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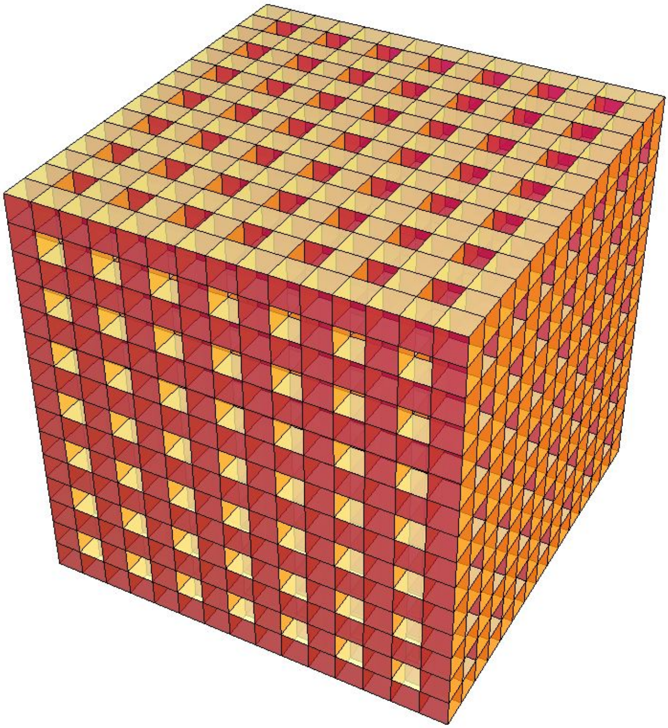
# M500 274



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**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: m500.org.uk.

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**The Revision Weekend** is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's web site.

**The Winter Weekend** is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's web site.

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# Resistance

## Tommy Moorhouse

This set of investigations is concerned with resistance and, in particular, combinations of resistors.

**The Y – Δ formula** There is a formula for converting an arrangement of resistors as three legs (Y) into an equivalent set of resistors arranged as the edges of a triangle (Δ). The resistance across each of the edges of the Δ is the same as that between the corresponding ends of the legs of Y.

Find the general formula for the resistances of the Δ in terms of those of the Y using the series and parallel resistance formulae. Apply the formula to find the resistances between any two vertices of a tetrahedron where all the resistors forming the edges have the same value. It is possible to generalize to the case where the resistors have different values, if you have the energy.

**Generalized Y – Δ rules?** One might wonder whether there is a higher dimensional Y – Δ rule relating the tetrahedron to four resistors branching from a common point? For the general case the answer is no and it is interesting to understand why.

To get an idea of the difficulty consider a set of four resistors arranged as the edges of a square, and a second set of four resistors branching from a point to form an X. What is wrong with this argument: both arrangements have four resistors and four vertices, so there must be a unique correspondence between the values in each arrangement. [Hint—how many ways are there of connecting two vertices, and how many independent conditions does this impose on the resistor values?]

**Cube** Consider a cubic net of resistors; that is, a set of twelve equal resistors (resistance $R$) forming the edges of a cube, with three resistors meeting at each vertex. If we connect the positive pole of a cell to a given vertex then there are three inequivalent locations for the negative pole. What is the resistance of each of these three configurations in terms of $R$?

**Further resistance** Consider the same problem with the cube replaced by another regular solid, such as a tetrahedron, octahedron and so on. How many configurations are possible for the relative locations of the two poles, and what is the resistance of each? Are there any useful tricks for finding the resistance of a sub-network of resistors? Try the simplest cases and see where they lead.

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***Q***: What is a proof? ***A***: 0.5 per cent alcohol.

# The Winter Weekend Hat Problems

Hat puzzles are logic problems that date back to as early as 1961. A number of players are each wearing a hat, which may be of various specified colours. Players can see the colours of at least some other players' hats, but not that of their own. As a promotional exercise we offered generous discounts for the 2017 M500 Winter Weekend to newcomers who correctly answered either (or both) of the following problems.

> **Puzzle 1** The simplest hat problem involves two players who can see each other's hats but not their own. Each hat may be either red or green. The task is for both players to simultaneously announce the colour of their hat (which of course they cannot see). What strategy should they adopt to ensure at least one of them gets the answer right?
>
> **Puzzle 2** Ten players stand in a line where they can see only the colours of the hats of the people standing in front of them. Once again each player is wearing either a red or green hat. The person at the back, who can see all the hats apart from his[1] own, announces his hat colour first. The person in front of him is the next to announce his hat colour and so on down the line until the player at the front, who can see no hats, calls last. If they adopt the optimum strategy, what is the percentage of correct guesses they will make?

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## Dick Boardman

The solution to the hat problems are as follows.

