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The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For full details and a booking form see m500.org.uk/may.

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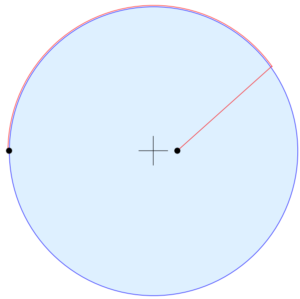
The M500 committee invites applications from members to join the Committee. Please apply to Secretary of the M500 Society not later than 1st October 2016.

Problem 270.1 – Pond

Tony Forbes



Imagine that you are an Open University mathematics tutor floating at coordinates $(1/6, 0)$ on the surface of a circular pond of radius 1 centered at $(0, 0)$. Your speed on land is nine times your speed in the water. What is your quickest route to point $(-1, 0)$?

After you have dried yourself off you remember that the students have recently learnt how to use differentiation to find maxima and minima. So you think it would be a good idea to set them an assignment question like the one above but with different parameters for the starting coordinates $(a, 0)$ and the land/water speed ratio r. Can you find numbers a and r such that the problem has a non-trivial solution which comes out nicely?

By 'comes out nicely' I mean that we avoid complicated expressions involving too many irrational numbers. I claim that my choice $a = 1/6$, $r = 9$ is amongst the nicest, and I would be interested if someone can prove otherwise. If you change r from 9 to 8 in our problem, the solution (but not necessarily the process of obtaining it) becomes trivial: swim directly to $(-1, 0)$. Similarly if you change a from $1/6$ to $1/7$.

Problem 270.2 – Bernoulli numbers

The *Bernoulli numbers* are defined by

$$B_n \;=\; \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^n.$$

Show that $2(2^n - 1)B_n$ is an integer.

Thanks to S. Ramanujan (*Notebook 2*, p. 53) for suggesting this problem.

Problem 270.3 – Cubes with even digits

(i) Positive integer cubes which have all of their digits even are quite common: 8, 64, 8000, 64000, 8000000, 8242408, 64000000, 6048464248, 6068404224, 6880682808, ... Show how to construct infinitely many.

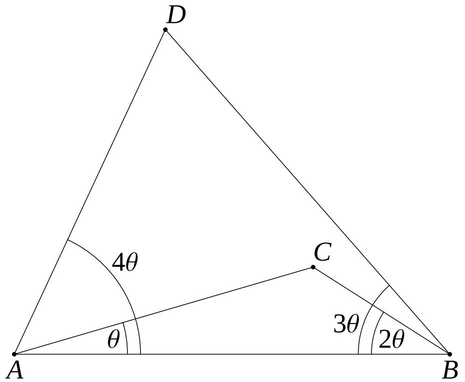
(ii) If we allow only non-zero even digits, the supply seems to dry up. Either show that 8 and 64 are the only examples, or find another one.

Solution 268.1 – Two triangles

Find two triangles ABC and ABD (so that they share a common base) such that

(i) angles CAB, CBA, DBA and DAB are in the ratio $1:2:3:4$, and

(ii) $|AB|$, $|AC|$, $|BC|$, $|DA|$ and $|DB|$ are positive integers.



Chris Pile

To find the sides of the two triangles consider them separately. Triangle ACB has sides in the ratio

$$\frac{|AC|}{|CB|} = \frac{\sin 2\theta}{\sin\theta} = 2\cos\theta.$$

The base is $|AB| = |AC|\cos\theta + |CB|\cos 2\theta$. For $\triangle ADC$, the sides are in the ratio

$$\frac{|DB|}{|AD|} = \frac{\sin 4\theta}{\sin 3\theta} = \frac{4(\cos\theta)(2\cos^2\theta - 1)}{4\cos^2\theta - 1},$$

and the base is $|AB| = |AD|\cos 4\theta + |DB|\cos 3\theta$.

For the triangles to exist, $7\theta < 180°$ and for the triangles to be acute, $\theta < 22.5°$. These ratios indicate that for integer sides, $\cos\theta$ should be rational and greater than about 0.9. Let $\cos\theta = s/t$, where s and t are integers. For $\triangle ACB$,

$$\frac{|AC|}{|CB|} = \frac{2s}{t} \quad\text{and}\quad |AB| = \frac{2s^2}{t} + \frac{t(2s^2 - t^2)}{t^2} = \frac{4s^2 - t^2}{t}.$$

For $\triangle ADC$,

$$\frac{|DB|}{|AD|} = \frac{a}{b} = \frac{4s/t(2(s/t)^2 - 1)}{4(s/t)^2 - 1} = \frac{4s(2s^2 - t^2)}{t(4s^2 - t^2)}.$$

Also

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \frac{4s^3 - 3st^2}{t^3},$$
$$\cos 4\theta = 1 - 8(\cos^2\theta)(1-\cos^2\theta) = \frac{t^4 - 8s^2t^2 + 8s^4}{t^4}.$$

Example 1: $\cos\theta = 11/12$; $\theta \approx 23.556°$.
We have

$$\frac{a}{b} = \frac{4s(2s^2-t^2)}{t(4s^2-t^2)} = \frac{4312}{4080} = \frac{539}{510},$$
$$d = a\cos 3\theta + b\cos 4\theta$$
$$= 359\,\frac{143}{432} - 510\,\frac{191}{2592} = \frac{30421}{216}.$$

Multiply all sides by 216 to give an integer triangle:

$$a = 116424, \quad b = 110160, \quad d = 30421.$$

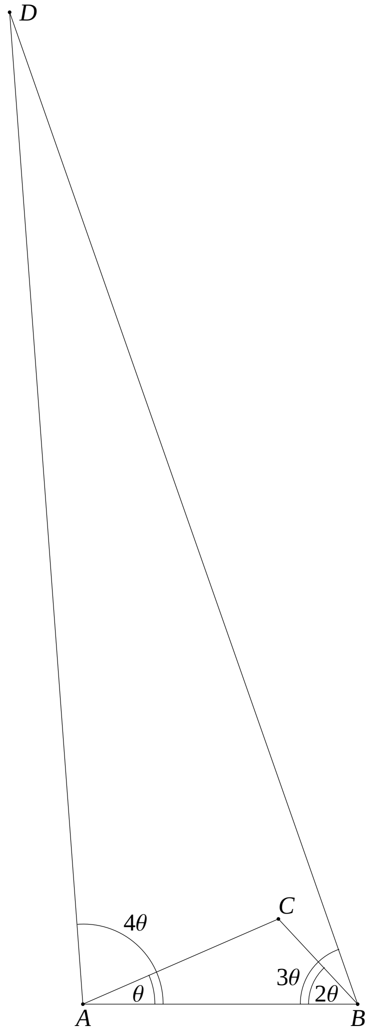
The sides of the small triangle are in the ratio $2s : t = 22 : 12$ and the base is $22 \cdot 11/12 + 98/12 = 85/3$. Multiply by 3 to give an integer triangle with sides 66, 36 and base 85.

