talk was interrupted for several minutes by a lively discussion about *Pick's theorem*, mainly because nobody could remember the formula—except me. However it turned out that the formula I offered, (2), below, was not the one Robin had in mind.

Pick's theorem tells you how to calculate the area of a simple polygon (such as S_1 in the illustration at the end) whose vertices are points on the integer lattice, $\mathbb{Z}^2$. The formula, (1), below, is quite remarkable because its only parameters are the numbers of lattice points inside and on the boundary of the polygon. For example, (1) gives the correct area, 1/2, for any triangle whose closure contains no lattice points apart from the three vertices. This is what Pick's theorem usually says.

Let S be a simple polygon in the plane whose vertices have integer coordinates. Then its area is given by

$$\mathrm{area}(S) \;=\; i + \frac{b}{2} - 1, \qquad (1)$$

where i and b are respectively the numbers of integer lattice points inside S and on the boundary of S.

Amazing for its simplicity and elegance, (1) is the expression you will find in *Wikipedia* and elsewhere, and indeed the formula was the one that our lecturer and his audience were eventually able to reconstruct. Although Georg Pick (1859–1942) is the name that has become associated with the formula for the area, it seems likely that anyone with access to a pencil and an infinite amount of squared paper will eventually deduce (1) for himself or herself. But in my opinion there is a much nicer formula, and moreover it applies with somewhat greater generality.

Let S be any plane shape bounded by straight lines that has a well-defined area and whose vertices have integer coordinates. Then

$$\mathrm{area}(S) \;=\; \sum_{P\in\mathbb{Z}^2} w(P), \qquad (2)$$

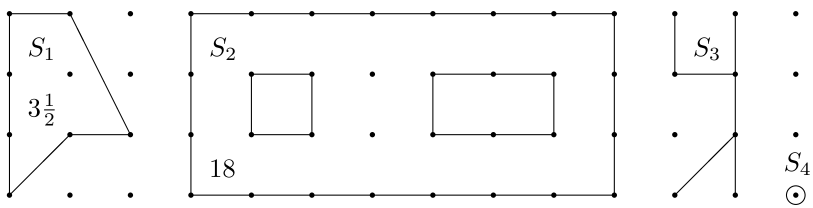
> *where $w(P)$, the weight of P, is calculated as follows. Let $D(P)$ be a disc with centre P which is so small that it does not intersect the boundary of S other than at edges of S containing P. Then $w(P)$ is the fraction of $D(P)$ that intersects S.*

Note that the sum is over all integer lattice points, but those outside S have zero weight anyway and so they can be ignored. Points inside S have weight 1 but points on the boundary have various weights in the interval $[0,\ 1]$.

Clearly, (2) implies the first formula if the contribution to (1) from v, the number of vertices, is $v/2-1$. But this follows because the edges meeting a given vertex, V, subtend an angle inside S of 180 degrees minus the exterior angle at V. So each vertex contributes $\frac{1}{2}$, and the -1 comes from the sum of the exterior angles, 360 degrees. Applied to the simple polygon S_1 both (1) and (2) agree. The weights of the vertices are $\arctan(2)/(2\pi)$, $\frac{1}{4}+\arctan(\frac{1}{2})/(2\pi)$, $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{5}{8}$, and the area is $3\frac{1}{2}$.

If you use formula (2) to compute the area of S_2, a 7×3 rectangle with two holes, you also get the correct answer. The two points inside S_2 have weight 1, a corner of the big rectangle has weight $\frac{1}{4}$, a corner of a hole has weight $\frac{3}{4}$ and all other points on the boundary have weight $\frac{1}{2}$. Putting these together gives $2+4\cdot\frac{1}{4}+8\cdot\frac{3}{4}+18\cdot\frac{1}{2}=18$. On the other hand, with formula (1) you have a choice. You can either (i) say that S_2 is not simple and give up, or (ii) apply (1) anyway to get the wrong answer, $2+30\cdot\frac{1}{2}-1=16$, or (iii) apply (1) to the big rectangle after removing the holes, then apply (1) to each hole separately and subtract.

This is most unsatisfactory, and we (Robin and I) were wondering if Pick's formula can be fudged. So we offer this problem to you. *Find an elementary extension to* (1) *for more general shapes.* Formula (2) also works in degenerate situations, such as S_3 or even a single point, S_4, where in each case (1) fails if you consider all the points to be on the boundary. Anyway, if you do find a suitable extension, we hope you will amuse yourself by trying it out on ever more bizarre shapes, at the same time confirming that (2) never fails to give the correct area.