(1) Let the two people be A and B. There are two possibilities: The hats are the same, or the hats are different.

The prior agreed strategy is that A will assume that the two hats are the same. He will therefore say that his hat is the same colour as the one he can see. Player B will assume that the two are different. He will therefore say that his hat is the opposite colour to the one he can see. One of the two must be right.

(2) We assume that the players 2 through 10 can hear what is announced but not see whether it is right or wrong. Amazingly, with the correct strategy at least nine will be correct and ten will be correct half the time. The strategy is as follows.

[1] Women are usually barred from this version of the game.

There will be $n$ red hats and $10 - n$ green hats. Here, $n$ may be either odd or even. Player 1 will announce that his hat is either red or green, so as to make the total number of red hats even. If player 2 hears red and can see an odd number of red hats, he will know his hat is green. If he hears red and can see an even number of red hats, his hat must be red. Either way his announcement is correct. Similarly, players 3 through 10 can correctly identify their own hat colours. Thus at least nine answers must be right. If player 1 is correct, all ten answers will be right; otherwise nine will be right.

Since I have been before, I can't claim a discount on the cost of the Winter Weekend.

# M337, or not M337, that is the question

## Dave Wild

If you are thinking of studying the OU's Complex Analysis module then there is a MOOC (Massive Open Online Course) which may help you decide whether to do so. On *Coursera*, the Wesleyan University runs a course entitled 'An Introduction to Complex Analysis'. This course requires eight weeks of study and their estimate is 6–12 hours of study per week. The course covers many of the same topics as M337 but in much less depth. If you do not like proofs then you will like this course. The course is free; so, if you are on broadband, it is worth a look. About 90 percent of the students drop out of these free courses; so nobody will care if you bale out. The videos should be watched online as they contain problems which you are expected to answer. If you download the videos then the problems do not appear.

# Problem 274.1 – Two dice

## Tony Forbes

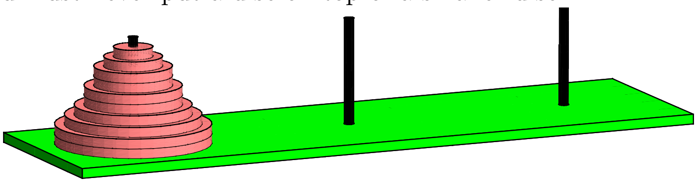
I offer you the chance to play the following game, which is repeated until one of us becomes bankrupt.

> I throw two dice.
> If no 6 appears, nothing happens.
> If precisely one 6 appears, you pay me £1.
> If double-6 appears, I pay you £9.

Excellent odds, you agree. Since we have eliminated the no-sixes case, the probability of getting the second 6 surely can't be less than 1/6. Well?

# Solution 267.2 – Hanoi revisited

There are three vertical pegs lined up in a row and $n$ discs. The discs have holes in their centres so that they can be threaded on to the pegs. Initially, all $n$ discs are placed on the left-hand peg in non-increasing order of radius to form a tower. The object of the game is to transfer the entire tower to the right-hand peg by moving discs from peg to peg, one at a time, according to the rules: (1) only a disc at the top of a tower may be moved; (2) you must never put a disc on top of a smaller disc.



(i) How many moves are needed?
(ii) How many distinct types of starting position are there?
(iii) Depending on the distribution of the radii, the number of moves varies from $n$ (when all discs have the same radius) to $2^n - 1$ (all different). Which numbers in this interval are represented?

## Tony Forbes

Our original version of the solution to this problem ended up with one of M500's rivals! However, the problem was stated in M500. So is appropriate that we should at the very least give brief answers to questions (i)–(iii) here.

Assume there are $t$ distinct radii, $1, 2, \ldots, t$, and that radius $i$ occurs $r_i$ times, so that $n = r_1 + r_2 + \cdots + r_t$. As is well known, the number of moves required to transfer a tower of $t$ *distinct* discs is $2^t - 1$, and after a bit of thought one can see that the $k$th disc, numbering from the top of the tower, gets moved $2^{t-k}$ times.

(i) When there are two identical discs at the top of a tower it is clearly[1] never a good idea to move them on to separate pegs. We can therefore treat a pile of $r_i$ discs of radius $i$ as if it were a single disc except that we count its moves with weight $r_i$. Hence, given a starting position $R = (r_1, r_2, \ldots, r_t)$,

[1] The word 'clearly' is there to avoid the difficult task of trying to justify the rest of the sentence. In [3] we adopt a different approach, where the stated assertion follows naturally and without the need for any explanation.

the number of moves is

$$N(R) = N((r_1, r_2, \ldots, r_t)) = \sum_{k=1}^{t} r_k 2^{t-k}.$$

For example, starting with 15 discs of radii $1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 5$, we have $t = 5$ and

$$N((5, 4, 3, 2, 1)) = 5 \cdot 2^4 + 4 \cdot 2^3 + 3 \cdot 2^2 + 2 \cdot 2 + 1 = 129.$$

(ii) The number of starting positions for $n$ discs is $2^{n-1}$, the same as the number of ordered partitions of $n$.