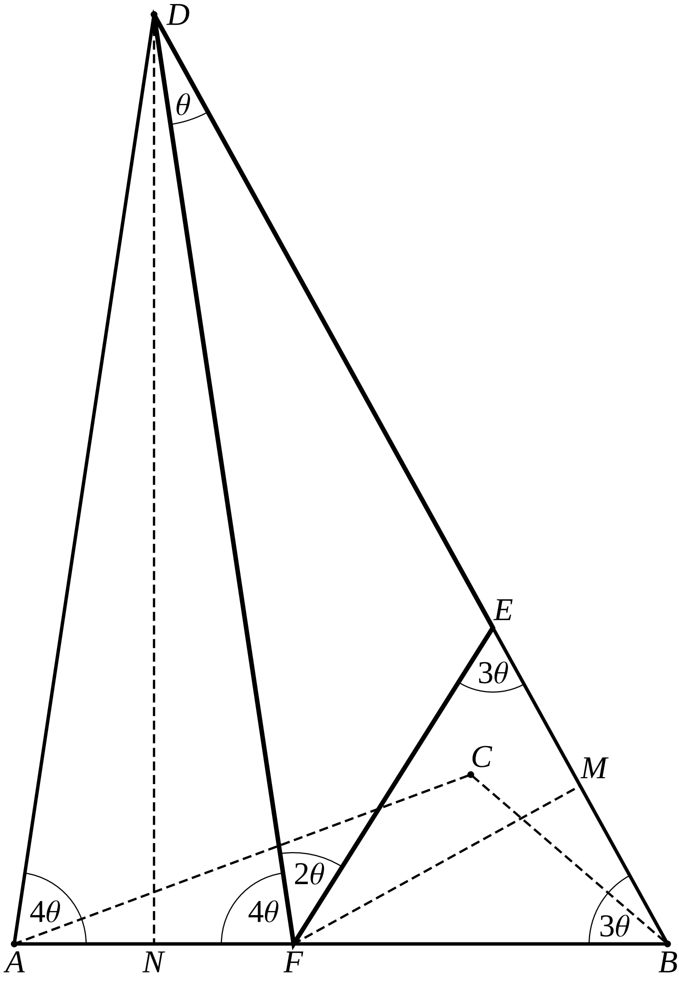
The bases of the two triangles have no common factors; so to give a common base, the sides of each triangle must be multiplied by the base of the other. Note that, as $4\theta \approx 94.2°$, $\triangle ADB$ is obtuse. The sides are

$$|AD| = 9363600, \quad |DB| = 9896040,$$
$$|AC| = 2007786, \quad |CB| = 1095156,$$

and the commonn base is

$$|AB| = 2585785.$$





Example 2: $\cos\theta = 15/16$.

We examine an alternative method for the construction of the triangle with angles 3θ and 4θ. In the diagram above, construct $\triangle DEF$ with angles θ and 2θ. Then sides DE and EF are in the ratio $(\sin 2\theta)/(\sin\theta) = 2\cos\theta$. That is,

$$\frac{|DE|}{|EF|} = \frac{30}{16} \quad \text{and hence} \quad |DF| = |DE|\cos\theta + |EF|\cos 2\theta = \frac{161}{4}.$$

Multiply all sides by 4 to give integers: $|DE| = 120$, $|EF| = 64$, $|DF| = 161$, and extend DE to B making $|EM| = |MB|$. Therefore $|FB| = |FE|$ and $\triangle DFB$ has angles θ and 3θ. Hence

$$|EM| \;=\; \frac{|DF|^2 - |EF|^2 - |DE|^2}{2|DE|} \;=\; \frac{495}{16}, \quad |EB| \;=\; \frac{495}{8}.$$

Multiply sides of $\triangle DBF$ by 8 to give integers: 1455, 512, 1288. Extend BF to A, making $|FN| = |NA|$. Then

$$|FN| \;=\; \frac{|DB|^2 - |DF|^2 - |BF|^2}{2|BF|} \;=\; \frac{195937}{1024}, \quad |AF| \;=\; \frac{195937}{512}.$$

Multiply by 512 to give integers: $|DB| = 744960$, $|DA| = 659456$, $|AB| = 458081$. The triangle can be combined with the original generating $\theta/2\theta$ triangle $(120 : 64 : 16)$ by multiplying each by the base of the other; i.e. the common base is $161 \cdot 458081 = 73751041$.

Example 3: $\cos\theta = 12/13$.

Similarly (to Example 1) we have

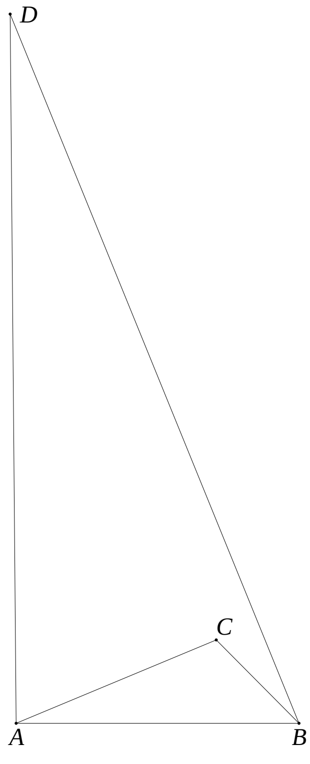
$$\begin{aligned}\frac{|AC|}{|CB|} &= 2\cos\theta = \frac{24}{13},\\ \frac{|DB|}{|DA|} &= \frac{\sin 4\theta}{\sin 3\theta} = \frac{5712}{5291}.\end{aligned}$$

Multiplying by 13 gives $\triangle ACB$ with integer sides $|AC| = 312$, $|CB| = 169$, $|AB| = 407$, and multiplying by 2197 gives $\triangle ADB$ with integer sides $|AD| = 11624327$, $|CB| = 12549264$, $|AB| = 4632263$. The two bases have no common factor; so the triangles must each be multiplied by the base of the other:

$$\begin{aligned}|AD| &= 4731101089, & |DB| &= 5107550448,\\ |AC| &= 1445266056, & |CB| &= 782852447,\end{aligned}$$

and the common base is

$$|AB| \;=\; 1885331041.$$



Vincent Lynch also found the solution given by Example 3, above.

