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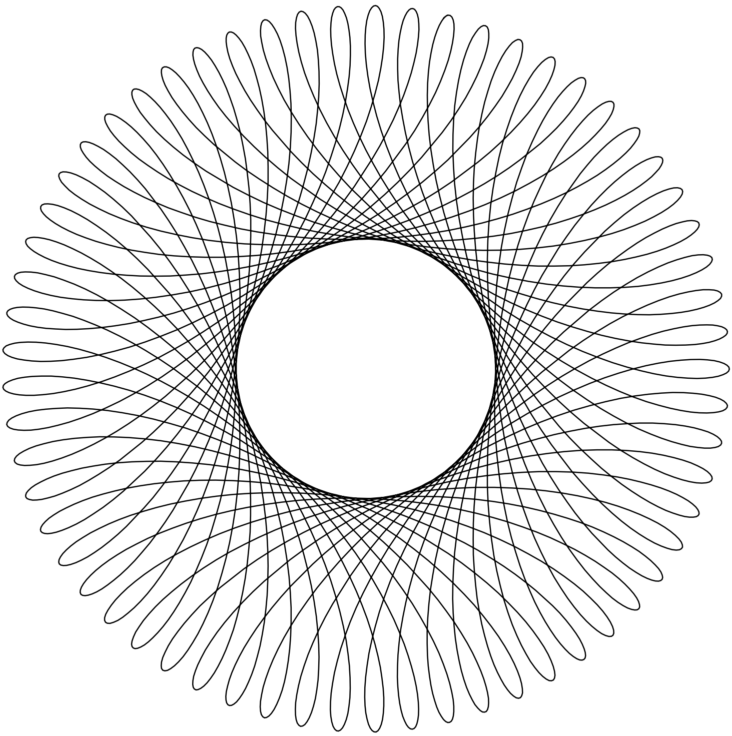

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Conway's and Kilminster's prime-producing fractions

Roger Thompson

Introduction

Here is a set of fractions with identifying letters.

A	B	D	H	E	F	I	R	P	S	T	L	M	N
$\frac{17}{91}$	$\frac{78}{85}$	$\frac{19}{51}$	$\frac{23}{38}$	$\frac{29}{33}$	$\frac{77}{29}$	$\frac{95}{23}$	$\frac{77}{19}$	$\frac{1}{17}$	$\frac{11}{13}$	$\frac{13}{11}$	$\frac{15}{14}$	$\frac{15}{2}$	$\frac{55}{1}$

These fractions have the amazing property that they can generate all the primes by following this simple algorithm. Start with $x = 2$. Scan the set of fractions from left to right until the first fraction f is found such that x is divisible by its denominator. Replace x by xf. If $x = 2^n$ for some integer n, write down n. Do this repeatedly. The numbers that you write down are the primes! To illustrate, here are the first twenty values of x, and the label of the fraction used to transform one value of x to its successor.

2 M 15 N 825 E 725 F 1925 T 2275 A 425 B 390 S 330 E 290 F
770 T 910 A 170 B 156 S 132 E 116 F 308 T 364 A 68 P 4

The twentieth value of x is $4 = 2^2$, so we write down 2. If we were patient enough, we would find the 70th value is 2^3, the 281st is 2^5, the 708th is 2^7, the 2364th is 2^{11}, and so on.

This set of fractions is an example of FracTran, a deliberately minimalistic computational model invented by John Conway. Its capabilities are equivalent to that of a Turing machine (just as all modern computers are). It demonstrates how the briefest of programs can achieve a relatively complex computational goal, provided we ignore the effort involved in writing, or indeed understanding the program, and we don't worry about its execution time.

The next section explains how the primes are produced (a very belated response to M500 Problem 226.1). We then go on to analyse another set of nine fractions invented by Devin Kilminster which also produces primes. These can be found on the Internet, along with an earlier version with ten fractions. Neither set seems to be analysed in a published paper.

How the primes are produced

The fraction used to transform a power of 2 is clearly M, since this is the only fraction whose denominator is a power of 2. We also find that the last fraction used to transform x to a power of 2 is always P. Clearly, the factors of x are conveying the information on which this prime generating 'machine' operates, with the factors of the numerators and denominators of the fractions dictating the transformations on successive x. Examining the factors of the fractions

A	B	D	H	E	F	I	R	P	S	T	L	M	N
$\frac{17}{7\cdot 13}$	$\frac{2\cdot 3\cdot 13}{5\cdot 17}$	$\frac{19}{3\cdot 17}$	$\frac{23}{2\cdot 19}$	$\frac{29}{3\cdot 11}$	$\frac{7\cdot 11}{29}$	$\frac{5\cdot 19}{23}$	$\frac{7\cdot 11}{19}$	$\frac{1}{17}$	$\frac{11}{13}$	$\frac{13}{11}$	$\frac{3\cdot 5}{2\cdot 7}$	$\frac{3\cdot 5}{2}$	$\frac{5\cdot 11}{1}$

we can see that each fraction apart from L and M has precisely one factor from the set $\{1, 11, 13, 17, 19, 23, 29\}$ in each of its numerator and denominator (where 1 is the entire value); so these values act as machine states (roughly, where the machine is in its program), with a particular fraction switching from one state to another. The fractions L and M are only invoked when we are in state 1 and x is even, and don't change state. Powers of $2, 3, 5, 7$ represent internal machine registers which can be incremented or decremented, and tested if they are zero or not. For instance, we will suppose we are in state 17, so x is divisible by 17. If the power of 5 in x is greater than 0, fraction B detects this, subtracts 1 from the power of 5, adds 1 to the powers of 2 and 3, and switches to state 13. If the power of 3 in x is greater than 0, fraction D detects this, subtracts 1 from the power of 3, and switches to state 19. If the powers of 3 and 5 in x are both zero, then fraction P switches to state 1.