(iii) Let $M_n$ denote the set of distinct values of $N(R)$ as $R$ ranges over all ordered partitions of $n$, and let $m(n) = |M_n|$, the cardinality of $M_n$. In [3] we show that

$$M_1 = \{1\}, \quad M_n = \{x + 1 : x \in M_{n-1}\} \cup \{2x + 1 : x \in M_{n-1}\}, \quad n > 1,$$

and by direct computation we get these values for $m(n)$:

$$m(n) = 1, 2, 4, 8, 15, 27, 47, 80, 134, 222, \ldots, \qquad n = 1, 2, \ldots.$$

The first thing to do with an unfamiliar sequence is look it up in [2], and, indeed, we find that our calculated values of $m(n)$ are the first ten terms of entry A000126. But [2, A000126] is actually $L(n, 4)$, where

$$L(n, r) = \sum_{i=0}^{r-1} \binom{n-1}{i} + \sum_{j=1}^{\lfloor (n-r)/2 \rfloor} \binom{n-j-1}{j+r-1},$$

described in [1] as a generalization of the Fibonacci sequence

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, \ldots, \qquad n = 1, 2, \ldots.$$

Moreover, it is proved in [1] that $L(n, 4) = F_{n+3} - n - 1$. So we appear to have the following interesting and remarkable result.

$$m(n) = F_{n+3} - n - 1, \qquad n = 1, 2, \ldots.$$

This is proved in [3].

### References

[1] D. A. Lind, On a class of nonlinear binomial sums, *Fibonacci Quarterly* **3** (1965), 292–298.

[2] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, oeis.org.

[3] Tamsin Forbes & Tony Forbes, Hanoi revisited, *The Mathematical Gazette* **100** (November 2016), 435–441.

# Solution 272.1 – Finite integral

Show that

$$\int_0^{2\pi} (\cos x)\left(\sin\frac{x}{2}\right)\left(\tan\frac{x}{3}\right)\,dx \;=\; \frac{-18\sqrt{3}}{5}.$$

## Bruce Roth

I look forward to seeing your elegant three-line solution. But until then here is my attempt. Let $x = 6\theta$. Then the integral is

$$6\int_0^{\pi/3} (\cos 6\theta)\,(\sin 3\theta)\,(\tan 2\theta)\,d\theta.$$

Using the identities

$$\tan 2\theta \;\equiv\; \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}, \qquad \sin 3\theta \;\equiv\; 3\sin\theta - 4\sin^3\theta$$

and, from de Moivre,

$$\cos 6\theta \;\equiv\; \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta,$$

which factorizes nicely as

$$\cos 6\theta \;\equiv\; (\cos^2\theta - \sin^2\theta)(\cos^4\theta - 14\cos^2\theta\sin^2\theta + \sin^4\theta),$$

gives us (letting $s = \sin\theta$ and $c = \cos\theta$)

$$6\int_0^{\pi/3} \left(c^4 - 14c^2s^2 + s^4)(3s - 4s^3\right)(2sc)\,d\theta,$$

which multiplies out to

$$12\int_0^{\pi/3} \left(3c^5s^2 - 42c^3s^4 + 3cs^6 - 4c^5s^4 + 56c^3s^6 - 4cs^8\right)\,d\theta.$$

Using $s^2 + c^2 \equiv 1$ we get

$$\begin{aligned} c^5s^2 &\equiv cs^2 - 2cs^4 + cs^6, & c^3s^4 &\equiv cs^4 - cs^6,\\ c^5s^4 &\equiv cs^4 - 2cs^6 + cs^8, & c^3s^6 &\equiv cs^6 - cs^8, \end{aligned}$$

which gives us

$$\begin{aligned} &12\int_0^{\pi/3} \left(3cs^2 - 52cs^4 + 112cs^6 - 64cs^8\right)\,d\theta\\ &\quad = 12\left[s^3 - \frac{52s^5}{5} + 16s^7 - \frac{64s^9}{9}\right]_0^{\pi/3} \;=\; \frac{-18\sqrt{3}}{5}. \end{aligned}$$

## Tony Forbes

I have no hand-made solution, let alone one that can be delivered in three lines. If you have suitable mathematical software, one of the things you can do to make your life slightly more exciting is compute

$$T(a,b,c) \;=\; \int_0^{2\pi} \left(\cos\frac{x}{a}\right)\left(\sin\frac{x}{b}\right)\left(\tan\frac{x}{c}\right) dx$$

for three distinct, smallish positive integers $a$, $b$ and $c$ chosen more or less at random. Amongst the various types of behaviour $T(a,b,c)$ can exhibit I can identify the following.