Solution 268.7 – Interest

> I deposit 50 pence in a bank account that offers interest at 2 per cent per annum. How much will I have after 200 years if the interest is calculated (i) annually, (ii) every six months?

Tony Forbes

I can't remember whether we did this stuff at high-school. Nevertheless, I expect most of you know that there is a simple formula for compound interest calculations. After y years at interest r per cent your initial investment amount A will have matured to $A(1+r/100)^y$. This is assuming the interest is compounded annually. For the every-six-months option, we halve the interest rate and double the time, $A(1+r/200)^{2y}$. Applying the formula then gives the straightforward answer (1), below.

However, real life does not quite work like that. Most banks I know of round their calculations by discarding fractions of a penny—with the unfortunate consequence that your 50p investment will remain 50p for ever when the interest is added half-yearly. I suppose there might exist banks which offer more generous terms. Possibly as a promotional device, or as a special service for long-term small savers, one might round *upwards*. Or perhaps the banks will just round, i.e. round to the nearest integer number of pennies, in one of its many variations depending on how you handle 0.5. Anyway, I thought it would be interesting to see how our 50 pence investment behaves under various schemes.

(1) Exact, using the formula, rounding down only at the end of the 200-year period: (i) £26.24, and (ii) £26.76.

(2) Round down each time the interest is added: (i) £15.28, (ii) £0.50.

(3) Round up each time the interest is added: (i) £39.42, (ii) £48.95.

(4) Round interest with $0.5 \to 0$: (i) £24.20, (ii) £0.50.

(5) Round interest with $0.5 \to 1$: (i) £24.67, (ii) £30.34.

(6) Round interest towards an even integer; that is, for integer n, all numbers in the closed interval $[2n-0.5,\ 2n+0.5]$ are rounded to $2n$ whereas numbers in the open interval $(2n+0.5,\ 2n+1.5)$ go to $2n+1$: (i) £24.46, (ii) £0.50.

(7) Round capital plus interest towards an even integer; this differs from (6) whenever the capital is an odd number of pence and the interest is an odd multiple of 0.5p: (i) £24.29, (ii) £0.50.

(8) Round interest towards an odd integer; a simple way of doing this is to add 1, round towards an even integer as described in (6) and then subtract 1: (i) £24.29, (ii) £30.50.

(9) Round capital plus interest towards an odd integer: (i) £24.46, (ii) £30.05.

(10) Round interest with $0.5 \to 0$ or 1 chosen at random with probability 1/2; here we quote results from 100000 trials: (i) mean £24.40, standard deviation £0.17, (ii) mean £29.91, standard deviation £0.44.

In view of (1)–(10), above, I am slightly amazed at the immense complexity of the rounding process. Determining the best way to handle all those annoying halves is evidently a major problem in elementary arithmetic which has no doubt occupied the brains of some of our great thinkers. Someone told me that financial organizations use (6), rounding towards an even integer. A bizarre choice, surely, as it introduces a significant bias against odd numbers (like 51). If you don't believe me, look again at (6)(ii) and (8)(ii). In fact all of (4)–(9) must be unsatisfactory for the same kind of reason. And if you compare (6) and (7), say, you will see that the final result of a calculation can depend on when you actually do the rounding.

The cause of the trouble is the parity of 10, the base of our number system. All we need to do is adopt an odd base, 9, say, and the problem goes away. One half in base 9 is the non-terminating sequence $0.444\ldots$ and hence the rounding of 1/2 never arises. Thus, for example, 8.4 rounds to 8 and 8.5 rounds to 10. Obviously I am aware that changing the number base will cause trouble. So, as a more sensible alternative, I strongly recommend that in the interest of fairness we all adopt the only one of the procedures we have considered that is truly unbiased, method (10). *Round to the nearest integer if there is one; otherwise toss a coin and then round up if it is a head, down if it is a tail.*

Chris Pile agrees with (1) and offers two more alternatives. Recall that we failed to make it clear what type of interest was to be used.

(11) Simple interest (calculated exactly): £2.50, and of course the frequency doesn't affect the final result.

(12) Continuous compounding (calculated exactly); this is the limit as $n \to \infty$ of computing the interest at regular intervals n times per year:

$$\pounds \lim_{n\to\infty} 0.5 \left(1 + \frac{0.02}{n}\right)^{200n} \;=\; \pounds 0.50 \cdot e^4 \;=\; \pounds 27.30,$$

a significant improvement on (1)(ii).

Vincent Lynch also gives (1) and (5) as well as the realistic answer to the problem. Our 1p and 2p coins are likely to be withdrawn before very long, certainly within the next hundred years. Probably also other coins under a pound could go. So, you could ask an economics expert about the rate of inflation over a 200-year period. He couldn't give you an answer.

Solution 267.4 – Bernoulli numbers

Let n be a positive even integer, let $b(n) = (2\,n!)^{1/n}/(2\pi)$ and denote the nth Bernoulli number by B_n. Show that

$$\left\lfloor b(n)^n \sum_{k=1}^{\lfloor b(n)\rfloor} \frac{1}{k^n} \right\rfloor \;=\; \lfloor |B_n| \rfloor \;\text{ or }\; \lfloor |B_n| \rfloor - 1,$$

or find a counter-example. See page 1 for the definition of B_n.

Roger Thompson

I have found these counter-examples less than 10^{17}: $n = 7907318552180658$, $n = 18138477272887374$, $n = 22197553586127894$, $n = 40209779143948206$, $n = 46068253987800486$, $n = 47486531600097486$, $n = 72367464814087326$, $n = 82440878149934382$. The following explains how they were discovered.

