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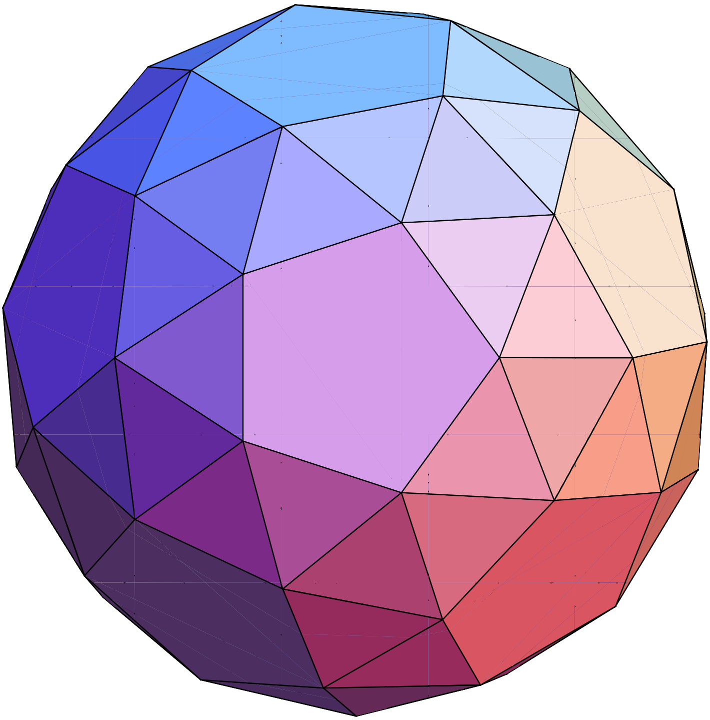

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The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

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What are real numbers and are they really real?

Sebastian Hayes

What is a real number? The set-theoretic definition is

> *A Cauchy equivalence class of Cauchy sequences of rational numbers.*

This sounds an unnecessarily abstruse definition. And is this really (*sic*) what a 'real number' is? Surely '3' is a real number but I don't envisage it as being a sequence, even less as an 'equivalence class' with an unlimited ('infinite') amount of members.

The difficulty, of course, is accommodating the irrationals.

Integers Although people like Dedekind and Russell manage to make quite a meal of the integers there is no great mystery involved. Numbering depends on the ability to recognize a 'one' when you see it, i.e. to distinguish between 'one' and 'more than one', and, secondly, on the ability to 'pair off' different sets of discrete objects. These abilities are very basic indeed since practically all adults have them and even, arguably, certain animal and insect species.

Once you have a generally accepted object, mark or sign for a 'one' (a single object) and the actual or hypothetical ability to copy or duplicate this chosen 'one' you have in effect 'all' the natural numbers. In set-theoretic parlance an integer is a *standard set* chosen from an equivalence class of discrete objects. The test for membership of a given (numerical) equivalence class is the possibility or not of carrying out a 'one-one correspondence' between the proposed set and the standard set.

All the rest, bases, cypherization, positional notation and so on is a matter of human convenience. If '1' is our chosen sign, then the symbol '4' is a shorthand way of noting '1 1 1 1' which is itself a standard manner of representing the numerically equivalent amount of objects, real or imaginary. '2 + 2 = 4' is not a logical truth as Russell and Whitehead would have us believe but a conventionalized representation of actual or possible states of affairs, i.e. it tells you what happens when you combine '1 1' objects with '1 1' other objects—always provided the objects do not merge when placed in close proximity. John Stuart Mill, who is always treated as an imbecile by professional mathematicians, is quite right when he says that '"2 + 2 = 4" is a matter of *fact*.'

Fractions Practically speaking, if you have a large discrete object of some sort and the ability to divide it up into *equal* portions as many times as you wish, you have in effect 'all' the fractions. As discrete objects considered irrespective of size, the portions are integers; considered in relation to the original whole they are 'fractions' (Lat. *frangere*, to break). As integers they have the Archimedean property 'forwards' and as fractions they have the Archimedean property 'backwards' for at every stage we have N objects each of size $1/N$ and we can (by hypothesis) carry on breaking for ever. Thus 'no greatest integer or smallest fraction'.

Fractions started disappearing from higher mathematics a long time ago: Euclid replaced them by geometric ratios while modern 'rational numbers' are not even ratios. Decimal notation in its perverse fashion prefers 0.125 to 1/8 and changes 1/3 into an endless series. For all that the 'man/woman in the street' remains obstinately attached to his or her fractions—about the only mathematical objects he/she actually uses on a daily basis apart from integers—hence the entrenched opposition to metrification and a certain instinctive distrust of mathematicians.

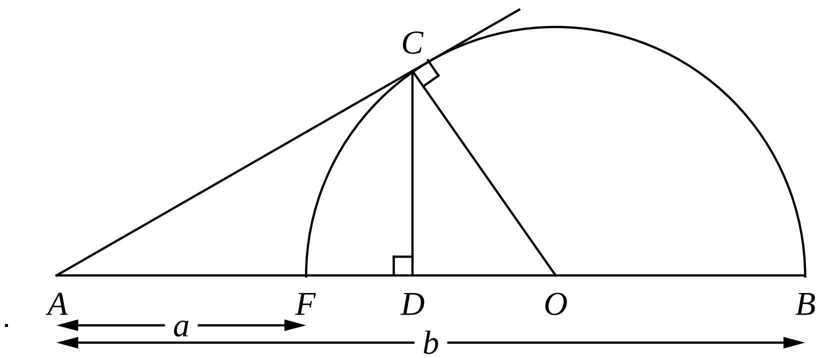
In set-theoretic terms, a 'rational number' is defined as 'an equivalence class of quotients a/b where a and b ($b \neq 0$) are integers'. Once again we need a standard set for each class and we choose the quotient which is 'in its lowest possible terms'. Membership of a class is given by the double prescription 'If a/b with $\gcd(a, b) = 1$ is a member, then c/d ($d \neq 0$) is a member if and only if $ad = cb$'.

Why the rule '$a/b = c/d$ iff $ad = cb$'? Ask a mathematician and he or she will say, 'Because of the axioms for fields'. Yes, I am aware of that but why do we have these particular axioms and not other ones? The answer is, of course, that we have this rule because it is indeed the case that, if the equation holds, a given object can be divided up in two completely different ways *without increasing or decreasing the amount of material in the designated part or in the whole*. Moreover, if ad is not numerically equivalent to cb, it will not be possible to transform a (equal) portions out of b into c (equal) portions out of d without increasing or decreasing material somewhere—though this is a bit more tricky to prove. Any algebraic system which did not include this criterion as an axiom or valid theorem might be of extreme interest but would be completely useless for calculation purposes because it would not square with the facts. It is distressing that this needs to be said.

Pythagoras and Pythagoreans Pythagoras, a shadowy figure who does seem to have nonetheless existed, is supposed to have taught that 'all is number'. What did he mean by this? Not that the solar system and our very existence in it depend on the precise values of certain constants such as the constant of gravitation or the fine structure constant—in this respect we are a good deal more 'number-orientated' than Pythagoras ever was. He seems to have meant that all phenomena could be interpreted in terms of simple whole number ratios. This wild conjecture has in part been brilliantly confirmed by modern physical discoveries (atomic number, periodic table &c.) but at the time the only real evidence came from the theory of sound.