(i) If $x/c$ includes $\pi/2$ or $3\pi/2$ as $x$ goes from 0 to $2\pi$, there will be trouble with $\tan(x/c)$ and the integral won't converge unless one of the other two trigonometric functions is zero at the same place. Thus, for example, the integral

$$T_\epsilon(4,8,3) \;=\; \int_0^{3\pi/2-\epsilon} + \int_{3\pi/2+\epsilon}^{2\pi} \left(\cos\frac{x}{4}\right)\left(\sin\frac{x}{8}\right)\left(\tan\frac{x}{3}\right) dx$$

does not converge when $\epsilon$ tends to zero.

(ii) The integral is computable but you would probably have to be insane if you even begin to think about doing it by hand. The result is usually a monstrous expression possibly involving hundreds of terms some of which can be quite complex. In a fairly typical case with small $a$, $b$, $c$, we have

$$\begin{aligned}
T(3,4,7) \;=\; & \frac{88}{7} + \frac{154}{25}(-1)^{2/21} + 112(-1)^{1/7} - 112(-1)^{4/21} \\
& - \frac{112}{25}(-1)^{5/21} - \frac{64}{7}(-1)^{1/3} - \frac{112}{25}(-1)^{3/7} - 154(-1)^{10/21} \\
& + 154(-1)^{11/21} + \frac{112}{25}(-1)^{4/7} + \frac{64}{7}(-1)^{2/3} + \frac{112}{25}(-1)^{16/21} \\
& + 112(-1)^{17/21} - 112(-1)^{6/7} - \frac{154}{25}(-1)^{19/21} \\
& + \frac{35}{4}(-1)^{7/24} \arctan\frac{\sin(\pi/42)}{(-1)^{1/24} - \cos(\pi/42)} \\
& + \cdots + \frac{7}{8}(-1)^{3/8}\sqrt{3}\log\left((-1)^{11/12} - 1 + 2(-1)^{23/24}\cos(\pi/42)\right)
\end{aligned}$$

with 244 terms omitted. The expression $(-1)^\alpha$ should be treated as short-hand for $e^{\pi i \alpha}$.

(iii) The integral comes out to a nice expression that is not too irrational, as in our example. Here's another, this time with a $\pi$ in it:

$$T(3,1,6) \;=\; \frac{63\sqrt{3}}{16} - 2\pi \;\approx\; 0.536765.$$

And a few more:

$$\begin{aligned}
T(3,1,8) &= \frac{3096}{455} - \frac{297\sqrt{3}}{80} \;\approx\; 0.374157, \\
T(3,1,9) &= \frac{9}{80}\left(-5\sqrt{3} + 18\sin\frac{2\pi}{9}\right) \;\approx\; 0.327366, \\
T(3,1,12) &= \frac{2109\sqrt{3}}{560} - 2\pi \;\approx\; 0.239842, \\
T(3,1,24) &= \frac{301328}{15015} - \frac{4419\sqrt{3}}{560} - 2\pi \;\approx\; 0.117543, \\
T(2,1,4) &= 2\pi.
\end{aligned}$$

Note that the integrand in $T(2,1,4)$ is finite at $x = 2\pi$.

(iv) Even better, the integral is actually rational:

$$\begin{aligned}
T(1,2,8) &= -\frac{8}{15}, \\
T(1,4,3) &= -\frac{832}{105}.
\end{aligned}$$

(v) The integral might be doable by hand but is not as nice as (iii):

$$\begin{aligned}
T(7,1,6) &= \frac{343}{48}\cos\frac{3\pi}{14} - \frac{441}{110}\sin\frac{\pi}{21} - \frac{441}{68}\cos\frac{5\pi}{42} \;\approx\; -1.04768, \\
T(1,4,7) &= -\frac{16}{15} + \frac{672}{95}\cos\frac{\pi}{7} - \frac{1120}{351}\sin\frac{\pi}{14} - \frac{224}{33}\sin\frac{3\pi}{14} \;\approx\; 0.364291.
\end{aligned}$$