Now x is always in the form $2^t 3^s 5^r 7^q p$, where p is one of {1, 11, 13, 17, 19, 23, 29}. Since x gets large very quickly, we will denote a particular x value by $(t, s, r, q)_p$, and refer to t as the value of the 2-register, s as the value of the 3-register, and so on. The divisibility test on a fraction's denominator is therefore replaced by comparison of the registers with prime power exponents in the denominators of the fractions.

A general program to print primes would look like this in pseudocode.

$n := 1$
loop
    $n := n + 1$
    if n is prime **then**
        Print n
    end if
end loop

Given the restricted capability of the machine, determining if n is prime could involve testing every divisor d from $n - 1$ downwards. If n is divisible by any $d > 1$, then n is not prime. Our program therefore looks like this.

```
n := 1
loop
    n := n + 1, d := n
    repeat
        d := d − 1
    until n is divisible by d
    d := d − 1
    if d = 0 then
        Print n
    end if
end loop
```

Since the machine cannot divide, we will need to achieve this by repeated subtraction.

```
procedure FindDivisor          • Sets d to the highest value < n such that
                               •   n is divisible by d
    d := n − 1
    repeat
        b := n                 • Finds b such that b < d, ad + b = n
        repeat
            b := b − d
        until b < d
        if b ≠ 0 then
            d := d − 1
        end if
    until b = 0
end procedure
```

The supplied fractions only have prime exponents of zero or 1, so cannot add or subtract anything other than 1, and can only test for something being zero, so we need to expand our routine even further. The idea is to keep two counters q, s with $q + s = d$, with s being incremented from 0 to d, and q being decremented from d to 0 at the same time as r is being decremented. Since we need to preserve n, we introduce another variable t such that $r + t = n$, so that when r is decremented, t is incremented. The t, s, r, q correspond to the 2-register, 3-register, 5-register and 7-register values defined earlier.

Before we look at the detailed manipulation of these registers, it might be helpful to look at an overall flow in terms of the fractions actually used at each step. For ease of understanding, the pseudocode uses n, d, b, a, with $n = ad + b$ as above, even though we really only have t, s, r, q. In the listing below, the register and state values are shown at various points. Fraction expressions such as $[(AB)^2C]^3$ mean successive fractions used are $ABABCABABCABABC$.

$d := 1$, $n := 1$
$(1, 0, 0, 0)_1$
loop
  $L^{d-1}M^{n-d+1}N$
  $(0, n-1, n, 0)_{11}$ • n has been incremented
  $d := n - 1$
  repeat
    $[(AB)^dS(EF)]^a$
    $(n-b, 0, b, d)_{13}$
    if $b \neq 0$ **then**
      $(AB)^bAD(HI)^nR(EF)^{b-1}T$
      $(0, 0, n, d)_{13}$ • d has been decremented
    end if
  until $b = 0$
end loop

Looking in more detail at register values, our program and FindDivisor procedure have had to go through extensive revision, but their foundations should hopefully still be visible. Both have been annotated with equivalent FracTran fractions and state numbers.

$r := 0$
$t := 1$ • $r + t = n = 1$, giving the initial value of $x = 2$
$q := 0$, $s := 0$
State1:
$s := t$
$r := t + 1$ • After t is cleared out below,
$q := 0$ • this effectively achieves (new) $n := n + 1$
$t := 0$ • $\equiv L^qM^{n-q}N : (n, 0, 0, q)_1 \to (0, n, n+1, q)_{11}$
• $(L, M$ achieves $q := 0$, $t := 0$, N increments $r)$,
• $q + s = d =$ (new) $n - 1$
Call FindDivisor
if $q = 0$ **then**
  Print t
else • n is divisible by $q + 1 > 1$, so n is composite,
• and x will be of the form 2^n7^q
end if
goto State1

```
procedure FindDivisor              • Sets q to the highest value < n
                                   •   such that n is divisible by q + 1
  State11:
  while s ≠ 0 do                   • Always entered with q = 0,
                                   •   so swaps q and s
    q := q + 1
    s := s − 1
  end while                        • ≡ (EF)^s T :
                                   •   (t, s, r, q)_11 → (t, 0, r, q + s)_13
  State13:
  if q ≠ 0 then
    q := q − 1                     • A
    State17:
    if r = 0 then
      if s ≠ 0 then
        s := s − 1                 • D
        q := q + 1                 • R (Compensates for q := q − 1 above,
                                   •   so that the net result is d := d − 1)
        repeat                     • Always entered with r = 0,
                                   •   so swaps r and t
          r := r + 1
          t := t − 1
        until t = 0                • Overall ≡ D(HI)^t R :
                                   •   (t, s, 0, q)_17 → (0, s − 1, t, q + 1)_11
        goto State11
      else                         • P
        exit procedure             • In State1
      end if
    else
      r := r − 1
      t := t + 1
      s := s + 1
      goto State13                 • B
    end if
  else
    goto State11                   • S
  end if
end procedure
```

Kilminster's fractions

Here is another set of fractions with identifying letters.

$$\begin{array}{ccccccccc} A & B & C & D & E & F & G & H & I \\ \dfrac{3}{11} & \dfrac{847}{45} & \dfrac{143}{6} & \dfrac{7}{3} & \dfrac{10}{91} & \dfrac{3}{7} & \dfrac{36}{325} & \dfrac{1}{2} & \dfrac{36}{5} \end{array}$$

These fractions can also generate all the primes as follows. Start with $x = 10$. Scan the set of fractions from left to right until the first fraction f is found such that x is divisible by its denominator. Replace x by xf. If $x = 10^n$ for some integer n, then n is prime. Do this repeatedly.