Solution 249.1 – Hypersphere

Show that the volume of the unit n-dimensional hypersphere is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{n/2}}\,dx_1\,dx_2\ldots dx_n. \qquad (1)$$

Stuart Walmsley

1 Introduction. The objective is to show that the integral (1) is numerically equal to the volume of an n-dimensional sphere of unit radius. In section 2, the result is demonstrated for the particular case $n = 2$.

In section 3, the result is proved for the general case in a compact manner. A transformation is made to generalized polar coordinates, consisting of r, the radial coordinate, and $n-1$ angles, giving an integral which is the product of an integral over r and an integral over the angles, A_n say. The result is proved without explicitly determining the value of A_n.

In section 4, the angular contribution A_n is determined, but using a method which only involves integration over r. Finally in section 5, the generalized n-dimensional polar coordinates are defined and the angular integrals determined directly. This serves to cross-check the forms obtained in section 4.

2 The two-dimensional case

Here

$$I_2 \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-(x_1^2+x_2^2)\right)dx_1\,dx_2. \qquad (2)$$

The integral may be transformed to polar coordinates, r and ϕ, for which

$$r^2 \;=\; x_1^2+x_2^2 \quad\text{and}\quad dx_1\,dx_2 \;=\; r\,dr\,d\phi;$$

r is positive and has the range 0 to ∞ and ϕ has the range 0 to 2π. Then

$$I_2 \;=\; \int_0^{\infty}\int_0^{2\pi}\exp(-r^2)\,r\,d\phi\,dr \;=\; \pi,$$

which is the area of a unit radius circle, thus proving the desired result. The Cartesian coordinate form (2) may be factored to give

$$\int_{-\infty}^{\infty}\exp(-x^2)dx \;=\; \sqrt{I_2} \;=\; \sqrt{\pi}, \qquad (3)$$

thus giving a direct evaluation of the well-known Gaussian integral.

3 The general case

In the general case we see that the integrand of I_n depends only on the radial coordinate r:

$$I_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp(-r^n)\, dx_1\, dx_2 \ldots dx_n.$$

In general, for a transformation to a new coordinate system $q_1, q_2, \ldots, q_n$,

$$dx_1\, dx_2 \ldots dx_n = |J(x,q)|\, dq_1\, dq_2 \ldots dq_n,$$

where $J(x,q)$ is the Jacobian of the transformation, being the determinant of the matrix whose elements are $(\partial x_j/\partial q_k)$.

In the generalized n-dimensional polar coordinates (which will be defined in section 5), q_1 is the radial coordinate r and the other coordinates are angles. Furthermore, each coordinate x_j is proportional to r, so that the elements in one column of the Jacobian matrix J are independent of r and the elements in all other columns are proportional to r. In this way, r^{n-1} may be factored out of J and the terms remaining in J depend only on the angles.

The integral I_n when transformed to polar coordinates thus takes the form

$$I_n = \int_0^{\infty} r^{n-1} \exp(-r^n)\, dr\, A_n,$$

in which A_n represents the value of the product of the integrals over the angular variables. Each individual integrand is the product of a dq and the terms in q from the Jacobian. The limits correspond to the range of q in its coordinate system.

The integral in r is readily evaluated:

$$\int_0^{\infty} r^{n-1} \exp(-r^n)\, dr = \left[\frac{-\exp(-r^n)}{n}\right]_0^{\infty} = \frac{1}{n}$$

so that

$$I_n = \frac{A_n}{n}.$$

The volume of the n-sphere, V_n, can be determined by integrating the volume element of the polar coordinates over the dimensions of the hypersphere. Thus r is integrated from 0 to r, the radius of the n-sphere. The angles are integrated over their appropriate ranges leading to the value A_n. In this way

$$V_n = \int_0^r r^{n-1}\, dr\, A_n = \frac{r^n}{n} A_n = r^n I_n$$

showing that I_n is equal to the volume of an n-sphere of unit radius and hence proving the desired result.