(vi) If $c = 2b$, the sin and tan factors combine to simplify the integral:

$$T\,(a,b,2b) \;=\; 2\int_0^{2\pi}\left(\cos\frac{x}{a}\right)\left(\sin^2\frac{x}{2b}\right)dx \;=\; a\sin\frac{2\pi}{a} + U(a,b) + U(b,a),$$

where

$$U(a,b) \;=\; \frac{ab^2\cos(2\pi/b)\sin(2\pi/a)}{a^2-b^2}.$$

# Fermat's Last Theorem: a possible explanation

**Peter L. Griffiths**

We assume $p$ and $q$ are positive integers with $p$ greater than or equal to $q$, and $n$ is an integer greater than 2. The binomial expansion of $(p+q)^n - (p-q)^n$ equals

$$\begin{aligned} &\binom{n}{0}p^n q^0 + \binom{n}{1}p^{n-1}q^1 + \binom{n}{2}p^{n-2}q^2 + \binom{n}{3}p^{n-3}q^3 + \dots \\ - &\binom{n}{0}p^n q^0 + \binom{n}{1}p^{n-1}q^1 - \binom{n}{2}p^{n-2}q^2 + \binom{n}{3}p^{n-3}q^3 - \dots \\ &\qquad = 2\binom{n}{1}p^{n-1}q^1 + 2\binom{n}{3}p^{n-3}q^3 + \dots . \end{aligned}$$

Let $q = p/r$ so that $r$ is greater than 1 and is rational; then

$$\left(p + \frac{p}{r}\right)^n - \left(p - \frac{p}{r}\right)^n = 2p^n\left(\binom{n}{1}\frac{1}{r} + \binom{n}{3}\frac{1}{r^3} + \dots\right).$$

Divide both sides by $p^n$,

$$\left(1 + \frac{1}{r}\right)^n - \left(1 - \frac{1}{r}\right)^n = 2\left(\binom{n}{1}\frac{1}{r} + \binom{n}{3}\frac{1}{r^3} + \dots\right).$$

Take the $n$th root of both sides

$$\left(\left(1 + \frac{1}{r}\right)^n - \left(1 - \frac{1}{r}\right)^n\right)^{1/n} = 2^{1/n}\left(\binom{n}{1}\frac{1}{r} + \binom{n}{3}\frac{1}{r^3} + \dots\right)^{1/n}.$$

Let $r$ equal 1, so that $p$ equals $q$, and the equation is at its maximum of 2, also so that $p$ equalling $q$ deliberately converts the three term assumption of FLT into two terms. This equality with maximum 2 can only occur if there are two terms not the three terms assumed in Fermat's Last Theorem.

This deliberate conversion of three terms into two terms takes the form

$$(2^n - 0)^{1/n} = 2 = 2^{1/n}\left(\binom{n}{1} + \binom{n}{3} + \dots\right)^{1/n}.$$

Hence when $r = 1$ and $p = q$ there are just two terms,

$$\left(\binom{n}{1} + \binom{n}{3} + \dots\right)^{1/n} = 2^{1-1/n}.$$

It also follows that provided $n > 2$ the Pythagorean triple power, and provided the three term assumption is restored, the equation with $r > 1$,

$$2^{1/n}\left(\binom{n}{1}\frac{1}{r}+\binom{n}{3}\frac{1}{r}+\dots\right)^{1/n} \;=\; \left(\left((1+\frac{1}{r}\right)^{n}-\left(1-\frac{1}{r}\right)^{n}\right)^{1/n}$$

will be less than 2. There can only be integer equality with two terms not the three terms assumed in Fermat's Last Theorem.

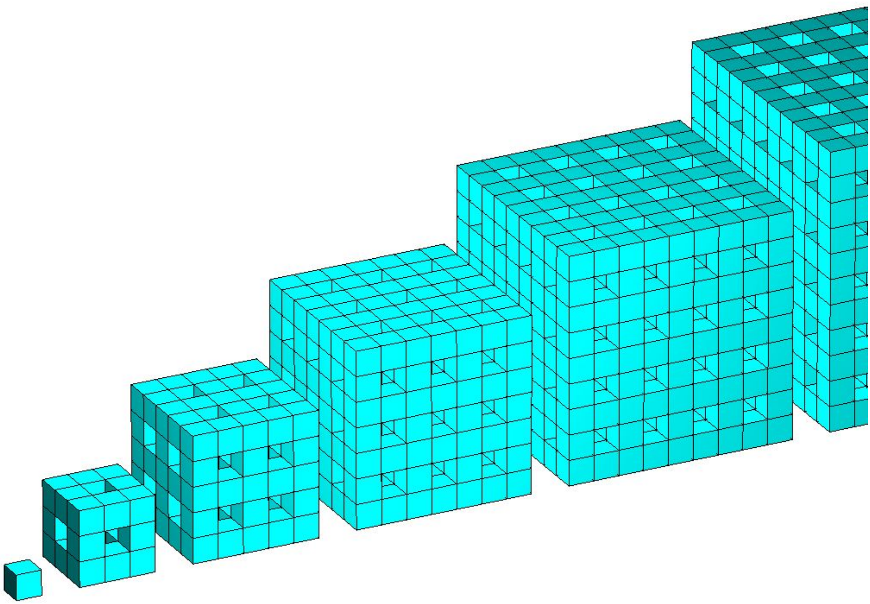
## Problem 274.2 – Holey cube

### Tony Forbes

A $(2h+1)\times(2h+1)\times(2h+1)$ cube has three orthogonal $h\times h$ arrays of $1\times 1$ holes running through it. Find a formula for $c(h)$, the number of little cubes used in its construction, and $f(h)$, the number of exposed facelets. Hence or otherwise compute the limit of $f(h)/c(h)$ as $h$ tends to infinity.