The eleventh value of x is 10^2, the 47th value is 10^3, the 197th is 10^5, the 501st is 10^7, the 1429th is 10^{11}, and so on. Examining the factors of the fractions:

$$\begin{array}{ccccccccc} A & B & C & D & E & F & G & H & I \\ \dfrac{3}{11} & \dfrac{7 \cdot 11^2}{3^2 \cdot 5} & \dfrac{11 \cdot 13}{2 \cdot 3} & \dfrac{7}{3} & \dfrac{2 \cdot 5}{7 \cdot 13} & \dfrac{3}{7} & \dfrac{2^2 \cdot 3^2}{5^2 \cdot 13} & \dfrac{1}{2} & \dfrac{2^2 \cdot 3^2}{5} \end{array}$$

we immediately see several differences from Conway's fractions. The first difference is that prime power exponents of up to 2 are present. The second is that the exponents of powers of 2 and 5 do not stay in step. When we examine the fractions in operation, we find that divisor d values start from 2 and increase, and that machine states are implicit, depending on which register combinations are zero. As we will see later, the 3- and 11- registers are used as counters in certain repetitive operations independent of n and d. We will therefore adopt a slightly different representation for x, namely (u, t, s, r, q, p), where u is the contents of the 2-register, …, p is the contents of the 13-register. The equivalent overall flow is as follows.

$n := 1$	
$(n, 0, n, 0, 0, 0)$	
loop	
$H^n I$	• n has been incremented
$(2, 2, n-2, 0, 0, 0)$	
$d := 1$	• d is incremented below,
$b := 0$	• so the first divisor is 2
repeat	
$(b+2, 2, n-2, 0, 0, d-b-1)$	• d, b from the previous iteration
$(BAA)^{n-2}(CA)^{b+2}D$	
$(0, 1, 0, n-1, 0, d)$	• d has been incremented to the
	• value for this iteration
if $d < n$ **then**	
$[DE^d F(CA)^d]^{a-1} DE^d$	• a, b determined from d
$(d, 0, ad, b, 0, 0)$	

if $b \neq 0$ **then**
$F(CA)^d DE^b G$
$(b+2, 2, n-2, 0, 0, d-b-1)$
else
DE^d
$(d, 0, n, 0, 0, 0)$
end if
else
DE^n
$(n, 0, n, 0, 0, 0)$ • n is prime
end if
until $b = 0$
end loop

The transformation $(BAA)^{n-2}(CA)^{b+2}D$ from $(b+2, 2, n-2, 0, 0, d-b-1)$ to $(0, 1, 0, n-1, 0, d)$ is purely mechanical, using the 3- and 11-registers to achieve a BAA cycle, terminating the repetition of BAA cycles when the 5-register reaches zero. The value of d at this point is held in the 13-register, so the C^{b+2} operations transform $d-b-1$ to $d+1$, the new value of d. The various operations after $(d, 0, ad, b, 0, 0)$ are also purely mechanical, so won't be considered further.

We will now consider how $[DE^d F(CA)^d]^{a-1} DE^d$ derives a, b, and how the transformation $(0, 1, 0, n-1, 0, d)$ to $(d, 0, ad, b, 0, 0)$ is accomplished. With the 2-, 5- and 11-registers all zero, i.e. $u = s = q = 0$, fractions A, B, C cannot be invoked. With the 3-register $t = 1$, D is invoked, setting $r = n$, and allowing E to decrement r and p repeatedly, counting iterations in u, s. If $d = n$, both r and p reach zero together, in which case we get the power of 10 from $u = s = n$, indicating a prime. If $d < n$, p will reach zero with $r \neq 0$ and $u = d$.

At this point, the registers are $(d, 0, md, n-md, 0, 0)$, where $m = 1$. F is invoked since $p = 0$, setting $r = n-d-1$, and invoking C and A alternately d times until u is reduced to zero again, and with $t-1$ from the last invocation of A. Another round of DE^d starts with $r - md$, where m is the number of times we've been round this loop. If n is divisible by r, then p and r will reach zero together when $m = a-1$. If $n = ad + b$ is not divisible by d, the registers after a iterations of $(DE)^d$ and $a-1$ iterations of $F(CA)^d$ will therefore be $(d, 0, ad, n-ad, 0, 0) = (d, 0, ad, b, 0, 0)$.

Reference

R. K. Guy, Conway's prime producing machine, *Mathematics Magazine*, Vol. 56, No. 1 (1983), 26–33.

Charged particles close to a modestly charged black hole

Tommy Moorhouse

In a previous article we set up the metric of a uniformly accelerated observer, the Rindler metric. Here we will show that the metric of the Schwarzschild black hole close to the event horizon can be approximated by the Rindler metric. We will generalize to a charged black hole metric and consider the motion of a charged particle initially at rest close to the event horizon, finding its energy and its speed when it is nudged away from the hole and accelerates towards infinity.