4 The angular term A_n in terms of integrals over the radial coordinate

A complete description of V_n requires the evaluation of A_n. This can be done using integrals over r only. The requirement is a function that can be integrated over x_j and over r. Such a function is

$$\exp\left(-(x_1^2 + x_2^2 + \cdots + x_n^2)\right) = \exp(-r^2).$$

Then factoring the integral over the Cartesian coordinates and noting the equality of the factors, we get

$$\left(\int_{-\infty}^{\infty} \exp(-x^2)\, dx\right)^n = \int_0^{\infty} r^{n-1} \exp(-r^2)\, dr\, A_n.$$

The integral on the left-hand side has been evaluated (3), so that

$$A_n = \frac{(\sqrt{\pi})^n}{\int_0^\infty r^{n-1} \exp(-r^2)\, dr} = \frac{(\sqrt{\pi})^n}{P_n}.$$

It is now required to evaluate the denominator denoted by P_n.

It will be convenient to consider even and odd values of n separately:

$$P_{2n} = \int_0^\infty r^{2n-1} \exp(-r^2)\, dr, \qquad P_{2n+1} = \int_0^\infty r^{2n} \exp(-r^2)\, dr.$$

For P_{2n}, the integral is simplified by a coordinate transformation:

$$s = r^2, \quad ds = 2r\, dr,$$

giving

$$P_{2n} = \frac{1}{2} \int_0^\infty s^{n-1} \exp(-s)\, ds.$$

This may be integrated by parts with

$$u = s^{n-1}, \quad dv = \exp(-s)\, ds, \quad du = (n-1)s^{n-2}\, ds, \quad v = -\exp(-s).$$

In $\int u\,dv = uv - \int v\,du$, uv vanishes when evaluated at the limits, leading to

$$P_{2n} \;=\; \frac{n-1}{2}\int_0^\infty s^{n-2}\exp(-s)\,ds \;=\; (n-1)P_{2n-2}.$$

The process is repeated until $P_2 = \frac{1}{2}\int_0^\infty \exp(-s)\,ds = \frac{1}{2}$ is reached, so that, finally

$$P_{2n} \;=\; \frac{(n-1)!}{2}.$$

For odd values of n, integration by parts is used directly:

$$P_{2n+1} \;=\; \int_0^\infty r^{2n}\exp(-r^2)\,dr,$$

$u = r^{2n-1}$, $dv = r\exp(-r^2)\,dr$, $du \;=\; (2n-1)r^{2n-2}\,du$, $v = -\frac{1}{2}\exp(-r^2)$.

The term in uv vanishes as before, leaving

$$P_{2n+1} \;=\; \frac{2n-1}{2}\,P_{2n-1}.$$

The process is repeated until $P_1 = \int_0^\infty \exp(-r^2)\,dr = \frac{1}{2}\sqrt{\pi}$, and

$$P_{2n+1} \;=\; \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\ldots\cdot\frac{1}{2}\,\frac{\sqrt{\pi}}{2},$$

and more compactly

$$P_{2n+1} \;=\; \frac{(2n-1)!!}{2^{n+1}}\,\sqrt{\pi}.$$

Putting these results together,

$$\begin{aligned} A_{2n} &= \frac{2\pi^n}{(n-1)!}, & I_{2n} &= \frac{\pi^n}{n}, \\ A_{2n+1} &= \frac{2^{n+1}\pi^n}{(2n-1)!!}, & I_{2n+1} &= \frac{2^{n+1}\pi^n}{(2n+1)!!}. \end{aligned} \tag{4}$$

5 Hyperspherical polar coordinates and the angular derivation of A_n

In two dimensions the polar coordinates are r, the distance of the point from the origin, and ϕ, the angle r makes with the x_1 axis.

In three dimensions, a coordinate θ_3 is added, the angle r makes with the x_3 axis. Its range is from 0 (along $+x_3$) to π (along $-x_3$). Explicitly,

$$x_1 \;=\; r\sin\theta_3\cos\phi, \qquad x_2 \;=\; r\sin\theta_3\sin\phi, \qquad x_3 \;=\; r\cos\theta_3.$$

The Jacobian has the form $J = r^2 \sin\theta_3$ and the integration element is

$$dx_1\, dx_2\, dx_3 \;=\; r^2\, dr \sin\theta_3\, d\theta_3\, d\phi.$$

When extended to higher dimensions, a new angle θ_n is added at each stage, being the angle r makes with the new Cartesian coordinate direction. The range is 0 to π. The coordinates $x_1, x_2, \ldots, x_{n-1}$ are augmented by $\sin\theta_n$ and the new coordinate is $r\cos\theta_n$. In this way

$$\begin{aligned}
x_n &= r\cos\theta_n,\\
x_{n-1} &= r\sin\theta_n\cos\theta_{n-1},\\
&\ldots,\\
x_3 &= r\sin\theta_n\sin\theta_{n-1}\ldots\cos\theta_3,\\
x_2 &= r\sin\theta_n\sin\theta_{n-1}\ldots\sin\theta_3\sin\phi,\\
x_1 &= r\sin\theta_n\sin\theta_{n-1}\ldots\sin\theta_3\cos\phi.
\end{aligned}$$

The Jacobian is extended by a factor $r\sin^{n-2}\theta_n$ at each stage.