The frequency of vibration of a stretched string is inversely proportional to its length, so if you halve the length you get the 'same note' at a higher pitch, i.e. the base note or tonic an octave higher. If you take the harmonic mean, $2ab/(a+b)$, of the two lengths so far defined, a and b, and pluck it you obtain the second most important musical interval, the *fifth* and if you take the take the arithmetic mean, $(a+b)/2$, you have the third most important musical interval, the *fourth*. Thus, for an original string of unit length, we have a length of 1/2 for the octave, 2/3 for the fifth, and 3/4 for the fourth. These are all simple whole number ratios, none simpler, and the strings, when plucked together, give a pleasing sound, i.e. they *harmonize*, whereas if the ratio of lengths was say 15 to 27, the result would be much less agreeable—so it is claimed at any rate.

Pythagoras realized that it was not the actual lengths that mattered so much as the *ratio* of the lengths and this naturally led on to the investigation of pleasing visual—as opposed to vocal—rapports which were likewise independent of change of scale, i.e. to the study of proportion. In architecture and design generally, it is the so-called geometric mean rather than the arithmetic or harmonic mean that is crucial. But the geometric mean proved to be the serpent in the Garden of Reason. In the drawing below if we call AF a and AB b, AO turns out to be the arithmetic mean of a and b and (using the theorem about the tangent squared) AC is the geometric mean since $(AC)^2 = ab$.

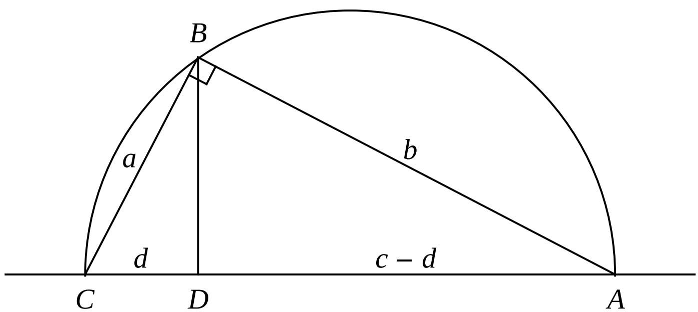


If we now set a at unity and $b = 2a$, the ratio AC/AF or AC/AB is clearly not expressible in *simple* whole numbers. This makes for a certain untidiness but the real blow comes with the realization that this simple geometric rapport cannot be represented as a ratio between whole numbers *at all*.

The famous *reductio ad absurdum* proof (not given in Euclid) that the side and diagonal of a unit square are incommensurable—or, as we would put it, that $2^{1/2}$ is irrational—is a strictly numerical proof. It rests on the argument that if a and c are in their lowest terms, a cannot be even—and yet we can show that it must be. But line segments are not 'odd' or 'even'—the very language belongs to arithmetic.

All this was a lot more puzzling to Greek eyes than to ours. In the isosceles right-angled triangle we generally take the side to be of unit size so the diagonal becomes root 2, an 'irrational'. But from a geometric point of view the diagonal is just as 'good' a line segment as the side and can just as well be set at unit length: it is not a matter of one line segment being 'normal' and the other 'peculiar'. The ratio is thus perfectly reversible

Daniel Shanks in his magnificent book *Solved and Unsolved Problems in Number Theory* suggests plausibly that the original 'proof' of Pythagoras' theorem was by way of the geometric mean—and to this day it is the proof given in elementary French textbooks.



Since triangles CBD, CBA are similar we have $d/a = a/c$. Thus $a^2 = cd$. Likewise from triangles CAD, CAB we have $(c - d)/b = b/c$ giving $b^2 = c(c - d)$. Adding, we obtain $a^2 + b^2 = cd + c(c - d) = c^2$.

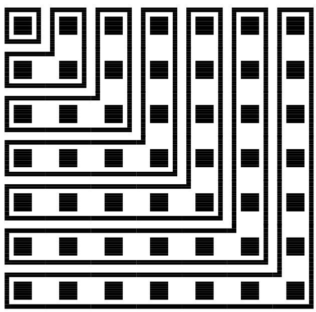
What is wrong with this proof? Not much. To keep to the style of Greek mathematics we would have to speak in terms of ratios and areas, not numbers, i.e. a^2 becomes 'the square on side BC' and so on. But the only questionable step is when we 'cross-multiply', i.e. move from the consideration of relative lengths to the direct evaluation of areas. The missing part is '$a^2 : 1 = cd : 1$'. But here's the rub: we are assuming there is necessarily a possible unit measure common to sides BC and CA, in other words that BC and AB are *commensurable*.

Now, prior to the development of geometry as *the* Greek science par excellence, there existed a figurative number theory which used counters (originally pebbles); hence the terms we still use today, *squares*, *cubes*, *triangular numbers* and so on. This was number given form but it differed from true geometry in that its methods and results were exact and empirically testable since it dealt only with the discrete. At first this older tradition did not clash with the development of geometry proper but the problem of incommensurables marked the parting of the ways.

From an empiricist atomic perspective—atomism was already a well-established theory in ancient Greece—the problem of incommensurables does not arise. All line segments are made up of so many atoms, so the hypotenuse of any right-angled triangle is bound to be 'commensurable' with the side and that is that. Which means that an isosceles right-angled triangle cannot exist—this seems to be a secret that has been more successfully guarded than the root 2 proof itself which supposedly cost a Pythagorean his life for divulging it.

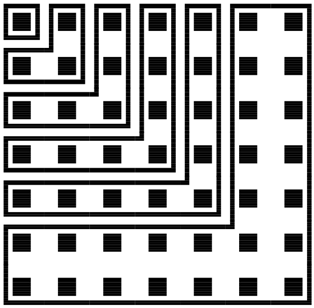
The Gnomon: scientific and mathematical instrument A gnomon was originally a small set-square used to measure the lengths of shadows—present-day sundials have a 'gnomon' on the top though the shape is more complicated. Thales is supposed to have used a gnomon to estimate the height of the Great Pyramid by employing properties of similar triangles: the gnomon was perhaps the first precision instrument of physical science.

Sets of gnomons put together—or drawings of them—become a kind of calculator once they are marked with regularly spaced dots



A marked gnomon always represents an odd number and the Pythagoreans realized that adding on a gnomon 'preserves the square form'. So the difference between two successive squares is the relevant odd number. Since a gnomon is twice the side of the preceding square plus a unit, if this gnomon is itself a square, we have a Pythagorean triple, i.e. $a^2 + (2a+1) = (a+1)^2$ where a is even and $(2a+1) = m^2$. And to find out what odd numbers are in fact squares you just have to consult an extended set of gnomons. The first case is $(2a+1) = 9$ giving us the original triple 3, 4, 5 (which the Egyptians only seem to have used to get a good right angle, not for the calculation of areas).

The procedure can be generalized if we allow a gnomon to be made up of more than one set square.



That is, we investigate $c^2 - a^2 = 2ra + r^2$, where $2ra + r^2$ is a square.

The root 2 problem turns out to be a special case of the above, namely where the 'gnomon' is exactly equal to the original square or $2ra + r^2 = a^2$ with $r = 1, 2, 3 \ldots$.