Approximating the Schwarzschild metric close to the horizon We will write the Schwarzschild metric, the metric of the space–time outside a spherical black hole, as

$$ds^2 \;=\; c^2K(r)dt^2 - K(r)^{-1}dr^2 - r^2 d\Omega^2.$$

Here $K(r) = 1 - 2GM/(rc^2)$ where G is the gravitational constant, M is the mass of the black hole, c is the speed of light, and $d\Omega^2$ is the volume element of a unit sphere, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

To analyse the situation close to the event horizon (following [1]) we set

$$r \;=\; r_S(1+y),$$

where r_S is the Schwarzschild radius $2GM/c^2$ and $y \ll 1$. Then $dr^2 = r_S^2 dy^2, K(r) \approx y$. Writing

$$\rho \;=\; \int_{r_S}^{r} \frac{dr}{\sqrt{K(r)}} \;\approx\; r_S \int_0^y \frac{dy}{\sqrt{y}}$$

we find the metric becomes

$$ds^2 \;\approx\; \kappa^2\rho^2 dt^2 - \rho^2 - \rho^2 d\Omega^2.$$

This is the Rindler metric we derived for a uniformly accelerated observer, and we see that close to the event horizon an observer feeling the gravitational pull of the black hole sees the same physics as if she were in a uniformly accelerated frame. Here $\kappa = c/2r_S$ is the 'surface gravity' of the black hole.

Introducing charge Next consider a charged particle in the gravitational field approximated by the Rindler metric. As we found in a previous article (or see [1, Chapter 2]) to the observer fixed in this frame the particle remains at rest, with its field lines penetrating the horizon at right angles. What is accelerating the charged particle (that is, how is it remaining at the same value of ρ)? Of course, this is an artificial question because we have simply postulated the existence of the accelerated particle, but suppose we give it meaning by requiring that a physical force is acting: the natural force in this case is due to an electric field. The field would have to be repulsive and strong enough to overcome the gravitational attraction. In contrast to the Newtonian model there is no combination of mass and charge such that the force on the charged 'test' particle cancels out everywhere (exercise for the reader—consider the Newtonian case and confirm that the forces balance everywhere for the right values of the masses and charges).

The Reissner–Nordstrom metric for a charged black hole There is a natural metric describing a charged black hole. This is the Reissner–Nordstrom metric (see, for example, [3] or [4]). Since gravity is so much weaker than the electromagnetic force (far from a black hole at least) we might be able to find a solution in which the charged particle is actually at rest outside the event horizon. Then our Rindler space analogy will have a natural home, although the field lines will no longer penetrate the horizon (they will be bent away by the electric field of the black hole).

The form of the Reissner–Nordstrom metric that we will use is

$$ds^2 = \frac{c^2\Delta(r)}{r^2}dt^2 - \frac{r^2}{\Delta(r)}dr^2 - r^2 d\Omega^2.$$

Here

$$\Delta(r) = r^2 - \frac{2GMr}{c^2} + \frac{GQ^2}{c^4}$$

and Q is the electric charge of the black hole, with the other constants keeping the units right. This is often written

$$\Delta(r) = r^2 - 2Mr + Q^2$$

in units ('geometric units'), such that $c = G = 1$, to streamline the notation. In these units both electric charge and mass have the dimensions of length. There are two 'horizons', at $r_\pm = M \pm \sqrt{M^2 - Q^2}$, and details can be found in [3] and [4].

Chandrasekhar's study [4] of the motion of charged particles in the Reissner–Nordstrom metric (also to be found in a slightly different form in [3]) centres on the Lagrangian (where e stands for the charge per unit mass of the test particle),

$$2L \;=\; 2L_g + 2\frac{eQ}{r}\frac{dt}{d\tau}$$

(where L_g is essentially the usual Lagrangian for the metric) because a charged particle does not fall freely (i.e. does not follow a geodesic). Defining the constant E, the energy per unit mass of the test particle, and considering only radial motion so that θ and ϕ are constant, we have

$$\begin{aligned}
\frac{\Delta(r)}{r^2}\dot{t} + \frac{eQ}{r} \;&=\; E, \\
\dot{r}^2 \;&=\; \left(E - \frac{eQ}{r}\right)^2 - \frac{\Delta(r)}{r^2} \;\equiv\; f(r,E), \\
\ddot{r} \;&=\; \frac{1}{2}\frac{df(r,E)}{dr}.
\end{aligned}$$

Here $\dot{t} = dt/d\tau$ and the last line follows by differentiating the line above with respect to the parameter τ. If an observer travelling with the particle measures a time interval $d\tau$, an observer far from the black hole will measure a time interval dt. The specific energy E of the test particle need not be equal to c^2, because, as we will see, the particle could be accelerated away from the black hole and have a non-zero speed 'at infinity'. In this case E represents the specific relativistic energy γc^2: more on this later. We note that if the particle is an electron, then e is very large, around 10^{21} in geometric units, while the mass of a solar mass black hole is around $M = 1.5 \times 10^3$m). We will also take $Q \ll M$. This simplifies the calculations while allowing a realistic non-extreme model to be developed. We would like to arrange that the particle is at rest just outside the outer horizon ($r_+ = M + \sqrt{M^2 - Q^2}$) by fixing Q to a suitable value. Let us parametrize Q in terms of M and e by setting $Q = \alpha M/e$ where α is a dimensionless parameter (note that e and E are also dimensionless in geometric units).

The electron at rest We will find values of E and r_0 such that $f(r_0, E) = 0$ and $f'(r_0, E) = 0$, for at these values the electron is at rest feeling no net acceleration. We can write

$$\dot{r}^2 \;=\; \left(E - \frac{\alpha M}{r}\right)^2 - 1 + \frac{2M}{r}.$$

Here we have used the fact that Q will be much smaller than M, and in fact we will have $Q \approx 10^{-18}$m for a solar mass black hole. The electron's charge is around 10^{-36}m, so Q represents around 10^{18} electrons or 0.1 coulomb.