The integrals over the angles A_n thus become

$$A_n \;=\; \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta_3\, d\theta_3 \int_0^{\pi} \sin^2\theta_4\, d\theta_4 \ldots \int_0^{\pi} \sin^{n-2}\theta_n\, d\theta_n.$$

The first factor is 2π and introducing

$$S_n \;=\; \int_0^{\pi} \sin^n\theta\, d\theta,$$

A_n becomes $A_n = 2\pi S_1 S_2 \ldots S_{n-2}$.

The integrals S_n may be evaluated using integration by parts with $u = \sin^{n-1}\theta$, $dv = \sin\theta\, d\theta$, $du = (n-1)\sin^{n-2}\theta\, d\theta$, $v = -\cos\theta$. The factor uv is zero over the integration limits giving

$$S_n \;=\; (n-1)\int_0^{\pi} \sin^{n-2}\theta\cos^2\theta\, d\theta \;=\; (n-1)S_{n-2} - (n-1)S_n,$$

yielding

$$S_n \;=\; \frac{n-1}{n}\, S_{n-2}.$$

The chain is terminated by $S_0 = \pi$ or $S_1 = 2$ as n is even or odd. Then

$$S_{2n} \;=\; \frac{\pi(2n-1)!!}{(2n)!!}, \qquad S_{2n+1} \;=\; \frac{2(2n)!!}{(2n+1)!!},$$

where

$$(2n)!! \;=\; 2n(2n-2)\dots 2, \qquad (2n+1)!! \;=\; (2n+1)(2n-1)\dots 1.$$

There is extensive cancellation in the product of the S factors leading to

$$A_{2n} \;=\; \frac{2^n \pi^n}{(2n-2)!!}, \qquad A_{2n+1} \;=\; \frac{2^{n+1}\pi^n}{(2n-1)!!}.$$

The second of these is exactly in the form (4). The two forms given here, however, emphasize the similarities between the even and odd cases.

The expression for A_{2n} derived without direct reference to the angles can be recovered by noting that

$$(2n-2)!! \;=\; (2n-2)(2n-4)\dots 2 \;=\; 2^{n-1}(n-1)!$$

and $A_{2n} = 2\pi^n/(n-1)!$, which shows that there is agreement between the two methods.

To round off this account, some numerical values of A_n and $I_n = A_n/n$, the volume of a hypersphere of unit radius are given.

n	A_n	n	A_n	n	I_n	n	I_n
2	2π	3	4π	2	π	3	$4\pi/3$
4	$2\pi^2$	5	$8\pi^2/3$	4	$\pi^2/2$	5	$8\pi^2/15$
6	π^3	7	$16\pi^3/15$	6	$\pi^3/6$	7	$16\pi^3/105$
8	$\pi^4/3$	9	$32\pi^4/105$	8	$\pi^4/24$	9	$32\pi^4/945$

Problem 253.1 – A Diophantine equation

Vincent Lynch

Given that $P = 2$, $Q = 1$, and $R = 7$ is a solution to the Diophantine equation

$$P^4 + 8P^2Q^2 + Q^4 \;=\; R^2,$$

use this to find further solutions.

Buy 2 shoes get 3rd $\frac{1}{2}$ price.

— Sign in a shoe shop window [sent by Ralph Hancock]

Solution 250.7 – Bernoulli numbers

Recall that the Bernoulli numbers, B_n, are defined for non-negative integers n as the coefficients of $x^n/n!$ in the Taylor expansion of $x/(e^x - 1)$:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

When is B_n a fraction with denominator 6?

Tommy Moorhouse

One way into this problem is the theorem proved independently by von Staudt and Clausen: for all $n > 0$

$$B_{2n} = I_n + \sum_{p-1|2n} \frac{1}{p}.$$

Here I_n is an integer (see T. M. Apostol, *An Introduction to Analytic Number Theory*, Chapter 12, Exercise 12 for a simple proof).