By trial and inspection it would have immediately been obvious that for small values of r this is impossible, e.g. for $(2a+1)$, $(4a+4)$ &c. But all is not lost numerically speaking since we can get very close to the Gorgon root 2 either by keeping the sides of a triangle equal with an angle slightly less than a right angle, $2a^2 > c^2$, or by keeping them equal with an angle slightly greater than a right angle, $2a^2 < c^2$. As a sort of number-theoretic riposte to the root 2 débacle, Theon of Smyrna hit upon a method of providing any amount of integer 'solutions'—all possible ones in fact. He almost certainly derived it by examining the gnomon diagram and considering the case where the relevant gnomon is very nearly equal to the inner square, either falling short by a counter or exceeding it by a counter.

In numbers the first cases are $2\cdot 1^2 - 1^2 = 1$, $3^2 - 2\cdot 2^2 = 1$, $2\cdot 5^2 - 7^2 = 1$.

odd terms	1	2	5	12	29	70	169	...
even terms	1	3	7	17	41	99	239	...

The series reminds one of the Fibonacci series—the difference comes with the 'doubling'. Note that all the even terms are odd and the odd terms alternately odd and even.

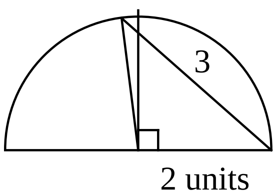
We can treat the series as a sum of two series each defined recursively but with a slightly different starting point, i.e.

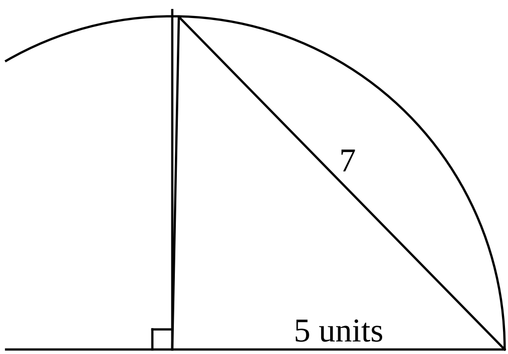
$$S_{\text{odd}}: \quad t_1 = 1, t_2 = 2, \quad t_{n+1} = 2t_n + t_{n-1},$$

$$S_{\text{even}}: \quad t_1 = 1, t_2 = 3, \quad t_{n+1} = 2t_n + t_{n-1},$$

though it is more usual to employ two variables, a and c. Then $a_{n+1} = a_n + c_n$, $c_{n+1} = 2a_n + c_n$ and $a_1 = 1$, $c_1 = 1$. Thus, $c_n^2 - 2a_n^2 = (-1)^n$ with proof by induction since $a_n^2 - 2c_n^2 = -1$ and $c_{n+1}^2 - 2a_{n+1}^2 = -(c_n^2 - 2a_n^2)$.

The algebra gives no indication of the reason for the alternating sign but if we refer back to the original problem we can visualize what is happening: the sum of the squares is either falling a unit short of the square on the 'hypotenuse', or exceeding it by a unit, and we are continually diminishing the size of the unit.





The ratios *even term / odd term* give better and better attempts at (not approximations to!) $2^{1/2}$ namely 1/1, 3/2, 7/5, 17/12, ... They are, of course, the convergents of the continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}.$$

Rather nicely, if we take the alternative approach and consider a right-angled triangle with sides very nearly equal, i.e. $c^2 = a^2 + (a+1)^2$, we get returned to the same set, since $2c^2 - (2a+1)^2 = 1$, so we are looking for odd 'hypotenuse' squares $(2a+1)^2$ and side squares c^2, thus reversing the role of the variables a and c.

In both these cases, we approach closer and closer to equality as we divide up the line segments into smaller and smaller units permitting a 'closer fit', i.e. we take $c^2 - 2a^2 = 1/N$ instead of 1. In our terminology $N \to \infty$, $c^2 \to 2a^2$.

And so, as a limit—in the precise modern sense of the term—Pythagoras' theorem is always true. This is typical of Euclidian geometry generally—viewed from a numerical perspective.

Number theory vs geometry The relation of the side of an isosceles triangle to the hypotenuse has a perfectly good *geometrical* existence but a numerical non-existence. So which does one choose? As we know, Greek mathematics decided in favour of geometric truth.

It is often stated that Eudoxus, who is considered responsible for the substance of Euclid's *Book V*, tidied up the earlier proofs (such as the 'French textbook' proof of Pythagoras) and developed a wholly rigorous theory. This is not entirely true.

Essentially what Eudoxus does is to enable one to sidestep the problem with a reasonably good conscience. Firstly, he extends the concept of ratio to cover the case of magnitudes of the same kind which do not have a base unit in common but which 'are capable, when multiplied, of exceeding one another' (Euclid, *Book V*). This is sensible enough but Eudoxus does not always distinguish clearly between the two types of ratios, those between commensurables and those between incommensurables.

Secondly, whenever possible, Eudoxus speaks of equality of ratios rather than equality of areas. Thus he will say that 'similar and similarly described figures will be to each other as their bases' which leaves it open as to whether the bases are commensurable or not.

But this, however, will not do in the case of Pythagoras' theorem because it is a theorem about equality of areas, not about similitude, and thus presupposes the existence of a base unit as all calculation must do. Eudoxus' improved version of the French textbook proof appears as Proposition 31 in *Book V*. It is a more carefully worded proof, also a more extensive one since it applies to any similar figures, not just rectangles, erected on the three sides. But it remains open to the same fatal objection. All Eudoxus manages to prove is that, referring back to the earlier diagram,

$$d + (c - d) \ : \ c = (\text{figure on } a) + (\text{figure on } b) : (\text{figure on } c).$$

But the LHS is a ratio of commensurable magnitudes—since every line segment is commensurable with itself—whereas the RHS is, in the unit square case, a ratio of *incommensurable* magnitudes and, although we may admit with Eudoxus that incommensurable magnitudes can have a 'ratio' in the new sense of the word, this ratio cannot properly be equated to a ratio in the old sense.

Euclid/Eudoxus could very easily have treated incommensurable 'quantities' as limits lying between two endless series, and if they did not do so it was probably because they viewed such an expedient as representing too much of a concession to the numerical opponent. Instead later Greek mathematics tried to expel number from geometry altogether, making number theory (the substance of Euclid, *Books VII–IX*) a mere appendage to the Theory of Proportion. Many people, flipping through Heath's translation, do not even realize Euclid is referring to numbers since he presents them as line segments (and not as arrays of dots) and even gives the formula for summing a geometric series in the form of a static ratio (*Book IX*, Prop. 35) which makes it useless for calculation purposes.

Now Euclidian geometry is an idealization: you will look in vain for the standard forms of circle, rectangle and even straight line in nature. Whole number theory is, however, not an idealization but a representation of fact. The decision to raise geometric truth above the level of arithmetic 'calculation' led straight on to Plato's transcendental realism which has so strongly affected mathematical and philosophical thought ever since. The irony of the situation, of course, is that Euclidian geometry itself eventually got swept away as being too 'this world' orientated. Not only do we get, in modern abstract geometry, theorems which are not strictly true if put to the test (such as Pythagoras' theorem for the unit square) but theorems which are completely fantastic and ridiculous such as the Banach-Tarski two sphere theorem which states that a sphere can be dissected in such a way as to be reassembled to give two spheres, each the size of the original one.