Since $b(n)^n \displaystyle\sum_{k=1}^{\lfloor b(n)\rfloor} \frac{1}{k^n} < 1$ for $n < 14$, we only need to consider $n \geq 14$. Now

$$\begin{aligned} b(n)^n \sum_{k=1}^{\lfloor b(n)\rfloor} \frac{1}{k^n} \;&=\; b(n)^n \left(\zeta(n) - \sum_{k=\lceil b(n)\rceil}^{\infty} \frac{1}{k^n} \right) \;=\; |B_n| - b(n)^n \sum_{k=\lceil b(n)\rceil}^{\infty} \frac{1}{k^n} \\ &=\; |B_n| - B(n), \quad \text{say.} \end{aligned}$$

Since $b(n)$ is strictly increasing for $n > 6$, $B(n)$ has local maxima at values of n such that $\lfloor b(n+1)\rfloor > \lfloor b(n)\rfloor$. Such values occur at $n = 30$, 47, 64, 81, 98, $\ldots$. We have

$$\begin{aligned} \log \frac{b(n)}{n} \;&=\; \frac{\log 2}{n} - \log(2\pi) - \log(n) \\ &\quad + \left[\log(n) - 1 + \frac{1}{2n}\log(2\pi n) + \frac{1}{12n^2} - \frac{1}{360n^4} + \frac{1}{1260n^6} - \frac{1}{1680n^8} + \ldots \right], \end{aligned}$$

where the bracketed terms are the standard expansion for $\dfrac{\log(n!)}{n}$. Simplifying, we get

$$\log \frac{n}{b(n)} \;=\; \log(2\pi e) - \frac{1}{2n}\log(8\pi n) + O\left(\frac{1}{n^2}\right),$$

i.e. $n/b(n) \to 2\pi e \approx 17.079468\ldots$ as $n \to \infty$.

Assuming the fractional part of $b(n)$ is uniformly distributed on $[0, 1)$, which is borne out empirically, then for any $\varepsilon > 0$, there exists some (probably large) n such that $b(n) + \delta = N$ for some integer N and some positive $\delta \le \varepsilon$. Denoting $2\pi e$ by C, we have for small δ, $N = \lceil b(n) \rceil$, $n \approx NC$; so

$$\begin{aligned} B(n) &= b(n)^n \sum_{k=N}^{\infty} \frac{1}{k^n} \approx \sum_{k=0}^{\infty} \Big(\frac{N-\delta}{N+k}\Big)^{NC} \approx \sum_{k=0}^{\infty} \Big(\frac{N}{N+k+\delta}\Big)^{NC} \\ &\approx \sum_{k=0}^{\infty} e^{-(k+\delta)C} = \frac{e^{-\delta C}}{1-e^{-C}} \approx e^{-\delta C}(1+e^{-C}) > 1 \end{aligned}$$

if $\delta < e^{-C}/C \approx 2.23875593 \times 10^{-9}$. The fractional part of $|B_n| = \Delta$ say, can be arbitrarily small (see Erdős and Wagstaff (1980)). However, n may be colossally large in such cases. Fortunately, we find that Δ for B_{417534} is equal to

$$\begin{aligned} &1-\text{the fractional part of} \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{79}+\frac{1}{107}+\frac{1}{607}+\frac{1}{7879}+\frac{1}{32119}\right) \\ &\quad = \frac{67157225}{54535834912799982} \approx 1.231433 \times 10^{-9}, \end{aligned}$$

which will do very nicely. If $B(n) - \Delta > 1$, then

$$\left\lfloor b(n)^n \sum_{k=1}^{\lfloor b(n) \rfloor} \frac{1}{k^n} \right\rfloor = \lfloor |B_n| - B(n) \rfloor = \lfloor \lfloor |B_n| \rfloor + \Delta - B(n) \rfloor < \lfloor |B_n| \rfloor - 1.$$

We therefore require $e^{-\delta C}(1 + e^{-C}) > 1 + \Delta$; i.e. we require

$$\begin{aligned} \text{the fractional part of } b(n) &= 1 - \delta > 1 - \frac{\log(1+e^{-C}) - \log(1+\Delta)}{C} \\ &\approx 1 - \frac{e^{-C} - \Delta}{C} \approx 0.99999999783334426. \end{aligned}$$

Since $b(417534) = 24447.018627\ldots$, δ is far too big in this case, so we need to search for $417534m$ for which no additional primes q are such that $417534m$ is divisible by $q - 1$ (so that their fractional part is also Δ), and attempt to find much smaller δ values.

For large m, this is best done by turning the criterion on its head, i.e. there must be no factor f of $417534m$ for which $f + 1$ is prime, other than 1, 2, 6, 78, 106, 606, 7878, 32118, the factors of 417534 for which $f + 1$ is prime. This task is made simpler by considering factors of $417534p$, where

p is prime. Since $417534 = 2 \times 3 \times 13 \times 53 \times 101$, we need to check $p \times$ each of the 31 factors of 417534. This excludes most small primes. For example, the only permissible primes < 1000 are 97, 149, 223, 349, 619, 811 and 929. If any prime factor of m is excluded, we can reject m immediately.

The number of $417534m$ in a given interval which pass the above test on factors increases, albeit slowly, as m increases. This occurs because the majority of acceptable m are prime. In this case, the factors of $417534m$ are the $2m\{f\}$, where the $\{f\}$ are the set of 16 odd factors of 417534. All of $2m\{f\}+1$ are more likely to be composite for large mf, outweighing the lower likelihood that m is prime. This all suggests the best place to search is for large values of m.

A quick search around $n = 10^{16}$ reveals that in an interval of 10^{12}, there are about 32 acceptable values of m with the fractional part of $b(417534m) > 0.999$. Assuming uniform distribution again, this suggests there should be one with fractional part $> 1 - 2.166 \times 10^{-9}$ in an interval of

$$\frac{10^{12}}{32} \cdot \frac{10^{-3}}{2.166 \times 10^{-9}} \;\approx\; 1.5 \times 10^{16},$$

in reasonable agreement with the number actually found $< 10^{17}$. The following method searches far more efficiently than just looking at each odd m.

We examine the continued fraction for $R_S = \dfrac{417534}{S/b(S)}$, where S is the start of the search, and $R_\infty = 417534/C$. For example, if we start searching at $S = 10^{15}$,

$$R_\infty \;=\; 24446.5453556750764396\ldots, \quad R_S \;=\; 24446.5453556755380264\ldots,$$

with R_X between these two values for all $X \geq S$.