This is a fairly modest charge for a black hole. Now, in this approximation we can solve for E and r to get the location and energy of the electron at rest ($\dot{r}^2 = 0$), and we find (exercise for the reader)

$$r_0 = \frac{2M\alpha^2}{\alpha^2 - 1}, \qquad E = \frac{\alpha^2 + 1}{2\alpha}.$$

Observe that α must be greater than 1, and that for large α the rest point is close to the horizon. The rest point is an unstable point, and any slight disturbance of the electron will result in it accelerating either into the black hole or towards infinity. If the electron is pushed gently away from the hole it will accelerate to the speed (another exercise—use the fact that the constant E can be written as $E = \gamma$ at large distances, and $\gamma = 1/\sqrt{(1 - v^2)}$ in geometric units)

$$v = c\,\frac{\alpha^2 - 1}{\alpha^2 + 1}$$

far away from the black hole. The higher the charge Q the closer the rest point to the horizon and the greater the final radial speed of the electron. The interested reader can check that while the electron is at its rest position its clock runs slowly, with $dt = \alpha d\tau$ (one second on the particle's clock lasts α seconds according to the distant observer).

Reissner–Nordstrom near-horizon coordinates The Rindler coordinates can be set up just outside the Reissner–Nordstrom outer horizon in a similar way to the Schwarzschild case, the main change being that the charge of the hole Q contributes to the surface gravity as well as to the electric field. The reader is invited to work out the details and find an expression for the surface gravity of the Reissner–Nordstrom black hole. In the foregoing approximation the contribution of Q can be neglected.

Energy extraction If an electron–positron pair is created (by vacuum fluctuations) in the region between the outer horizon and the radius $2M\alpha^2/(\alpha^2 - 1)$ such that the electron reaches the rest point with energy $E \geq (\alpha^2 + 1)/2\alpha$ then the electron can be accelerated to great distances from the black hole where its energy can be harvested. The positron would fall into the black hole with, in effect, negative energy, and energy would have been extracted from the black hole.

Conclusion It is possible to have a charged particle at rest and feeling no net force close to the horizon of a charged black hole, but the particle will be in a precarious state: a push outward and it will race off to infinity; inward and it will plunge into the black hole. Energy can be extracted from the black hole by this mechanism.

References

[1] V. P. Frolov and A. Zelnikov, *Black Hole Physics*, Cambridge, 2011. This book explores many of the core topics in black hole theory. The SI system is not used.

[2] B. Schutz, *A First Course in General Relativity*, Cambridge, 2009. A solid text for anyone wanting to get a practical introduction to GR, but does not cover the Reissner–Nordstrom metric.

[3] K. Thorne, C. Misner and J. Wheeler, *Gravitation*, Freeman, 1973. This is a comprehensive set of courses in GR, again with a non-SI choice of units.

[4] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford, 1983. The notation can overwhelm, and some of the topics are specialized, but a little patience can uncover a treasury of results here.

Solution 267.3 – Floor and ceiling

Show that for integer $n > 0$, these two functions are identical:

$$f(n) = 2^k \left(n - \frac{k(k-1)}{2} - 1\right), \quad \text{where} \quad k = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor,$$

$$c(n) = 2^r \left(n - \frac{r(r-1)}{2} - 1\right), \quad \text{where} \quad r = \left\lceil \frac{\sqrt{8n+1}-3}{2} \right\rceil.$$

Tony Forbes

This came about while I was trying to obtain

$$T_4(n) = c(n) + 1, \tag{1}$$

the conjectured number of moves required to solve the *Tower of Saigon* puzzle (recall that this is like the Tower of Hanoi except that it has four pegs instead of three [M500 **267**, 8–13]). I got as far as $f(n) + 1$ but could not immediately see how it would lead to (1). Most likely I had made a mathematical deviation on the way. Eventually I became enlightened; $f(n)$ is indeed equal to $c(n)$.

It is clear that $r = k$ except possibly when $8n + 1$ is an odd square, in which case the thing being floored or ceilinged will be an integer. So let us assume $r = (\sqrt{8n+1} - 3)/2$ is an integer. Then $k = r + 1$ and $n = k(k + 1)/2 = (r + 1)(r + 2)/2$. Consequently $f(n) = c(n)$.

Problem 271.1 – Complex exponential sums

Tony Forbes

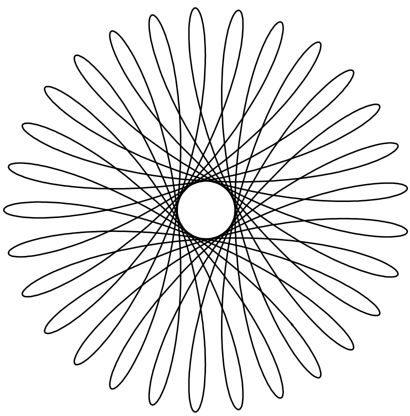
Consider the functions

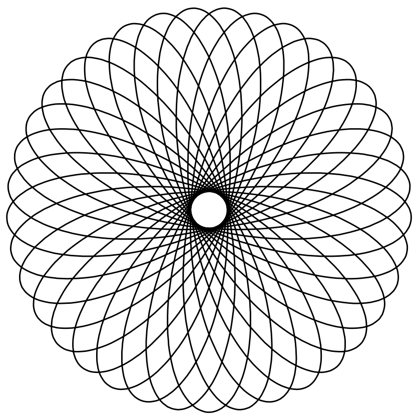
$$\begin{aligned} f(t) &= \frac{1}{2}e^{-18it} - \frac{2i}{3}e^{11it}, \\ g(t) &= ie^{23it} - \frac{5}{6}e^{-14it} \quad \text{and} \\ h(t) &= \left(\frac{1}{3} + i\right)e^{-17it} + \frac{i}{2}e^{44it}. \end{aligned}$$

In the left-hand picture, below, $f(t)$ is plotted on the complex plane for $0 \le t \le 2\pi$. As you can see, the result is a nice, symmetric, floral pattern with 29 petals. Show that the graph really does have 29-fold rotational symmetry.