The legalization of irrational numbers Although engineers and practical people have been using rough values for $2^{1/2}$ and π from the time of the Babylonians, it was only in the late nineteenth century that root 2 and its companions acquired a truly respectable *arithmetical* (as opposed to geometrical) existence. The first person to have tackled the problem head on seems to have been Dedekind. His solution was to introduce a completely new axiom into number theory which a contemporary textbook (Burkill's *First Course in Analysis*) summarizes as follows.

> *Dedekind's Axiom. Suppose that the system of all real numbers is divided into two classes, L, R, every member l of L being less than every member r of R (and neither class being empty). Then there is a dividing number ζ with the properties that every number less than ζ belongs to L and every number greater than ζ belongs to R. The number ζ itself may belong either to L or to R. If it is in L, it is the greatest member of L; if it is in R, it is the least member of R.*

This is all very well up to a point, but treating the constructed intermediary 'number' as 'a member of either class' is very inelegant and betrays a certain indecision as to the real nature of the rogue number. But the real trouble is of course that all Dedekind's procedure actually defines is not a 'quantity' as such but a gap. Instead of talking about 'Dedekind's cut' we should talk about Dedekind's gap—but this sounds rather less impressive already and we have not acquired any new knowledge since we knew the 'number line was gapped' already, or the Greeks did anyway. Contemporary mathematicians endlessly repeat and even seem to believe that the number line is 'everywhere dense' but if it really were 'everywhere dense' our 'numbers' would be an indistinguishable mass of mud. The 'everywhere dense' optic conceals a 'Platonic' prejudice: the real numbers, all of them, are somehow 'already there', independently of our discovery of them. If on the other hand we adopt a 'constructive' approach to mathematics, the 'number line' starts off as everywhere gapped, gets gradually filled up as mathematicians define or calculate with specific numbers, but is never completely full.

To me there is a world of difference between a positive rational number and an irrational. A rational number represents, or at any rate *can* represent, a specific length or other quantity which, within acceptable limits of technical exactitude, actually does exist, while the second represents something which not only does not exist in the real world, but cannot exist there. For there are no irrational quantities in the real world and all calculations that are, have been or ever will be made, employ only rationals.

There are other ways of creating irrationals but all depend on a similar *parti pris* backed up by a similar sleight of hand. 'Every increasing sequence of rational numbers converges to a limit'—the axiom of completeness. Who says it does? The author of the textbook and behind him/her the entire mathematical establishment. But if I ask to see, hear, read, calculate with, this 'limit' I find I am fobbed off with something which it is not, namely a so-called rational 'approximation'.

It is possible to sidestep the issue in much the same way as Eudoxus sidestepped the issue of incommensurable ratios. We can say that an irrational number is 'given' by (1) a mathematical formula or procedure (of a certain type) and (2) an initial 'store of numbers'—domain if you like—to which the formula is to be applied. Thus ϕ is 'given' by 't_{n+1}/t_n with $t_1 = 1$, $t_2 = 1$, $t_{n+1} = t_n + t_{n-1}$'. This tells us what to do while making no explicit commitment to the existence or not of the implied limit. However, it is really a coward's way out: I want to know not only what 'gives' ϕ but what ϕ actually 'is'.

Set theoretic vs analytical definition What is a Cauchy sequence? A sequence of rational numbers $\{x\}$ is Cauchy iff given some positive rational number ϵ we can find a positive integer N such that for all $m, n > N$, $x_m - x_n < \epsilon$.

In other words, the tail of the sequence (assumed to be indefinitely extendable) dwindles away towards nothing (though in most cases remaining a positive quantity).

To speak the language of traditional analysis for the moment, all Cauchy series converge and it can be proved that every convergent series must be Cauchy. In point of fact the Cauchy criterion is a good deal more basic than 'convergence' and was in effect used by the Greeks while 'convergence to a limit' never was. Also, the man or woman in the street accepts quite readily the idea behind the Cauchy criterion. Why are we quite happy to use 3.14159 as a value for π even though π has been taken to hundreds of thousands of places? Because, as mathematical students, we have learned that 'every increasing set of real numbers that is bounded above has a least upper bound'? I think not. We are not bothered because we believe that what follows 3.14159 is of little account since it can at most affect a decimal place no matter how long we carry on taking partial sums—in other words that $3 + 1/10 + 4/100 + 1/1000 + 5/10000 + 9/100000 + \ldots$ is Cauchy.

The set-theoretic definition speaks of an 'equivalence class'. I was for a long time puzzled as to why we need to have such a large assembly. Then I recalled that there are, for example, a multitude of different formulae leading to such a 'number' as π, as David Singmaster reminds us in M500 **168**, 'A history of π'. We have for example Leibnitz's beautiful

$$4(1/1 - 1/3 + 1/5 - 1/7 + \dots) \qquad (*)$$

or Wallis's

$$2(2/1 \cdot 2/3 \cdot 4/3 \cdot 4/5 \cdot \dots)$$

and many other bizarre concoctions from celebrated mathematicians. All such formulae give Cauchy sequences which may for a considerable number of terms differ appreciably but which are 'Cauchy equivalent', i.e. we can find an N such that the difference between t_m and t_n for any $m, n > N$ from each of the two sequences is less than any specified positive quantity.

(The most dramatic example of the 'all paths lead to Rome' syndrome is ϕ since it has been proved that the ratio of successive terms in a Fibonacci series converges to ϕ no matter what starting points, t_1, t_2, are used. (**))

The devastation caused by the forcible introduction of irrationals into our numeric system is considerable: it is a case of 'Jack's the only boy in step'. We find we have to transform integers and fractions into 'infinite series' with constant terms before they qualify as 'real numbers'; **1** itself, a 'one' if ever there were such a thing, becomes $1, 1, 1, 1, 1, \dots$ and so on for ever—and even then **1** is no more than a chosen representative of the unimaginably huge class of Cauchy series which includes what you get when you apply the formula $(n+1)/n$ or $(n-1)/n$ or sum $1/2 + 1/4 + 1/8 + \dots$.

Irrationals as limits In practice though no one ever bothers with the set-theoretic definition. What we all do is adopt the analytical approach which views a real number as the *limit* of an equivalence class of Cauchy sequences not as the equivalence class itself. This is shown by our very language—which even I find it impossible to avoid. We speak of 'approximations to' root 2 or ϕ.

This is the sense employed in M332, Unit 0, 'Real Analysis'.

> Theorem 12 (Nested Intervals Theorem)
>
> Let $\{S_n\}$ be a sequence of nested closed intervals ... where the lengths converge to zero. Then
>
> (i) S is non-empty and contains a unique real number x_0 ...

Note that this language and the way of thinking it embodies imply that the 'limit' in question is actually attained, which in the vast majority of cases it cannot be. Even quite respectable authors insist on using the wretched phrase 'sum to infinity' making, for example, **2** the 'sum to infinity' of $1 + 1/2 + 1/4 + 1/8 + \dots$ even though every partial sum to n terms is $2 - 1/2^n$, $n = 0, 1, 2, \dots$. Worse still, I have even come across one or two writers on mathematics who state that a convergent series 'converges to itself'—so the serpent eats its tail and our end is our beginning. Such nonsense would never be accepted in any subject other than mathematics.