The fractional part of R_S has the continued fraction 1, 1, 5, 83, 2, 2, 4, 1, 43, …, and the fractional part of R_∞ has the continued fraction 1, 1, 5, 83, 2, 2, 5, 3, 1, 1, 1, …. The first four convergents for both are 1/1, 1/2, 6/11, 499/915, …. We use the denominators of these convergents to see how to predict the fractional parts of $b(417534m)$ for different m. The table on the next page gives the fractional part of dR_∞, dR_S for relevant d, showing where values agree in bold. We will need these values in a moment, where we will also use the convention $[a \pm b]$ to indicate the range of values in the first digit that does not agree for the two R values. Suppose $n = 417354m$, and $b(n) = N - \varepsilon$, where N is an integer, and $\varepsilon < 0.0012$ (we will see why this value is chosen in a moment). Then $mR_n = N - \varepsilon$. By definition,

$$b(417534(m+a)) \;=\; (m+a)R_{n+417534a} \;\approx\; N - \varepsilon + aR_{n+417534a},$$

with fractional part $-\varepsilon + A$, where A is the fractional part of $aR_{n+417534a}$, or $1 - \varepsilon + A$ if $A < \varepsilon$.

d	dR_∞ fractional part	dR_S fractional part
1	**0.545355675**07643963...	**0.545355675**53802644...
2	**0.09071135**015287927...	**0.09071135**107605289...
11	**0.9989124**2584083603...	**0.9989124**3091829092...
904	**0.001530**26910143421...	**0.001530**68637590896...
915	**0.00044**269494227024...	**0.00044**311729419989...
926	**0.999355**12078310628...	**0.999355**54821249081...

For $a = 1$, the fractional part α is much less than $1 - 0.0012 = 0.9988$.

For $a = 2$, α is $0.09071135[0 \pm 1]$, so is much less than 0.9988.

For $a = 11$, α is $0.9989124[3 \pm 1] - \varepsilon$, with $1 - \alpha = \varepsilon + 0.0010875[7 \pm 1]$, i.e. worse than ε. For successive multiples of 11, this worsening continues.

For $a = 915 - 11$, α is $0.001530[5 \pm 3] - \varepsilon$, $1 - \alpha = \varepsilon - 0.001530[5 \pm 3] < 0$, so gives a small value of the fractional part of $b(417534m)$.

For $a = 915$, α is $0.00044[3 \pm 1] - \varepsilon$, $1 - \alpha = \varepsilon - 0.00044[3 \pm 1]$, i.e. better than ε if $\varepsilon > 0.00044[3 \pm 1]$, but gives a small value of the fractional part of $b(417534m)$ otherwise.

For $a = 915 + 11$, α is $0.999355[3 \pm 3] - \varepsilon$, $1 - \alpha = \varepsilon + 0.000644[7 \pm 3]$.

We therefore start searching by finding an acceptable m with the fractional part ν of $b(417534m) > 0.9988$. Since $m - 11$ has a fractional part $\nu + 0.0010875[7 \pm 1] < 1$ for $\nu < 0.9989114$, we need to add 904 to m in this case, otherwise we add 915 to m. If the resulting $b(417534m)$ has a low fractional part, we add 11 to m then use whatever ε value this gives. This may be worse than the last ε used, but will always be less than 0.0012, since the increment for 11 is $0.0010875[7 \pm 1]$. We continue by repeatedly adding 915 or $915 + 11$ as appropriate. If we keep track of the predicted value of the fractional part of $b(n)$ using estimates based on R_S, R_∞, we don't need to evaluate $b(n)$ unless the estimates differ by more than some small threshold (5×10^{-5} seems adequate). Of course, if two estimates straddle an integer boundary, their difference will exceed the threshold. Once the threshold has been exceeded, we evaluate $b(n)$, resetting the estimated values to the actual value, then check if m is acceptable.

The first four convergents of R_S agree for $S > 10^{12}$. For n between 10^{12} and 10^{15}, it's best to choose some upper limit F, and use R_F instead of R_∞. For $n < 10^{12}$, checking all odd m is probably necessary.

We already showed that if $B(n) - \Delta > 1$, then

$$\left\lfloor b(n)^n \sum_{k=1}^{\lfloor b(n) \rfloor} \frac{1}{k^n} \right\rfloor \; < \; \lfloor |B_n| \rfloor - 1.$$

All the n values found are prime multiples of 417534. The following tables show the value of these primes, $B(n) - \Delta$ and $b(n)$.

n	p	$B(n) - \Delta$
7907318552180658	18938142887	1.0000000297068976...
18138477272887374	43441916761	1.0000000015677401...
22197553586127894	53163463541	1.0000000298645131...
40209779143948206	96303005609	1.0000000337175813...
46068253987800486	110334138029	1.0000000269527712...
47486531600097486	113730933529	1.0000000136774489...
72367464814087326	173321130289	1.0000000128670188...
82440878149934382	197447101673	1.0000000021692892...

n	$b(n)$
7907318552180658	462972169039301.9999999995726780...
18138477272887374	1062004788435248.9999999979251351...
22197553586127894	1299663022719835.99999999958190639...
40209779143948206	2354275794508250.99999999980750288...
46068253987800486	2697288509605263.99999999941142440...
47486531600097486	2780328424859966.99999999863415654...
72367464814087326	4237102872706908.99999999858670599...
82440878149934382	4826899526395583.99999999796035573...

Recall that $417534 = 2 \times 3 \times 13 \times 53 \times 101$. So we also need to check that none of the numbers in the first column of each of the two tables on the next page are prime. For each p, the smallest divisor is shown for each entry. Since none of the numbers are prime, the fractional part for each B_n is exactly 67157225/54535834912799982. However, even the lowest B_n is approximately $1.76561657 \times 10^{115965214218311873}$. The only other $n < 4 \times 10^7$ for which the fractional part of $|B_n| < e^{-C}$ is 9265494, with

$$1 \; - \; \text{the fractional part of} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{2339} + \frac{1}{7927} \right)$$
$$= \; \frac{857}{33485502918} \; \approx \; 2.5593 \times 10^{-8}.$$

This leads to an even more stringent requirement for δ, so this has not been investigated further.