Since $2 - 1 = 1$ and $3 - 1 = 2$ we always have the terms $\frac{1}{2} + \frac{1}{3}$ in the sum. This means that if there are no other primes p such that $p - 1$ divides $2n$ then the denominator of B_{2n} is 6.

Inspecting the list of indices $2k$ we see that in most cases this is of the form $2p$ where p is a prime of the form $p = 6m + 1$. We can prove that such a term is divisible by $q - 1$, where q is a prime, only if $q = 2$ or $q = 3$. This follows since the only prime factors of $2p$ are 2 and p, but if $q - 1 = p$ then q must be divisible by 2 and hence not prime. Thus, since the only factors of $2p$ are $2, p$ and $2p$, if $q - 1|2p$ then $q - 1 = 2p$. But $2p + 1 = 12m + 3$ is divisible by 3 and so q cannot be prime. There is no prime $q > 3$ such that $q - 1|2p$. More generally, if p is prime but $2p + 1$ is composite then the denominator of B_{2p} is 6. We can thus say that p cannot be a Sophie Germain (SG) prime (a prime p such that $2p + 1$ is also prime).

Note that since $5 - 1 = 4$ the Bernoulli numbers B_{4k} cannot have denominator equal to 6 for any k. Considering primes of the form $p = Am + B$ where the greatest common divisor $(A, B) = 1$ we see that if $(Am, 2B + 1) \neq 1$ then the only primes q such that $q - 1|2k$ are 2 and 3. Dirichlet's result that any such sequence has an infinite number of primes shows that the list of Bernoulli numbers with denominator equal to 6 is infinite. However, a prime can lie in more than one such sequence, and not all the entries in the list of indices are of the form $2p$ where p is prime, so the question of ordering the indices $2p$ to get a formula for the graph seems difficult.

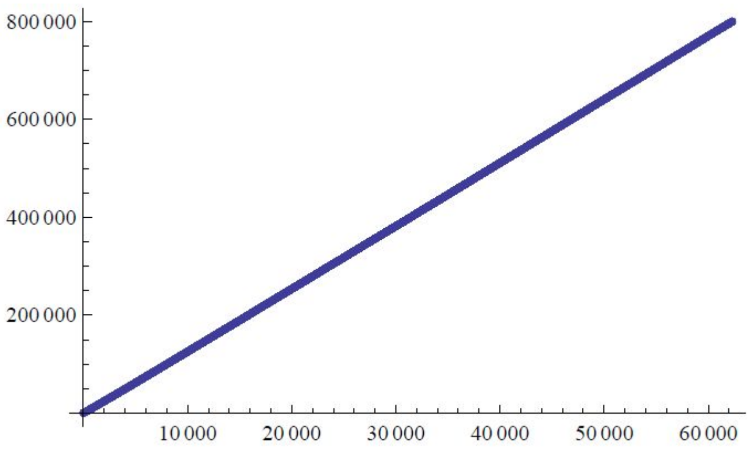
Other indices also work. For example, if p and Q are both not SG and $2pQ+1$ is composite then B_{2pQ} has denominator 6. For example, the denominators of $B_{98} = B_{2.7.7}$ and $B_{182} = B_{2.7.13}$ are 6.

To get a very rough idea of the behaviour of the true graph, let us take all the indices in the list to be of the form $2p_m$ where p_m is the mth prime. The prime number theorem gives (see Apostol, p. 80)

$$p_n \sim n \log n.$$

Then at nth position in the list we have $2p_n \sim 2n \log n$ so we plot $(n, 2n \log n)$. The gradient tends to a constant as $n \to \infty$, but not as rapidly as the full sequence as demonstrated in the graph in M500. By considering more carefully the conditions on the primes $p = An + B$, $(An, 2B+1) \neq 1$ we can hope to improve this first guess.

Tony Forbes writes. The graph mentioned above is the one that plots a point at $(x,\ y)$ if y is the x^{th} entry in the list of Bernoulli numbers that have denominator 6. In M500 **251** I stated that it looks almost as if it could be linear with slope about 11.8. Here it is again with more data. The slope is now about 12.9.



---

pizza

Pi revisited

Colin Davies

Re: Pi [Letters, M500 **250**, 14–16]. Edward Hoppus was a Victorian gentleman who devised a system for calculating the volume of a tree using a girth tape and a book of tables. We had a book of Hoppus's measure in the office of the timber company where I worked in the 1950s. That copy of the book has long gone, but Google have put it on a Google Books website, and I have copied two examples of 'how to find the volume of a round log by girting it.'