What's in a name? Quite a lot apparently. Primitive peoples avoided telling members of other tribes their name, believing that knowing someone's name gave one a certain leverage over the person in question. In just such a primitive way we—and that even includes myself—feel that the 'ratio' of the radius to half the circumference of a circle really exists because we have a symbol for it (only since the eighteenth century). And somehow this extends to $1 + 1/2^2 + 1/3^2 + \ldots$ since it 'sums to $\pi^2/6$'. But this is more a matter of human psychology than scientific fact. There are any amount of series which qualify as convergent—because, say, they are monotonic and bounded above or below—but whose limits, though calculable to any number of decimal places, do not correspond to anything we are familiar with. In such a case does one really feel that the formula or procedure given defines a *single* number? I don't—it just defines a series.

After years of pondering the issue I find that 'I came out by the same door as in I went' (Omar Khayyam). My final conclusion is that there is no such quantity as $2^{1/2}$ but that it is extremely convenient to pretend that there is.

The quantification of the reality of real numbers Mathematics, a large part of it at any rate, is quantification so, in principle at least, it should be possible to quantify the degree of reality which numbers and other mathematical entities possess ranging from 1 for complete reality to 0 for a completely fictitious existence.

As a first bash I suggest something along the lines of

Positive integers	1
Negative integers	1/10
Zero	1/2
Positive fractions (proper or improper)	3/4
Irrationals	$n \mapsto 1/10^n$
Transfinite cardinals/ordinals	0

The above calls perhaps for some brief comment. The integers I take to be real because they are direct representations of actual or possible entities. If there are four sheep in a field there are four sheep in the field, not 'approximately' or 'in the limit' but actually, which is to say exactly. Moreover, a sheep really is a distinct entity, a 'one'. It cannot be chopped up without losing its identity—this is the point about the judgment of Solomon.

I give proper fractions a reality value somewhat less than 1 since their existence supposes that it is possible to divide up an object absolutely equally, while clearly this is not the case. We have thus moved already some distance away from actuality towards ideal conditions—but reality is still at arm's reach.

What about -3? -2? There has always been strong cultural resistance to negative numbers essentially because people feel, quite rightly, that there are no such things as 'negative entities' — a thing either exists or it does not. (Those inclined to scoff would do well to bear in mind that no less a person than Newton regarded negative numbers with suspicion.) In practice negative numbers only come in when we are comparing positions relative to a fixed point which is usually itself arbitrary, or perhaps when we are referring to imaginary 'quantities' like overdrafts. A position is less real than a thing, hence the depleted reality value.

Zero rates fairly well in my estimation since I do feel it to be in some sense part of our experience — 'nothing' is what remains when we have removed all the objects within a certain space. Certain Eastern philosophies make 'nothing' more basic than 'something', of course, but this is perhaps going too far, at any rate, mathematically. See Kaplan, *The Nothing that Is*.

Irrationals just about keep a grip on the cliff by their fingernails since they can be, as it were, approached via better founded numbers.

The infinite, *a fortiori* the transfinite, is not part of our sense experience (***). Infinity, it is finally agreed even by mathematicians, is not a number but it is still used in a way that suggests to the unwary that it is. In the majority of cases we could dispense with the infinity sign altogether and just use an arrow pointing in one direction ($\rightarrow$) meaning 'carry on increasing as much as you see fit'. There are no sets that can actually be put in one-one correspondence with one of their own proper subsets: any attempt to do such a thing in the case of an indefinitely extendable sequence would be an interminable procedure akin to measuring a permanently expanding object.

(*) We assume that Leibnitz derived his series from $\tan^{-1} x = x - x^3/3 + x^5/5 + \dots$, setting $x = 1$ for an angle of $\pi/4$. The general series had already been given by Gregory though whether Leibnitz knew this or not remains a bone of contention.

I had always thought, remarkable though the series is, that it was a bit of a cheat deriving a result in pure number theory by way of Taylor series and calculus and that an 'elementary proof' should be possible. I was both surprised and delighted to find that Daniel Shanks shares my misgivings and provides a strictly number-theoretic proof. However, the argument is very intricate and depends on a theorem Shanks states without proof, namely that

If $n \geq 1$, has A positive divisors $\equiv 1 \pmod 4$ and B positive divisors $\equiv -1 \pmod 4$, then $r(n) = 4(A - B)$, where $r(n)$ is the number of representations $n = x^2 + y^2$ in integers x and y, which are positive, negative or zero, the representations being considered distinct even if the xs and ys differ only in sign or order.

Shanks then considers the number of Cartesian lattice points (a, b) in or on a circle $x^2 + y^2 = N$ and shows that they equal $r(N)$. He then shows that as N increases, $r(N) \to \pi N$ and so on. (*Solved and Unsolved Problems in Number Theory*, pp. 162–6).

(**) The proof that the ratio of successive terms in any sequence $t_{n+1} = t_n + t_{n-1}$ 'converges' to the golden section irrespective of the starting point is quite simple and may be worth giving here for the benefit of those who don't know it.

Consider the sequence defined by the relation

$$u_n = a\phi^{n-1} + b(-1/\phi)^{n-1},$$

where a and b are constants. Then

$$u_{n+1} = a\phi^n + b(-1/\phi)^n,$$

and adding to u_n we obtain

$$\begin{aligned}
&a\phi^{n-1}(1+\phi) + b(-1/\phi)^{n-1}(1 - 1/\phi) \\
&\quad = a\phi^{n+1} + b(-1/\phi)^{n+1} \quad (\text{because } 1 + \phi = \phi^2 \text{ and } 1 - 1/\phi = \phi^{-2}) \\
&\quad = u_{n+2}.
\end{aligned}$$

Now, this relation holds whatever real values we give to a and b. As n increases without bound the second part of u_n, namely $b(-1/\phi)^n$, and similarly of u_{n+1}, namely $b(-1/\phi)^{n+1}$, both go to zero and so we only need to take into account the first parts $a\phi^{n-1}$ and $a\phi^n$. Thus, as n increases without limit, the ratio u_{n+1}/u_n approaches (though never actually attains) $\phi^n/\phi^{n-1} = \phi$.

The usual Fibonacci sequence beginning $1, 1, 2, 3, 5, \ldots$ is produced by setting $a = \phi^2/(\phi^2 + 1)$, $b = 1/(\phi^2 + 1)$.

(***) Mystics might quarrel with this. But what seems to be an essential feature of the mystic vision is a sense of the 'oneness' of everything. In such a case, the whole concept of dividing things up into bits is bypassed or obliterated and if we are to give a numerical value to such an 'infinite' vision it can only be **1**. Galileo, of all people, has a curious passage where he says that the 'conditions of infinite existence are to be met with in the case of unity' or something to this effect.