$p =$	18938142887	43441916761	53163463541	96303005609
$2p + 1$	5	3	13	117703
$2p \times 3 + 1$	23	2689	1613	5
$2p \times 13 + 1$	277	3	31	5
$2p \times 13 \times 3 + 1$	7	17	7	7
$2p \times 53 + 1$	3	19	3	3
$2p \times 53 \times 3 + 1$	67	11	11	97
$2p \times 53 \times 13 + 1$	3	9511	3	3
$2p \times 53 \times 13 \times 3 + 1$	239	5	5	19973
$2p \times 101 + 1$	3	23	3	3
$2p \times 101 \times 3 + 1$	419	7	600577	5
$2p \times 101 \times 13 + 1$	3	4057	3	3
$2p \times 101 \times 13 \times 3 + 1$	89	9753853	17029	19
$2p \times 101 \times 53 + 1$	641	3	53951	5
$2p \times 101 \times 53 \times 3 + 1$	1523	8747	1217	11486347
$2p \times 101 \times 53 \times 13 + 1$	47	3	43	23
$2p \times 101 \times 53 \times 13 \times 3 + 1$	79	5	5	223

$p =$	110334138029	113730933529	173321130289	197447101673
$2p + 1$	7	3	3	7
$2p \times 3 + 1$	5	5	5	43
$2p \times 13 + 1$	5	3	3	11
$2p \times 13 \times 3 + 1$	1723	122663	7	5
$2p \times 53 + 1$	3	5	5	3
$2p \times 53 \times 3 + 1$	31	97	31	5
$2p \times 53 \times 13 + 1$	3	1249817	1361	3
$2p \times 53 \times 13 \times 3 + 1$	131	54601	149	19
$2p \times 101 + 1$	3	61	113	3
$2p \times 101 \times 3 + 1$	5	5	5	29
$2p \times 101 \times 13 + 1$	3	5	5	3
$2p \times 101 \times 13 \times 3 + 1$	11	23	23	5
$2p \times 101 \times 53 + 1$	5	3	3	3001
$2p \times 101 \times 53 \times 3 + 1$	7	7	13	5
$2p \times 101 \times 53 \times 13 + 1$	89	3	3	5
$2p \times 101 \times 53 \times 13 \times 3 + 1$	5347	116247049	16787077	127

Reference

P. Erdős and S. S. Wagstaff Jr (1980), 'The fractional parts of the Bernoulli numbers', *Illinois Journal of Mathematics*, vol. 24, no. 1, pp. 104–112.

Postscript

There should be primes p such that

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{79}+\frac{1}{107}+\frac{1}{607}+\frac{1}{7879}+\frac{1}{32119}+\frac{1}{417534p+1}\right) \;<\; 1,$$

$417534p+1$ is prime, but that there is no other factor f of $417534p$ for which $f+1$ is prime, other than 1, 2, 6, 78, 106, 606, 7878, 32118. Acceptable p are 2113, 2213, 2633, 2749, ... The first of these gives Δ for

$$B_{882249342} \;=\; \frac{4713582721153193}{48114204521774246409911826} \approx 9.796655\times 10^{-11}.$$

The following $n = 882249342p$ with p prime, with $B(n)-\Delta > 1$, with B_n having the above fractional part, and there is no factor f of n for which $f+1$ is prime, other than 1, 2, 6, 78, 106, 606, 7878, 32118, 882249342.

$$\begin{aligned} n &= 1951614983613709401546,\\ p &= 2212090041563,\\ b(n) &= 1142667284909195849\,43.9999999999740517\ldots,\\ B(n) &= 1.0000000376956147\ldots. \end{aligned}$$

The above $B(n)-\Delta-1$ is more than 98.5% of its maximum value, $e^{-C} = 0.0000000382367613\,3978$.

Problem 270.4 – Limit

Show that

$$\frac{(2\,n!)^{1/n}}{n} \;\to\; \frac{1}{e} \qquad \text{as } n\to\infty.$$

Problem 270.5 – Binomial coefficient sum

Integers n and k satisfy $k > n \geq 0$. Show that

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n \;=\; 0.$$

One way of getting this result is via the recursion formula for Stirling numbers of the second kind. However, we are really interested in a simple, direct proof, assuming such a thing exists.

An artist encounters mathematics

Eddie Kent

Emmanuel Radnitzky, known as Man Ray, was an American painter who moved to Paris in 1921. Living in the artist's quarter he knew Alice (Kiki) de Montparnasse, a model and character. His experiments in photography led to early work with the moving image – now well represented on You Tube. In 1929 he began an affair with Lee Miller, a surrealist photographer, who became his assistant. Since most of the artists and poets around sat for his portraits he became very well known; indeed some of his pictures of Kiki have become iconic.

Thus in 1934, Christian Zervos, critic and historian, commissioned him to photograph a collection of three-dimensional mathematical models at the Institut Henri Poincaré. These were of course nineteenth-century teaching aids but Zervos admired their interesting and beautiful shapes, and he published the pictures in *Cahiers d'Art* in 1936, as 'Crisis of the Object.' Man Ray, however, began entering the original photographs in Surrealist exhibitions. This was against the advice of his friend André Breton who had strict ideas about what constitutes surrealism.

With the arrival of the Nazis Man Ray returned home, to work in Hollywood which he said was the most surreal place on earth. While in America he reinvented himself, married a dancer and published a manifesto declaring that *La Photographie n'est pas l'Art*. On leaving Paris he had abandoned all his work, but in 1947 he returned for it. And now he began to see those mathematical pictures in a new light.

Breton suggested titles for them: 'The rose penitent,' 'Pursued by her hoop,' etc., but Man Ray went in a different direction, and used titles of Shakespeare plays. The exhibition he put on was called *Man Ray—Human Equations: A Journey from Mathematics to Shakespeare*.

In the Sotheby's catalogue Andrew Strauss wrote that Man Ray had wedded himself to the idea of working in different media in a new and exciting combination. 'Man Ray felt that refreshing new titles in English could add to their potential popularity and commercial appeal in his new environment.' The cover had a flap designed by Man Ray with 'To Be' written on it. You lifted the flap to read 'Continued Unnoticed.'

Now, long after Man Ray's death, this exhibition has been revived. The Phillips Collection of Washington DC have called it *Man Ray–Human Equations* and toured it extensively. There is a book *Man Ray: Human Equations* by Wendy Grossman (Hatje Cantz 2015) which contains all the photographs, as well as the paintings suggested by them.