Similarly, show that the graphs of $g(t)$ (right-hand picture) and $h(t)$ (front cover) have 37-fold and 61-fold rotational symmetry respectively. I did not include the axes in any of the diagrams because I was of the opinion that they would interfere excessively with the overall prettiness of the things.

The inspiration for this problem came from Sarah Hart's review in the *London Mathematical Society Newsletter*, issue 455, of *Creating Symmetry* by Frank Farris. The functions presented here are the results of an idle afternoon spent experimenting with MATHEMATICA.





Solution 269.5 – Coins

Simplify

$$\sum_{i=0}^{n} |n-2i| \binom{n}{i},$$

where n is a positive integer. Hence or otherwise determine

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\, \mathbb{E}[\Delta(n)], \quad \text{where} \quad \mathbb{E}[\Delta(n)] \;=\; \frac{1}{2^n} \sum_{i=0}^{n} |n-2i| \binom{n}{i}.$$

Coin tossers might recognize $\mathbb{E}[\Delta(n)]$ as the expected value of the discrepancy (i.e. |(number of heads) − (number of tails)|) resulting from randomizing the orientations of n coins.

Tommy Moorhouse

Outline

We will use a kind of accounting trick to show that

$$S_n \;=\; \sum_{k=0}^{n} |n-2k| \binom{n}{k} \;=\; n \binom{n}{n/2}$$

when n is even, and

$$S_n \;=\; \sum_{k=0}^{n} |n-2k| \binom{n}{k} \;=\; 2n \binom{n-1}{(n-1)/2}$$

when n is odd. We will then use Stirling's identity for the gamma function to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}[\Delta(n)] \;=\; \sqrt{\frac{2}{\pi}}.$$

A trick

The 'trick' we will use is to consider the expression

$$P(x,\theta) \;=\; (1+x\theta)^n \;=\; \sum_{k=0}^{n} x^k \theta^k \binom{n}{k},$$

where $\theta^m = 0$ for some m to be determined. We are not too concerned what kind of object θ may be (for our purposes it will simply be assigned the properties we need to simplify the work), just with the fact that it truncates the expansion of our expression. If we set $x = 1$ then $(1 + x)^n = 2^n$. We will use this below.

Two cases

Now we reduce the sum to something we can work with without having to worry about the modulus sign. A little trial suggests that taking the n odd and n even cases separately will simplify things. First we take n to be even: the case of n odd is simpler and will be left to the reader. We take $\theta^{n/2} = 0$, and the sum S_n becomes

$$S_n \;=\; 2 \sum_{k=0}^{n/2-1} n \binom{n}{k} - 4 \sum_{k=0}^{n/2-1} k \binom{n}{k}.$$

The first part of the sum does not require any tricks, and is

$$2n \left(2^n - \binom{n}{n/2} \right) \frac{1}{2} \;=\; n \left(2^n - \binom{n}{n/2} \right).$$

For the second sum we note that

$$\frac{d}{dx}(1 + x\theta)^n \;=\; \sum_{k=1}^{n/2-1} k x^{k-1} \theta^k \binom{n}{k},$$

which will give the sum we want if we set the terms $x^a \theta^b$ equal to 1. The derivative can also be expressed as

$$n\theta(1 + x\theta)^{n-1} \;=\; n \sum_{k=0}^{n/2-2} \binom{n-1}{k} x^k \theta^{k+1}.$$

Here we have used the fact that $\theta^{n/2} = 0$, and as above we will continue our analysis by setting all the terms $x^a \theta^b$ to 1. In this way we see that the second part of the sum is

$$2n \left(2^{n-1} - 2\binom{n-1}{n/2} \right),$$

using the fact that

$$\binom{n-1}{n/2} \;=\; \binom{n-1}{n/2-1}.$$

Putting this all together we have

$$S_n \;=\; n \left(2^n - \binom{n}{n/2} - 2^n + 4\binom{n-1}{n/2} \right) \;=\; n \left(4\binom{n-1}{n/2} - \binom{n}{n/2} \right).$$

This expression simplifies (a good exercise in combinatorial arithmetic) to give, for even n,

$$S_n \;=\; n\binom{n}{n/2}.$$

The result for odd n is

$$S_n \;=\; 2n\binom{n-1}{(n-1)/2}.$$

Stirling's asymptotic expression for $\Gamma(z)$

We use the fact that, for large n, when we can treat odd and even cases as essentially the same,

$$\binom{n}{n/2} = \frac{4}{n}\frac{\Gamma(n)}{\Gamma(n/2)^2} \text{ for even } n, \quad 2\binom{n-1}{(n-1)/2} \sim \frac{4}{n}\frac{\Gamma(n)}{\Gamma(n/2)^2} \text{ for odd } n,$$

to allow us to use Stirling's formula

$$\Gamma(z) \;\sim\; z^z e^{-z}\sqrt{2\pi/z},$$

$$\frac{1}{\sqrt{n}}\,\mathbb{E}[\Delta(n)] \;\sim\; \frac{n}{2^n\sqrt{n}}\;\frac{4}{n}\;\frac{n^n e^{-n} n\sqrt{2\pi}}{2(n/2)^n e^{-n}(2\pi\sqrt{n})}.$$

The n-dependent terms cancel in this limit to give

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\,\mathbb{E}[\Delta(n)] \;=\; \sqrt{\frac{2}{\pi}}.$$

Reference

[1] John W. Dettman, *Applied Complex Variables*, Dover, 1965. Includes a derivation of Stirling's asymptotic formula for $\Gamma(z)$, as do many other standard texts.