For small $h$, we have the following values.

| | | | | |
|---|---|---|---|---|
| $h$ | 0 | 1 | 2 | 3 |
| $c(h)$ | 1 | 20 | 81 | 208 |
| $f(h)$ | 6 | 72 | quite a lot | many more |

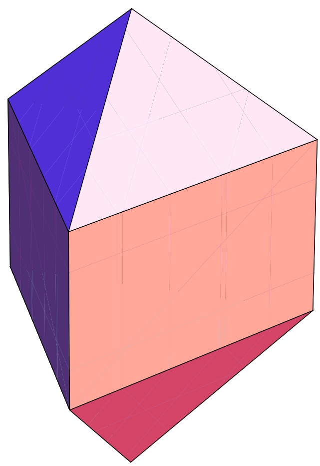


# Solution 269.4 – Three-sided dice

> Is it possible to make a three-sided die. Can it be done for any odd number of sides?
>
> We leave it for your imagination to decide exactly what constitutes a valid die. However, we insist that the probabilities of landing on the various sides must be calculable. And to save time, let us rule out blatant forms of cheating—such as a cube with the faces numbered in pairs.

## Tony Forbes

Taking the wording of the question at face value, the answer has surely got to be 'no', unless the die is allowed to have faces that are not flat.



So we bend the rules, but only slightly. We now require the die to be a polyhedron where the probabilities of landing on its various faces can be calculated to have the values 1/3 for three of them and 0 for the others.

Now the problem is easily solved. Take a prism based on an equilateral triangle and stick a regular tetrahedron on each end. If the prism is sufficiently long, the probability of the thing landing on a triangular face will be zero.

This leads to an interesting question, which we formally state as Problem 274.3. How long is sufficiently long? For example, in the illustration all the edges have the same length, and you can confirm, perhaps by building a model, that it will never land on a triangle.

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# Problem 274.3 – A nine-sided die

A 9-sided die is made using stuff of uniform density by sticking a regular tetrahedron of side 1 on to each end of a prism based on an equilateral triangle of side 1. How long must the prism be for the die to have zero probability of landing on a triangular face?

Also solve the problem for 12-sided and 15-sided dice constructed in a similar manner from equilateral triangles and a 4- or 5-sided prism.

# Solution 268.2 – Induction

What's wrong with this argument? We wish to prove that

$$\sum_{k=1}^{n} k \;=\; O(n). \qquad (*)$$

Clearly $\sum_{k=1}^{1} k = 1 = O(1)$. Using induction, we assume $(*)$ is true for some $n \geq 1$. Then we have

$$\sum_{k=1}^{n+1} k \;=\; O(n) + n + 1 \;=\; O(n+1).$$

Hence $(*)$ is true for all $n \geq 1$. $\square$

A nice result except that it is actually false. On the other hand, it is true that $\sum_{k=1}^{n} k = O(n^2)$.

## Reinhardt Messerschmidt

The statement $(*)$ is shorthand for

there exist $C > 0$ and an integer $N$

$$\text{such that } \sum_{k=1}^{n} k \leq Cn \text{ for every } n \geq N. \qquad (**)$$

Suppose we have chosen values for $C$, $N$ for which we want to attempt a proof of $(**)$ by induction on $n$. We have $\sum_{k=1}^{N} k = N(N+1)/2$, therefore the base case holds if and only if $C \geq N(N+1)/2$. For the inductive case, suppose $n$ is such that $\sum_{k=1}^{n} k \leq Cn$; then

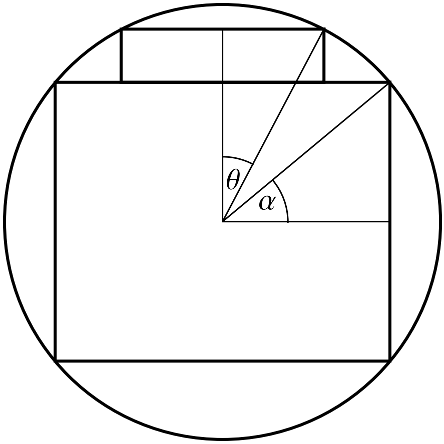
$$\sum_{k=1}^{n+1} k \;=\; \sum_{k=1}^{n} k + (n+1) \;\leq\; Cn + n + 1 \;=\; C(n+1) + (n - C + 1).$$