(There might be some equations to upset mathematicians: for instance written on the blackboard that is shown in Shakespearean Equation, Julius Caesar, one can see $2+2 = 22$; also $a : A = b : B$ and $a : b = A : B$ but these must be thought of as an interface between mathematics and surrealism.)

Uniform acceleration

Tommy Moorhouse

A uniformly accelerated observer forms a crucial link between special and general relativity. This is because a uniformly accelerated observer is postulated to be equivalent to an observer situated in an homogenous gravitational field, and physics as seen by one is the same physics seen by the other.

In this article we will try to unravel the secrets of the uniformly accelerated observer, and find some curious insights into her world. Part of the problem is making clear which reference frames we are using, and understanding this unlocks a wide range of results.

The first question to get to grips with is: in whose world is the acceleration 'uniform'? To answer this, and give our investigation a solid framework, we define some key ideas. Our uniformly accelerated observer, $\mathcal{O}_a$, must feel the same acceleration at all times. The time in question is the time according to the observer's watch, or the observer's 'proper' time.

We also need to consider a second, unaccelerated observer, $\mathcal{O}_f$. This observer may be taken to be at rest at all times. One possible scenario we could develop is a trip to colonise some distant planetary system. A fixed observer remains on Earth while the colonists travel through space using a propulsion system that gives a constant acceleration g, the acceleration due to gravity at the Earth's surface.

At this point we must be clear that the acceleration is constant according to the accelerating observer. This means that in an interval $\Delta\tau$ of proper time the speed of the observer increases by $g\Delta\tau$, as measured by the accelerating observer by comparison with a sequence of inertial frames. The effect is the same (in a sense) as if the observer were at rest in a gravitational field.

We can use the relativistic law of addition of velocities (in this case speeds in the x-direction) in the rest frame of the unaccelerated observer. This observer sees $\mathcal{O}_a$ moving with an initial speed v, say. The accelerating observer agrees with this, and due to the constant acceleration says that $g\Delta\tau$ is added to the observed speed in the time $\Delta\tau$:

$$v + \delta v \;=\; \frac{v + g\Delta\tau}{1 + \dfrac{vg\Delta\tau}{c^2}}$$

so that, to first order in small quantities,

$$\Delta v = g\Delta\tau\left(1 - \frac{v^2}{c^2}\right).$$

Taking the limit as $\Delta\tau \to 0$ we have

$$\frac{dv}{1 - v^2/c^2} = g\tau + g\tau_0,$$

which is solved (taking $\tau_0 = 0$) by $v(\tau) = c\tanh(g\tau/c)$. Thus, according to the moving observer's proper time, the constant acceleration leads to a limiting speed of c. The world velocity of the accelerated observer can be found in terms of the proper time τ using

$$v(t) = \frac{dx}{d\tau}\Big/\frac{dt}{d\tau}$$

and $u^\mu u_\mu = 1$. Here $u^\mu = (c\,dt/d\tau, dx/d\tau, 0, 0)$. (You can find the components of u^μ explicitly, and the answer is used in the next paragraph.)

According to the fixed observer the coordinates of the accelerating observer, expressed in terms of that observer's proper time and with obvious initial conditions, are

$$\left(c^2 g^{-1}\sinh\left(\frac{g\tau}{c}\right), c^2 g^{-1}\cosh\left(\frac{g\tau}{c}\right), y, z\right).$$

This may be plotted on the fixed (ct, x)-plane. Drawing on the lines $x = \pm ct$ you can deduce that no information from the negative x-axis (in fact from a larger space–time region—you could try to characterize it) can ever reach the accelerating observer. There is an 'horizon' beyond which lies a region about which nothing can be discovered by the accelerating observer.

The fixed observer sees that the t-coordinate and x-coordinates of $\mathcal{O}_a$ are related by

$$x = c^2 g^{-1}\cosh\left(g^{-1}\sinh^{-1}\left(\frac{g\tau}{c}\right)\right) = c^2 g^{-1}\sqrt{1 + \frac{g^2t^2}{c^2}}.$$

The speed of the moving observer, as discerned by the fixed observer, is therefore

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1 + g^2t^2/c^2}}.$$

In the fixed frame the acceleration is not perceived to be constant, although at small times we can expand the denominator and differentiate with respect to t to find that the acceleration is approximately given by g. The speed asymptotically approaches c, as relativity demands.

Now we consider a coordinate frame for $\mathcal{O}_a$. Since u^μ has components $(\cosh(g\tau/c), \sinh(g\tau/c), 0, 0)$ we see that the vector

$$w^\mu = \left(\sinh\left(\frac{g\tau}{c}\right), \cosh\left(\frac{g\tau}{c}\right), 0, 0\right)$$

is orthogonal to u^μ (using the standard Minkowski metric). The line ρw^μ is orthogonal to the world-line of the moving observer, and so the coordnates (τ, ρ, y, z) are good coordinates in this frame. The moving coordinates are known as Rindler coordinates, and ρ may be interpreted as inverse acceleration. In these coordinates the accelerated observer asigns herself a fixed position in 'space', say (ρ, y, z), and the $\mathcal{O}_f$ coordinates are related to hers by

$$\begin{aligned}
ct &= \rho \sinh\left(\frac{g\tau}{c}\right), \\
x &= \rho \cosh\left(\frac{g\tau}{c}\right), \\
y &= y, \\
z &= z.
\end{aligned}$$

To find the metric in Rindler coordinates we note that

$$\begin{aligned}
cdt &= \sinh\left(\frac{g\tau}{c}\right) d\rho + \frac{g\rho}{c} \cosh\left(\frac{g\tau}{c}\right) d\tau, \\
dx &= \cosh\left(\frac{g\tau}{c}\right) d\rho + \frac{g\rho}{c} \sinh\left(\frac{g\tau}{c}\right) d\tau, \\
c^2dt^2 - dx^2 - dy^2 - dz^2 &= \frac{g^2\rho^2}{c^2} d\tau^2 - d\rho^2 - dy^2 - dz^2.
\end{aligned}$$

From now on we shall choose units such that $c = 1$ but hopefully the work above shows how c can be explicitly reinstated. In the Rindler coordinates the metric is diagonal:

$$g_{00} = g^2\rho^2, \quad g^{00} = (g\rho)^{-2}$$

and so on, and $\sqrt{-\det g_{\mu\nu}} = g\rho$. It is important to understand that this frame is not obtained from the fixed frame by a Lorenz transformation: it is a coordinate transformation of a more general kind.