Example I Let the circumference of a tree, or piece of round timber (found by girting it), be 36 inches, one quarter of which is nine inches, the side of the square, and let the length of the piece be 40 feet: – what quantity of timber is in this piece?

Find 9 inches, the side of the square, and over against 40 feet stands 22 feet 6 inches, which is the quantity of timber contained in a piece that is 40 feet long and 36 inches round.

Example II Let the girt of a piece of timber be 75 inches, $\frac{1}{4}$ thereof is $18\frac{3}{4}$, for the side of the square. And let the length of the piece be 45 feet: – how much solid timber does this piece contain?

Find $18\frac{3}{4}$ inches the side of the square, and over-against 45 feet, the length of the piece, stands the solid content, viz. 109 feet, 10 inches, and 4 twelfth parts or 1/3 of an inch.

It took me a long time to understand how this method of measuring could work. The '22 feet 6 inches' and the '109 feet, 10 inches and $4/12 = 1/3$ of an inch' seem to be cubic quantities.

But apart from that confusion, what Hoppus appears to be saying is that a square whose circumference is the same as the circumference of the (presumably circular) log will have the same surface area as the cross section of the log. Which, if correct, means that he has found a way a squaring the circle.

Then I met a man in the Timber Trade who I, and everyone else, knew as 'Victor Serry' of Phoenix Timber. He pointed out to me that Hoppus's measure was based on the assumption that $\pi = 4$.

Outside the timber trade, Victor Serry used his original name as Victor Serebriakoff, and became well known as an early member of Mensa. See *Wikipedia* on Victor Serebriakoff. One of Serebriakoff's more memorable utterings was 'being intelligent is no guarantee against being stupid.'

Problem 253.2 – Quadratic

This is like finding Pythagorean triples but slightly different. Solve the quadratic $3x^2 + y^2 = z^2$ for positive integers x, y, z.

Problem 253.3 – Four logs

This interesting problem appeared on the internet forum NMBRTHRY some time ago. Let a, b, c, d be integers greater than 2 such that $(\log a)(\log b) = (\log c)(\log d)$. Is it always the case that at least one of $(\log a)/(\log c)$ and $(\log a)/(\log d)$ must be rational?

Problem 253.4 – Two colours

Is it possible to colour each point of $\mathbb{R}^2$ red or blue in such a manner that no continuous curve containing more than one point is monochromatic?

Problem 253.5 – Integral

Compute $\displaystyle\int \frac{d\theta}{\sin^6\theta + \cos^6\theta}$.

Problem 253.6 – Four sums

Prove the following

$$\begin{aligned}
\frac{1}{1\cdot 4} + \frac{1}{6\cdot 9} + \frac{1}{11\cdot 14} + \frac{1}{16\cdot 19} + \dots &= \frac{\pi}{15}\sqrt{1 + \frac{2}{\sqrt{5}}},\\
\frac{1}{2\cdot 3} + \frac{1}{7\cdot 8} + \frac{1}{12\cdot 13} + \frac{1}{17\cdot 18} + \dots &= \frac{\pi}{5}\sqrt{1 - \frac{2}{\sqrt{5}}},\\
\frac{1}{1\cdot 11} + \frac{1}{13\cdot 23} + \frac{1}{25\cdot 35} + \frac{1}{37\cdot 47} + \dots &= \frac{(2+\sqrt{3})\pi}{120},\\
\frac{1}{5\cdot 7} + \frac{1}{17\cdot 19} + \frac{1}{29\cdot 31} + \frac{1}{41\cdot 43} + \dots &= \frac{(2-\sqrt{3})\pi}{24}.
\end{aligned}$$

Problem 253.7 – Quintic roots

Show that the 27th powers of the roots of $x^5 + ax + b$ add up to $90(ab)^3$.

A popular Italian-style dish has radius z and height a. What is its volume? (Answer elsewhere in this issue.)

Contents M500 253 – August 2013

04.01

Front cover: The fifty-six 5-block partial Steiner triple systems.

M500 Winter Weekend 2014

The **thirty-third M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 3$^{\text{rd}}$ – Sunday 5$^{\text{th}}$ January 2014.

The theme is to be decided. Cost: £205 to M500 members, £210 to non-members. You can obtain a booking form from the M500 web site.

http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, please send a stamped addressed envelope to Diana Maxwell.

As well as a complete programme of mathematical entertainments we will have the usual extras. On Friday we will be running a pub quiz with Valuable Prizes, and for the ceilidh on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.