Problem 271.2 – Two-digit squares

(i) Squares which are composed of only two distinct digits seem to be very common: 16, 25, 36, 49, 64, 81, 100, 121, 144, 225, 400, 441, 484, ... Show how to construct infinitely many.

(ii) Squares which are composed of only two distinct non-zero digits are somewhat rarer. Here they are up to 10^{14}: 16, 25, 36, 49, 64, 81, 121, 144, 225, 441, 484, 676, 1444, 7744, 11881, 29929, 44944, 55225, 69696, 9696996, 6661661161. Are there infinitely many?

Reversed cheque

Andrew Pettit

Problem

[Martin Gardner, *My Best Mathematical and Logic Puzzles*] Mr Brown cashes a cheque but the clerk reverses the dollars and cents. After spending 5 cents (on a newspaper) Mr Brown has twice the amount of his cheque. How much was the cheque for?

Solution

Let Mr Brown's cheque be for $\$d.c = (100d + c)$ cents. Hence Mr Brown is given $\$c.d$ by the teller $= (100c + d)$ cents. After buying the newspaper he has $(100c + d - 5)$ cents which is twice $(100d + c)$ cents; i.e. $98c - 199d = 5$.

Applying the Euclidean Algorithm to find $\gcd(199, 98)$,

$$98 \;=\; 32 \cdot 3 + 2, \quad 3 \;=\; 2 \cdot 1 + 1.$$

Hence 199 and 98 are co-prime. Reversing the process:

$$1 \;=\; 3 - 2, \quad 2 \;=\; 98 - 32 \cdot 3, \quad 3 \;=\; 199 - 98 \cdot 2.$$

Hence

$$1 \;=\; 3 - (98 - 32 \cdot 3) \;=\; 33 \cdot 3 - 98 \;=\; 33(199 - 2 \cdot 98) - 98 \;=\; 33 \cdot 199 - 67 \cdot 98$$

$$\Rightarrow\; 165 \cdot 199 - 335 \cdot 98 \;=\; 5$$

is a particular solution to the equation $ac + bd = 5$. Let $a = 98$ and $b = 199$; then $98c - 199d = 5$ with $c = -355$ and $d = 165$. By Theorem 2.9 in *M381*,

$$98(-335 + 199t) + 199(165 - 98t) \;=\; 5, \quad t \in \mathbb{Z},$$

is a general solution to the same equation.

The cents must clearly be positive and less than 100; so the only valid value of c is 63, which has $t = 2$. From this d can be deduced as 31 to give an answer of \$31.63. But why does d not come out of the general solution to the Diophantine equation?

Erratum

In M500 **270** the expression for $\log(n/b(n))$ on page 8 should read

$$\log \frac{n}{b(n)} \;=\; \log(2\pi e) - \frac{1}{2n} \log(8\pi n) + O\left(\frac{1}{n^2}\right).$$

Solution 269.8 – Integral

Show that

$$I_0 = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\,dx = \frac{\pi\left(\cos(1/\sqrt{2}) + \sin(1/\sqrt{2})\right)}{\sqrt{2}\,e^{1/\sqrt{2}}} \approx 1.54428.$$

Just in case it might be relevant, recall that in Problem 242.1 we asked you for a proof that $\int_{-\infty}^{\infty} (\cos x)/(1+x^2)\,dx = \pi/e$.

Tommy Moorhouse

We wish to find the integral I_0. The most straightforward way, discussed in standard texts such as [1], seems to be to integrate

$$I = \int_C \frac{e^{iz}}{1+z^4} dz$$

over a semicircular contour C in the complex z-plane, noting that the poles of the integral occur at the zeros of z^4+1 and are

$$\rho = \exp(i\pi/4), \quad -\rho, \quad \omega = \exp(3\pi i/4) \quad \text{and} \quad -\omega,$$

and the denominator can be written as $(z-\rho)(z+\rho)(z-\omega)(z+\omega)$. The real part of I will be the solution we seek.

We close the contour, which runs from $-R$ to R along the real axis, in the upper half-plane, so that the integral over the semicircle tends to zero as R goes to infinity. The value of the integral is then $2\pi i$ times the sum of the residues of the integrand at the poles in the upper half plane, namely ρ and ω. To calculate the residues we use the cover-up rule to find that

$$I = 2\pi i \left(\frac{e^{i\rho}}{2\rho(\rho^2-\omega^2)} + \frac{e^{i\omega}}{2\omega(\omega^2-\rho^2)}\right).$$

Using $\rho\omega = -1$, $\rho^2 = i$ and $\omega^2 = -i$ we find $I = \pi/2\,(\rho e^{i\omega} - \omega e^{i\rho})$. Using

$$\rho = \frac{1}{\sqrt{2}}(1+i), \quad \omega = \frac{1}{\sqrt{2}}(-1+i)$$

we arrive at

$$I = \frac{\pi e^{-1/\sqrt{2}}}{\sqrt{2}} \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right).$$

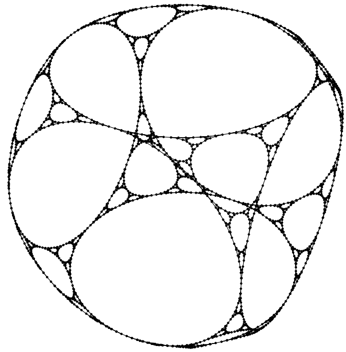
Since I is real we have also calculated I_0.

Reference

[1] M. J. Ablowitz and A. S. Fokas, *Complex Variables*, Cambridge, 2003. The essentials of complex variable theory can be found here, clearly explained.

Problem 271.3 – Truncation

Ralph Hancock



M500 **244** had a truncated truncated truncated truncated truncated cube on the cover, which suggests a problem.