This is where the suggested proof goes wrong: it effectively throws away the $n - C + 1$ term and concludes that $\sum_{k=1}^{n+1} k \leq C(n+1)$. There is of course no way of fixing the proof, because no matter how $C$, $N$ have been chosen, if $n$ is such that $n \geq N$ and $(n+2)/2 > C$, then

$$\sum_{k=1}^{n+1} k \;=\; (n+1)(n+2)/2 \;>\; C(n+1).$$

# Solution 269.2 – Two rectangles

Two rectangles are packed inside a circle of radius 1. What is the largest area they can occupy?




## Ted Gore

I took a slightly different tack but got the same result as Tony Forbes [M500 **272**].

Let $\alpha$ and $\theta$ denote the angles indicated in the diagram. Let $A$ be the total area of the rectangles. Then

$$\begin{aligned} A &= 4\sin\alpha\cos\alpha + 2\sin\theta(\cos\theta - \sin\alpha), \\ \frac{\partial A}{\partial\theta} &= 2\cos^2\theta - 2\sin^2\theta - 2\sin\alpha\cos\theta, \end{aligned}$$

and $A$ is at a maximum when $\partial A/\partial\theta = 0$; that is, when

$$\sin\alpha = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta}. \qquad (1)$$

Moreover,

$$\begin{aligned} \frac{\partial A}{\partial\alpha} &= 4(\cos^2\alpha - \sin^2\alpha) - 2\sin\theta\cos\alpha \\ &= 8\cos^2\alpha - 2\sin\theta\cos\alpha - 4, \end{aligned}$$

and $A$ is at a maximum when $\partial A/\partial\alpha = 0$; that is, when

$$\cos\alpha = \frac{\sin\theta \pm \sqrt{\sin^2\theta + 32}}{8}. \qquad (2)$$

Now $\sin^2\alpha + \cos^2\alpha = 1$. So (1) and (2) gives us

$$\cos^2\theta\left(\sin\theta + \sqrt{\sin^2\theta + 32}\right)^2 + 64(\cos^2\theta - \sin^2\theta)^2 - 64\cos^2\theta = 0.$$

Solving this numerically gives

$$\theta \approx 0.4837543, \quad \alpha \approx 0.6956597, \quad A \approx 2.195184.$$

# Solution 272.2 – Infinite integral

Show that

$$\int_0^\infty e^{-x}(\log x)\,dx \;=\; -\gamma \;=\; -0.5772156649\ldots.$$

## Bruce Roth

From the integral definition of the gamma function,

$$\Gamma(z) \;=\; \int_0^\infty x^{z-1}e^{-x}\,dx.$$

Differentiating with respect to $z$,

$$\Gamma'(z) \;=\; \int_0^\infty x^{z-1}(\ln x)e^{-x}\,dx,$$

because

$$y \;=\; x^{z-1} \quad\Rightarrow\quad \ln y \;=\; (z-1)(\ln x),$$

and differentiating this last expression with respect to $z$,

$$\frac{1}{y}\frac{dy}{dz} \;=\; \ln x \quad\Rightarrow\quad \frac{dy}{dz} \;=\; y(\ln x) \;=\; x^{z-1}(\ln x).$$

Hence

$$\Gamma'(1) \;=\; \int_0^\infty (\ln x)e^{-x}\,dx.$$

From *Advanced Mathematical Methods with Maple* by Derek Richards, the set book for M833, page 835, we have this equation for the digamma function:

$$\psi(z) \;=\; \frac{\Gamma'(z)}{\Gamma(z)}.$$

Now, as $\Gamma(1) = 1$, we have

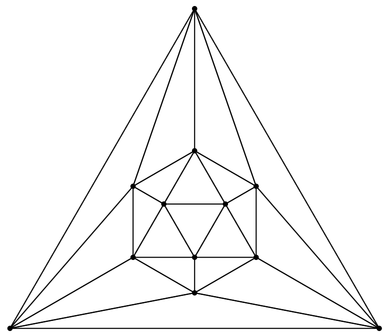
$$\psi(1) \;=\; \Gamma'(1), \quad \text{or} \quad \int_0^\infty e^{-x}(\ln x)\,dx \;=\; \psi(1),$$

and at the bottom of page 835 some special values of $\psi(z)$ are given, including $\psi(1) = -\gamma$. Therefore