We can calculate the Christoffel symbols quite easily since $g^{\mu\nu}$ is diagonal. We fnd

$$\Gamma^0_{01} \;=\; \rho^{-1}, \quad \Gamma^1_{00} \;=\; g\rho,$$

with all the others not related by symmetry vanishing. The covariant derivative can now be constructed, and we will use it shortly.

The above work can be applied to physical situations, and the electric field of a uniformly accelerated charge is considered in [1]. It turns out that the electric field lines emerge radially from the charge, but enter the 'infinite acceleration' horizon ($\rho = 0$) at right angles, so that the horizon behaves like the surface of an electrical conductor. There is an analogous situation close to a black hole.

References

[1] This article was inspired by a desire to understand the material in Chapter 2 of *Black Hole Physics* by V. P. Frolov and A. Zelnikov (Cambridge, 2011). This is a fascinating, modern exploration of the subject, with some frustrating typos.

[2] A good introduction to general relativity and coordinate transformations, using similar terminology to [1] is *A First Course in General Relativity* by B. Schutz (Cambridge, 2009).

Solution 268.5 – Factorization

> Factorize $2^{4n+2} + 1$. Hence or otherwise completely factorize $2^{58} + 1$.

Vincent Lynch

I thought this problem was going to be difficult, but then I remembered the factorization of $x^4 + 4$. And this is similar:

$$2^{4n+2} + 1 \;=\; \left(2^{2n+1} + 1\right)^2 - 2^{2n+2} \;=\; \left(2^{2n+1} + 1\right)^2 - \left(2^{n+1}\right)^2.$$

This is now the difference of two squares and hence factorises as

$$2^{4n+2} + 1 \;=\; \left(2^{2n+1} - 2^{n+1} + 1\right)\left(2^{2n+1} + 2^{n+1} + 1\right).$$

So

$$2^{58} + 1 \;=\; (2^{29} - 2^{15} + 1)(2^{29} + 2^{15} + 1) \;=\; 536838145 \times 536903681.$$

Solution 268.6 – Squares with even digits

Dave Wild

(i) Squares which have all of their digits even are very common: 0, 4, 64, 400, 484, 4624, 6084, 6400, 8464, 26244, 28224, 40000, ... Show how to construct infinitely many.

(ii) If we restrict ourselves to non-zero even digits, there still appear to be plenty: 4, 64, 484, 4624, 8464, 26244, 28224, 68644, 228484, 446224, 824464, 868624, 2862864, 8282884, 8868484, 22448644, 26646244, 44462224, 82228624, 82664464, ... Are there infinitely many?

(iii) On the other hand, squares with all digits odd seem to be very rare. Either show that 1 and 9 are the only examples, or find another one.

(a) If n is a non-negative integer, then $4 \cdot 10^{2n}$ are squares with all their digits even.

(b) If a digit d is repeated n times, where n is a non-negative integer, then this will be written as d_n. So, for example, $56_0 7_2 = 577$.

As $8^2 = 64$, $68^2 = 4624$, $668^2 = 446224$, $6668^2 = 44462224$, and $66668^2 = 4444622224$, it appears squares of numbers of the form $6_n 8$ may have the desired property. For $n > 0$,

$$\begin{aligned} 6_n 8^2 &= \left(\frac{20_{n+1} + 4}{3}\right)^2 = \frac{40_{2n+2} + 160_{n+1} + 16}{9} \\ &= \frac{40_{n-1}160_{n-1}16}{9} = 4_n 62_n 4. \end{aligned}$$

It may be helpful to look at a couple of examples. When $n = 1$,

$$68^2 = \frac{(200+4)^2}{9} = \frac{41616}{9} = 4624.$$

When $n = 4$,

$$66668^2 = \frac{(200000+4)^2}{9} = \frac{40001600016}{9} = 4444622224.$$

Therefore numbers of the form $4_n 62_n 4$ are squares which contain only non-zero even digits.

(c) If the square only contains odd digits then it must be the square of an odd number. We shall show that for any odd number greater than 3 the tens digit is even.

As $5^2 = 25$, $7^2 = 49$ and $9^2 = 81$ this is true for the remaining odd numbers less than 10. When a number is squared the only digits which affect the value of the tens digit are the last two digits. Therefore we only have to check the squares of the odd numbers between 10 and 100. These are of the form $10d + o$, where o is an odd digit.

But $(10d + o)^2 = 10(10d^2 + 2do) + o^2$. Since $10d^2 + 2do$ is even and any carried value from o^2 is even, the tens digit is even. Therefore 1 and 9 are the only squares with all odd digits.

Prime k-tuplets

Tony Forbes

Prime k-tuplets are like prime twins except that there might be more than two of them. More formally, given an integer $k \geq 2$, a set S of positive integers is a *prime k-tuplet* if (i) S consists of k primes, (ii) for every prime q, there exists m such that $p \not\equiv m \pmod q$ for all $p \in S$, and (iii) the difference between the largest and smallest elements of S is as small as possible. For small k, it is common practice to use its Latin equivalent rather than the number k itself; thus, for example, $\{11, 13, 17\}$ is a prime triplet, $\{11, 13, 17, 19\}$ is a prime quadruplet, and so on.

Older readers of M500 will be aware that I am always keen to promote activity concerned with the discovery of these things. Well, it happens that two important milestones have been passed. Last year saw the discovery by **Raanan Chermoni** and **Jaroslaw Wroblewski** of the first ever non-trivial prime 21-tuplet:

$$\{394338677302163715754576643 99 + d :$$
$$d = 0, 2, 8, 12, 14, 18, 24, 30, 32, 38, 42, 44, 50, 54, 60, 68, 72, 74, 78, 80, 84\}.$$

And more recently, the 1000-digit barrier was broken for $k = 6$. In March this year **Norman Luhn** found the titanic prime sextuplet:

$$\left\{ 28993093368077 \prod_{p \leq 2400,\ p \text{ prime}} p + 19417 + d : d = 0, 4, 6, 10, 12, 16 \right\}$$

(1037 digits). The primes were verified using PRIMO, an implementation by **Marcel Martin** of the elliptic curve primality proving method. In keeping with tradition (see M500 **146**, **154**, **189**, **220** and **226**), we have expanded Norman's discovery into decimal notation and put it on the front cover.

Contents

04.02

Front cover Six primes; see page 21.