Re: ONE + TWELVE = TWO + ELEVEN

David Singmaster

Colin Davies's question,

$$\text{How is } \texttt{ONE} + \texttt{TWELVE} = \texttt{TWO} + \texttt{ELEVEN}$$

[M500 **191**, p. 10] was posed about three years ago by Victor Bryant on *Puzzle Panel*, R4. My immediate response was it must be an alphametic, but it turned out to be simply an ingenious anagram. Somehow the idea recurred to me and I decided to see whether there were any solutions to the problem as an alphametic. Obviously there is some difficulty about the left hand three places of the longer words and a little thought shows that we must have $\texttt{TWE} = \texttt{ELE}$, digit by digit. So $\texttt{T} = \texttt{E}$ and $\texttt{W} = \texttt{L}$ are necessary. This reduces to five letters, two of which depend on the other three. A simple program finds 133 solutions. I took $\texttt{E}$, $\texttt{O}$, $\texttt{V}$ as the independent variables. Some of these have $\texttt{O} = 0$; there are 21 such, so 112 without any leading zeroes. This is not a very satisfactory alphametic, but most of the solutions form groups of four to six related solutions with the same value of $\texttt{ONE}$. There are 36 groups. There are just five examples where the value of $\texttt{ONE}$ occurs just once. These are numbered 67, 84, 85, 101 and 116 (two of these have $\texttt{O} = 0$). Though not very satisfactory as an alphametic, it is interesting as the closest(?) possible version of a triply-true alphametic: the English statement is true; it is a correct anagram; and it is arithmetically true. As a result, I entered this in my *Sources in Recreational Mathematics*. So I'd like to know where the original version came from.

After writing the above, I remembered, perhaps incorrectly, that Victor had said that

$$\texttt{ONE} + \texttt{TWELVE} = \texttt{TWO} + \texttt{ELEVEN}$$

was unique in being anagrammatic, but I've just been thinking about it and I find

$$\texttt{FOUR} + \texttt{SIXTEEN} = \texttt{SIX} + \texttt{FOURTEEN}$$

and there are six such examples, letting 4, 6 be replaced by 4, 6; 4, 7; 4, 9; 6, 7; 6, 9; 7, 9.

I've now tried these examples to see if they give alphametics as above. In all but one case, the lengths differ and this rapidly leads to a contradiction. E.g. for the first case cited, we have to have $\texttt{SIX} = 999$, $\texttt{FOUR} = 1000$ and then the units digits lead to $\texttt{X} = \texttt{R}$, which is a contradiction. (This is making the assumption that the numbers do not have leading zeroes.) But for

$$\texttt{FOUR} + \texttt{NINETEEN} = \texttt{NINE} + \texttt{FOURTEEN},$$

the units digits give us $\mathtt{R} = \mathtt{E}$ and this forces $\mathtt{FOUR} = \mathtt{NINE}$ and the problem reduces to

$$\mathtt{NINE} + \mathtt{NINETEEN} = \mathtt{NINE} + \mathtt{NINETEEN},$$

which is trivial, with $(10)_7 = 10 \cdot 9 \cdot 8 \cdot 7$ solutions (this includes the cases with leading zeroes, but replacing the 10 on the right by a 9 gives the number without leading zeroes). So this isn't really satisfactory, but again it seems to be the best one can do.

Clearly,

$$\mathtt{SIXTY\text{-}SEVEN} \text{ and } \mathtt{SEVENTY\text{-}SIX}$$

are anagrams, as are 69 & 96, 79 & 97, but I can't see how to make use of these to make a doubly-true example.

We also have quite a number of examples like

$$\mathtt{TWENTY\text{-}ONE} + \mathtt{THIRTY\text{-}TWO} = \mathtt{TWENTY\text{-}TWO} + \mathtt{THIRTY\text{-}ONE}.$$

I thought at first that these led to trivial alphametics with the letters in each column of one sum being a permutation of the letters in the corresponding column of the other sum. But the lengths of the words can vary and this makes the situation more complex. I write the word forms without hyphens or spaces. For an early interesting example,

$$\mathtt{TWENTYONE} + \mathtt{THIRTYTHREE} = \mathtt{TWENTYTHREE} + \mathtt{THIRTYONE}.$$

The 0th, 1st, 2nd, 3rd, 4th columns from the right give no information, but the 5th column (appropriately) gives us $\mathtt{N} + \mathtt{Y} = \mathtt{R} + \mathtt{Y}$, so $\mathtt{N} = \mathtt{R}$. Proceeding to the left and using previous results, we eventually get $\mathtt{E} = \mathtt{I}$, $\mathtt{H} = \mathtt{W}$, $\mathtt{N} = \mathtt{R} = \mathtt{Y}$ and both sides reduce to

$$\mathtt{THENTNONE} + \mathtt{THENTNTHNEE},$$

with $(10)_5$ solutions. (Again, multiply by 9/10 to get the number without leading zeroes.)

After doing a lot of examples, I recognized that solvability depends on the lengths of the words rather than the exact forms. So consider the problem as being of the form $AC + BD = AD + BC$, where the first example above would be $21 + 32 = 22 + 31$, or $A = \mathtt{TWENTY}$, $C = \mathtt{ONE}$, $B = \mathtt{THIRTY}$, $D = \mathtt{TWO}$. Let $|X|$ be the number of letters in the English word for X; so $|A| = |B| = 6$, $|C| = |D| = 3$. We can assume $|D| \geq |C|$. It is easily seen that any assignment of values to letters gives an alphametic solution when $|C| = |D|$. But if $|D| > |C|$, then we can get an alphametic if and only if $|A| = |B|$. These alphametics will generally have some different letters having the same value.

There are also possibilities of the form $20 + 31 = 21 + 30$, i.e. $|C| = 0$.

Similar analysis shows this gives an alphametic solution if and only if $|A| = |B|$.

More elaborately, we have $67+79+96 = 76+69+97$ and $679+796+967 = 697+976+769$. We write this out as

$$\begin{aligned} & \texttt{SIXTYSEVEN} + \texttt{SEVENTYNINE} + \texttt{NINETYSIX} \\ = \; & \texttt{SEVENTYSIX} + \texttt{SIXTYNINE} + \texttt{NINETYSEVEN}. \end{aligned}$$

The 0th, 1st, 2nd, 3rd and 4th columns from the right give no information, but the 5th column (again appropriately!) gives us $\texttt{Y} + \texttt{T} + \texttt{E} = \texttt{N} + \texttt{T} + \texttt{Y}$, whence we must have $\texttt{E} = \texttt{N}$. The 6th column gives $\texttt{T} + \texttt{N} + \texttt{N} = \texttt{E} + \texttt{X} + \texttt{T}$, but $\texttt{E} = \texttt{N}$ forces $\texttt{E} = \texttt{X}$. Carrying on, we get $\texttt{E} = \texttt{I} = \texttt{N} = \texttt{S} = \texttt{V} = \texttt{X}$ and both sides reduce to

$$\texttt{EEETYEEEEE} + \texttt{EEEEETYEEEE} + \texttt{EEEETYEEE},$$

which has $(10)_3$ solutions. Considering the hyphen as a character only shifts the argument a bit and we get both sides reducing to

$$\texttt{EEETY-EEEEE} + \texttt{EEEEETY-EEEE} + \texttt{EEEETY-EEE},$$

with $(10)_4$ solutions. In the second case, we get the same letter identifications and both sides of the problem reduce to

$$\begin{aligned} & \texttt{EEEHUEDREDEEEETYEEEEE} \\ + \; & \texttt{EEEEHUEDREDEEEEETYEEE} \\ + \; & \texttt{EEEEEHUEDREDEEETYEEEE}, \end{aligned}$$

with $(10)_7$ solutions.