Vlad the Truncator presented each of his courtiers with a perfect cube of sardonyx weighing 100 vrk and a small file, and gave them this order.

'You are to truncate this cube according to the rule that the first cuts shall turn the square sides into regular octagons, and subsequent ones shall maintain these sides as regular polygons. I require you to answer these questions in advance.

1. What weight of sardonyx will be left after 5 such truncations?

2. If the truncation is continued infinitely, towards what figure will the decreasing weights converge?

If the weight of your piece does not conform to your prediction, you will incur the attentions of the Court Truncator.'

One courtier survived the test unscathed. What were his answers?

Tony Forbes writes. As there is quite a lot at stake I had better clarify the truncation methodology. Let $t(E)$ denote the proportion of polyhedron edge E that is removed from each end of E. We assume always that the same amount is taken from both ends to leave a reduced edge of length $|E|(1-2\,t(E))$. Looking at the MATHEMATICA code what I wrote to create the picture on the cover of M500 **244**, I appear to have set

$$t(E) \;=\; \frac{2}{4+\sqrt{2+2\cos(2\pi/p_E)}+\sqrt{2+2\cos(2\pi/q_E)}},$$

where p_E and q_E are the numbers of sides of the two polygons that have edge E in common. Unfortunately that doesn't satisfy the requirements of Vlad the Truncator. So instead I suggest you work with

$$t(E) \;=\; \frac{1}{2+\sqrt{2+2\cos(2\pi/\max\{p_E,q_E\})}}.$$

The (truncated)^5 cube so constructed is illustrated above. Initially we have $p_E = q_E = 4$; hence $t(E) = 1/(2+\sqrt{2})$, the correct value for the six octagons and eight triangles of the Archimedean truncated cube.

Problems for Metagrobologists

A Collection of Puzzles with Real Mathematical, Logical or Scientific Content

by David Singmaster

World Scientific, 2016, ISBN 9789814663649, £18.00 (soft cover)

TF writes. The book is a collection of 221 problems collected by the author since 1987. Some have already appeared in various publications, but in his book David usually manages to add a wealth of further interesting material. Many problems have not been published previously. Although the chatty style might suggest that the intended reader is the person in the street, most of the problems actually have a significantly non-trivial mathematical or scientific content. Indeed, a fairly tricky conundrum involving Diophantine equations begins 'Jessica and Sophie were playing together with matchsticks.' Anyway, I am sure *Problems for Metagrobologists* will appeal to readers of this magazine.

The thirteen chapters group the problems under headings covering arithmetic (including base 10 puzzles), geometry, logic, the subtle interplay between numbers and (English) words, combinatorics, and various forms of applied mathematics such as physics and geography.

An important feature is the provision of complete *solutions*. This is really useful. Some of the problems I found truly baffling and I have to admit that I gained a great deal of enlightenment by cheating. And to save you the trouble of looking up that long word, David explains all in his Introduction.

Problem 271.4 – Fractions

I (TF) have been told that a surprising (i.e. positive) number of undergraduates doing mathematics courses confuse the Farey mean, $(a+c)/(b+d)$, of two fractions, a/b and c/d, with their sum, $a/b + c/d$. Of course if $a = -b^2c/d^2$, the two are the same. However, when a, b, c and d are positive they are nothing like each other.

On the other hand, if we compare the Farey mean with the ordinary mean, the two are usually much closer. How close? For example, if the fractions are 3/4 and 6/5, the means are respectively 1 and 39/40. Is there a simple criterion to indicate when one mean is greater than, less than or equal to the other?

Problem 271.5 – Fibonacci's geese

Ralph Hancock

A pair of Egyptian geese arrived in Kensington Gardens twelve years ago. There are now just over 100 of them, numbers fluctuating as early chicks are hatched and mostly eaten by gulls. They are long-lived birds, and the original pair are still alive.

Suppose there were no casualties. We start with two adults. Egyptian geese take two years to reach breeding age. Then they breed twice a year, at any time of year, and have six chicks per brood. If there are equal numbers of male and female offspring and all pair up as soon as they can, how many will there be after twelve years?

Roots and the division of angles

Peter L. Griffiths

To obtain the two square roots of i. From the Cotes Formula $\cos 90° + i \sin 90° = 0 + i = i$, for the first square root, divide the angle 90° by 2:

$$\cos(90°/2) + i\sin(90°/2) \;=\; i^{1/2},$$

which is the first square root.

For the second square root, you add 360° to the first angle, 90°, and then divide the sum by 2: $(90 + 360)/2 = 225$. This gives $\cos 225° + i \sin 225° = i^{1/2}$ as the second square root.

For investigating the possibility of a third square root, $(90 + 720)/2$ exceeds 360; so there will be no third square root. The excess over 360 is 45, which is a return to the angle of the first square root.

I can repeat this exercise for cube roots as follows. For the first cube root divide angle 90° by 3.

$$\cos(90°/3) + i\sin(90°/3) \;=\; i^{1/3},$$

which is the first cube root. For the second cube root you add 360 to the first angle 90, and then divide by 3: $(90 + 360)/3 = 150$. This gives $\cos 150° + i \sin 150° = i^{1/3}$ as the second cube root. For the third cube root, you add 720 to the first angle 90, and then divide by 3: $(90 + 720)/3 = 270$. This gives $\cos 270° + i \sin 270° = i^{1/3}$ as the third cube root. For investigating the possibility of a fourth cube root, $(90 + 1080)/3$ exceeds 360, so there will be no fourth cube root. The excess over 360 is 30 which is a return to the angle of the first cube root.

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04.00

Front cover Graph of $\left(\frac{1}{3}+i\right)e^{-17it}+\frac{i}{2}e^{44it}$ (see page 13)