$$\int_0^\infty e^{-x}(\log x)\,dx \;=\; -\gamma.$$

# Problem 274.4 – Cocktail party decomposition

## Tony Forbes

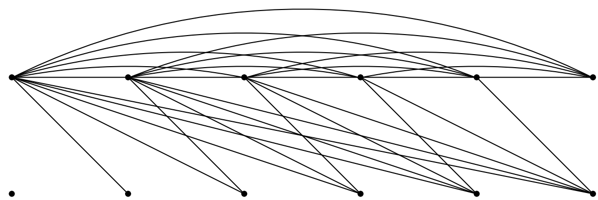


The cocktail party graph $K_{2,2,2,2,2,2}$ has twelve vertices partitioned into six pairs. Two vertices are adjacent if and only if they belong to distinct pairs. So the graph has 60 edges. We wish to partition $K_{2,2,2,2,2,2}$ into two edge-disjoint isomorphic copies of a specified graph $G$ that has 12 vertices and 30 edges. We also insist that $G$ is regular, meaning that each vertex has the same number of neighbours.

As is well known, this can be done if $G$ is the icosahedron. Take the icosahedron graph, above, and add an edge $\{x, y\}$ whenever vertices $x$ and $y$ are at distance 2. Then (i) the 30 new edges form another icosahedron and (ii) the old edges together with the new edges form a $K_{2,2,2,2,2,2}$.

Is there another way to partition the edges of $K_{2,2,2,2,2,2}$ into two 12-vertex, 5-regular graphs?

Soon after setting this problem I discovered that the answer is indeed 'yes'. All I did was use my graph-decomposition program to check all of the 7849 5-regular 12-vertex graphs. The result is that exactly 321 of them (including the icosahedron) work. Nevertheless, we would be interested if you can find a suitable construction by hand. If we drop the regularity condition, we can answer the question with the following, a $K_6$ with some extra bits added.



I have to admit that I have never been to a cocktail party. Moreover, I cannot recall ever being invited to one. So I really don't know what goes on at these events. However, I understand that (i) only married couples may attend, and (ii) one must talk to everybody at the gathering except one's spouse.

## Problem 274.5 – 27 cubes

There are 27 cubes each face of which is coloured either red, blue or green. Moreover, the 27 cubes can be assembled in three ways to form either a red, blue or green $3 \times 3 \times 3$ cube. Interestingly, this can be achieved in essentially only one way. How?

Thanks to Carrie Rutherford for suggesting the problem.

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## Problem 274.6 – Tan integral

Show that

$$\int_0^{\pi} \frac{\tan(t/6)\tan(t/3)}{\tan(t/2)}\,dt \;=\; \log\frac{3^6}{2^9}.$$

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## Problem 274.7 – Double-sided printing

Suppose you want to print two copies of a document. In the interests of the Environment one should try to use both sides of the paper. So, if your printer doesn't naturally do this, try the following procedure.

(i) Print the document normally (using only one side of the paper).

(ii) Place the printed result upside-down in the paper tray.

(iii) Print it again.

This works fine—so long as the original has precisely two pages. What's so special about 2? Why doesn't it work for one, or three or more pages?

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## Problem 274.8 – Binomial coefficients

Let $n_1, n_2, \ldots, n_r$ be $r \geq 0$ numbers. Show that

$$\sum_{i=1}^{r}\binom{n_i}{2} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} n_i n_j \;=\; \binom{n_1+n_2+\cdots+n_r}{2}.$$

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## Problem 274.9 – Even planar graphs

Suppose each face of a planar graph is a polygon with an even number of sides. Prove that the graph must be bipartite. Or find a counter-example.

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# M500 Mathematics Revision Weekend 2017

The forty-third M500 Revision Weekend will be held at

**Kents Hill Park Training and Conference Centre,**
**Milton Keynes, MK7 6BZ**
**from Friday 12th to Sunday 14th May 2017.**

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £265 for single occupancy, or £230 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £150.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2017 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 11th April 2017. The Late Booking Fee for applications received after 11th April 2017 is £20, with no membership discount applicable.

There is free on-site parking for those travelling by private transport. For full details and an application form please go to the Society's web site at www.m500.org.uk.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

*Please note that the venue differs from last year.*

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**WANTED: more contributions** It is possible we might be in some small danger of running out of suitable material. In the pipeline there is something by me about dissecting trapeziums as well as a detailed discussion of the Number Field Sieve by Roger Thompson. After that it could be a little bleak.

As usual, we are always interested in articles of, say, 2–6 (or even 7, 8 or 9) pages on mathematical topics, preferably stuff that can be readily understood by first-year mathematics undergraduates. We are also keen on problems and the solutions thereof, short mathematical notes, personal reminiscences, etc. And don't forget that there is no time limit for submitting an answer to an M500 Problem. — **TF**

04.00

**Front cover** Cube with holes; see page 10.