Problem 196.1 – 47 primes

ADF

Let S_n denote the smallest prime p such that $(n+1)^p - n^p$ is also prime. By a relatively straightforward computation, the sequence S_n for $n = 1, 2, ..., 46$ is as follows: (2, 2, 2, 3, 2; 2, 7, 2, 2, 3; 2, 17, 3, 2, 2; 5, 3, 2, 5, 2; 2, 229, 2, 3, 3; 2, 3, 3, 2, 2; 5, 3, 2, 3, 2; 2, 3, 3, 2, 7; 2, 3, 37, 2, 3; 5). Thus, for example, $5^2 - 4^2 = 9$ is composite whereas $5^3 - 4^3 = 61$ is prime—therefore $S_4 = 3$, the fourth entry in the list.

All we want you to do is determine S_{47}.

As you can see, S_n takes small values for n up to 46 although there is a bit of a blip at 22 and to a lesser extent at 43. But something strange seems to happen when you get to 47. If it exists, S_{47} exceeds 20000. The qualifier is relevant because, as far as I am aware, nobody has yet found a single example of a p for which $48^p - 47^p$ is prime.

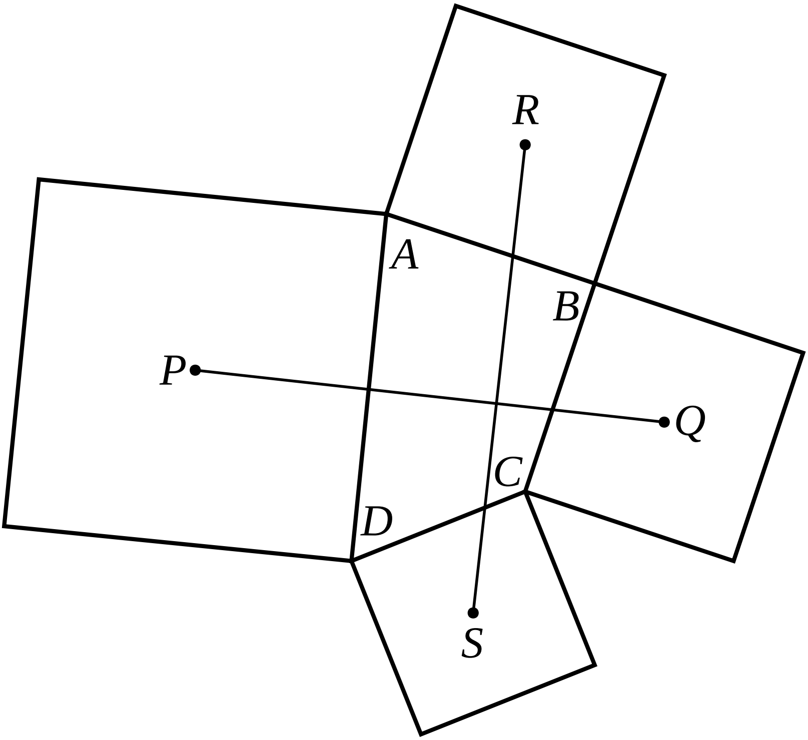
Problem 196.2 – Quadrilateral

Barbara Lee

In any quadrilateral $ABCD$, draw squares on its sides. Join the centres of the squares of opposite sides with PQ and RS. Prove that PQ is perpendicular to RS.

When you have done that, prove that PQ and RS are equal in length. This is more difficult.

It is helpful to obtain expressions for $(PQ)^2$ and $(RS)^2$ and equate them.



Anne Robinson – Which mathematical term translates from Latin as 'by the hundred'?

Contestant – Pythagoras' theorem. [JRH]

Solution 193.4 – Factorial inequality

Show that for positive integer n, $n! \le \left(\dfrac{n(n+1)^3}{8}\right)^{n/4}$.

Basil Thompson

By trial it can be seen that the inequality is true for $n = 1, 2, 3, 4, 5$, and it appears that the complicated factor $(n+1)^3/8$ is there only to cover these first few cases. For $n \ge 6$ we shall prove the simpler inequality,

$$n! < \frac{n^n}{2^n}. \tag{1}$$

This implies the original inequality since $n^n/2^n = (n^4/16)^{n/4}$ and $n^4/16 < n(n+1)^3/8$.

Write (1) as

$$\frac{n^n}{n!} > 2^n. \tag{2}$$

We prove (2) for $n \ge 6$ by induction. Clearly, (2) is true for $n = 6$. Suppose (2) is true for some $n \ge 6$, and consider $n+1$. Then

$$\begin{aligned}\frac{(n+1)^{n+1}}{(n+1)!} &= \frac{(n+1)^n}{n!} = \frac{n^n}{n!}\left(1+\frac{1}{n}\right)^n \\ &> 2^n\left(1+\frac{1}{n}\right)^n \qquad \text{(by the induction hypothesis)} \\ &> 2^{n+1},\end{aligned}$$

since $(1+1/n)^n$ can be expanded as

$$1 + \frac{n}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots > 2. \tag{3}$$

Hence (2) is true for $n+1$.

Tony Forbes

This is interesting. Observe that the series (3) tends to $e = 2.718281828\ldots$ as n tends to infinity. So we should be able to obtain a tighter inequality than (1), at least for sufficiently large n. For example, $612! < 612^{612}/(2.7)^{612}$ and then the above proof is quite easily modified to show that $n! < n^n/(2.7)^n$ holds for all $n \ge 612$.

John Bull

Define the arithmetic and geometric means A and G by

$$A \;=\; \frac{1}{n}(a_1 + a_2 + ... + a_n), \quad G \;=\; (a_1 a_2 \ldots a_n)^{1/n}.$$

We then use the arithmetic–geometric mean inequality, $A \geq G$.

When $a_1, a_2, \ldots, a_n$ happen to be the positive integers, we have $A = ((n+1)/2)$, $G = (n!)^{1/n}$ and it follows from $A \geq G$ that

$$n! \;\leq\; \left(\frac{n+1}{2}\right)^n.$$

Furthermore, since $n > 0$, we have

$$0 \leq n - 1 \;\Rightarrow\; n + 1 \leq 2n \;\Rightarrow\; \left(\frac{n+1}{2}\right)^4 \leq \frac{n(n+1)^3}{8}.$$

Hence

$$n! \;\leq\; \left(\frac{n+1}{2}\right)^n \;\leq\; \left(\frac{n(n+1)^3}{8}\right)^{n/4}.$$

Problem 196.3 – Combinatorial index

Tony Forbes

Let b be a number base. Imagine a list of all those t-digit numbers in base b which have increasing digits when read from left to right. The list is in numerical order. Let

$$N \;=\; d_t d_{t-1} \ldots d_2 d_1$$

be one of these numbers. Then the position of N in the list is given by

$$I(N) \;=\; \binom{b}{t} \;-\; \sum_{i=1}^{t} \binom{b-1-d_i}{i}.$$

To see how it works, let $b = 7$, $t = 3$. The list consists of the 35 numbers

012, 013, 014, 015, 016, 023, 024, 025, 026, 034, 035, 036,
045, 046, 056, 123, 124, 125, 126, 134, 135, 136, 145, 146,
156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456,

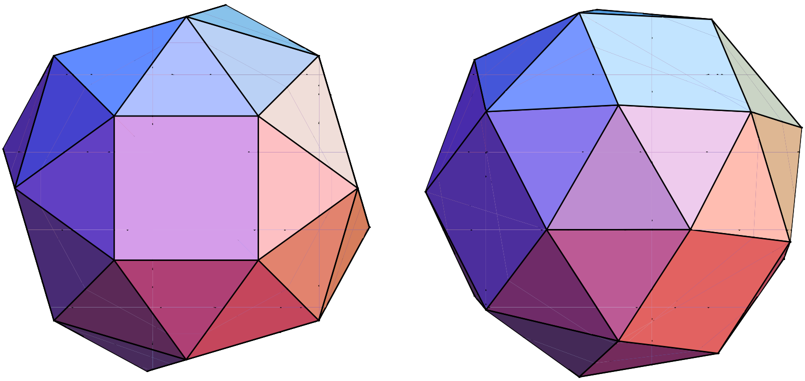
and, for example, if you apply the formula to 235, you should get 27.

Problem: Find a formula for the inverse function of $I(N)$. That is, given b and t, we want a function $J(n)$ which maps n to the number at position n in the list.

Problem 196.4 – Snub cube

Tony Forbes

Behold, two views of a snub cube, the Archimedean solid consisting of six squares and 32 equilateral triangles.



It is a chiral object and therefore it exists in two forms. The left-hand diagram shows the view looking down the z-axis on to a square. You can see that the four squares whose centres are in the (x, y)-plane have been rotated by an angle α, say, about the z-axis in a counter-clockwise direction. Alternatively, the squares individually have each undergone a clockwise rotation through α from their original 'cube' orientation. Reversing the angle to $-\alpha$ gives the other form of the object. You can see it by viewing the picture in a mirror.

I am curious. What is α?

Also I would like to know the angles between adjacent faces and the distance between two opposite squares. They are given approximately by Cundy & Rollett, *Mathematical Models*:

$$\begin{array}{lcl} \text{square–triangle dihedral angle} & \delta_1 & \approx\ 142^\circ\,59', \\ \text{triangle–triangle dihedral angle} & \delta_2 & \approx\ 153^\circ\,14', \\ \text{distance between opposite squares} & d_s & \approx\ 1/0.438\ \approx\ 16/7. \end{array}$$

However, this is not the type of answer that interests me. I am hoping that someone can obtain expressions giving δ_1, δ_2 and d_s exactly.

And if you are feeling really energetic, do something similar for the *snub dodecahedron*, the object featured on the front cover.

Dot products and determinants

John Spencer

In his interesting and informative article 'Dot products and determinants – is there a connection?' [M500 **193**], Robin Marks says that the area of a parallelogram whose sides are two vectors f and g is $\|f\|\|g\|\cos\theta$, the dot product of the vectors. In fact the area of the parallelogram is given by $\|f\|\|g\|\sin\theta$, the cross-product. The dot product is the product of those parts of the vectors which have a common orientation, the cross-product multiplies the vectors' orthogonal (perpendicular) parts. But $\|g\|\sin\theta$ is orthogonal to f, whereas $\|g\|\cos\theta$ is in the same direction as f, so cannot contribute to a measure of area.

It's not hard to see that for $0 < \theta < \pi/2$, the area enclosed by the vectors *increases* as θ increases, so sin rather than cos is the appropriate function. For proof, consider vectors f and g at angles of $\theta+\phi$ and ϕ to the x-axis of a plane. Without altering the area between them, f and g can be rotated about the origin through an angle $-\phi$ so that g lies along the x-axis.

In terms of the components of the two vectors, the rotating matrix is

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \frac{1}{\|g\|}\begin{bmatrix} g_0 & g_1 \\ -g_1 & g_0 \end{bmatrix}.$$

Acting on f, this gives

$$\frac{1}{\|g\|}\begin{bmatrix} g_0 & g_1 \\ -g_1 & g_0 \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \frac{1}{\|g\|}\begin{bmatrix} g_0 f_0 + g_1 f_1 \\ g_0 f_1 - g_1 f_0 \end{bmatrix}.$$

The rotating matrix moves f from its original position to $\|f\|\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$. So we can equate the two expressions

$$\|f\|\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \frac{1}{\|g\|}\begin{bmatrix} g_0 f_0 + g_1 f_1 \\ g_0 f_1 - g_1 f_0 \end{bmatrix}$$

to get

$$f_0 g_0 + f_1 g_1 = \|f\|\|g\|\cos\theta \quad \text{and} \quad g_0 f_1 - g_1 f_0 = \|f\|\|g\|\sin\theta.$$

The latter expression is the absolute value of $\det\begin{bmatrix} f_0 & g_0 \\ f_1 & g_1 \end{bmatrix}$.

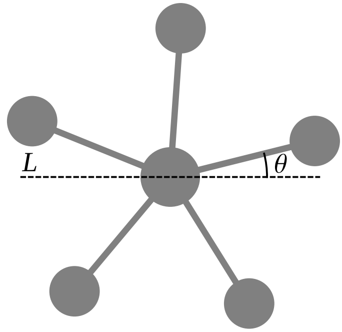
I don't think this affects the rest of Robin's argument, because he has based it on equating areas (volumes) and determinants, rather than on dot products.

Cosines and chandeliers

Dilwyn Edwards

While gazing at the ceiling of our committee room during an extremely long discussion, my colleague Mike Bramwell became convinced that cosine formulae of the form

$$\begin{aligned}\cos q + \cos(q + 2\pi/n)\\ + \cos(q + 4\pi/n) + \ldots\\ + \cos(q + 2(n-1)\pi/n)\\ = 0\end{aligned}$$



are caused by chandeliers.

The chandelier in question hangs so that its five arms are horizontal. Assume that the arms are weightless and of length r and that each light fitting has mass m. Taking moments about a horizontal line L, we have for equilibrium

$$\begin{aligned} 0 &= mgr\cos q + mgr\cos(q + 2\pi/5) + mgr\cos(q + 4\pi/5)\\ &\qquad + mgr\cos(q + 6\pi/5) + mgr\cos(q + 8\pi/5).\end{aligned}$$

Of course, as the chandelier has five arms the proof is for $n = 5$. To obtain the result for some other n we just need to persuade the university to fit a new chandelier.

Problem 196.5 – Three more friends

David Kerr

I have three friends, Alan, Bert and Curt. I write an integer greater than zero on the forehead of each of them and I tell them that one of the numbers is the sum of the other two. They take it in turns in alphabetical order to attempt to deduce their own number. The conversation goes as follows.

Alan: “I cannot deduce my number.”

Bert: “I cannot deduce my number.”

Curt: “I cannot deduce my number.”

Alan: “My number is 50.”

What are Bert’s and Curt’s numbers?

[Not quite the same as Problem 189.6. See David’s letter on page 28.]

Solution 193.2 Thirteen tarts

> There are thirteen tarts. All weigh the same, with one exception. Is it possible to identify the odd tart with three weighings?

Dick Boardman

For the problem to be soluble, the number of possible observations must not be less than the number of answers. This is true, not just at the start, but at all stages of the solution.

Now suppose the first weighing balances four against four, with five tarts off the scales. Suppose this weighing balances. Then the odd one is one of the five off the scales; that is 10 possible answers. But there are only two weighings left—only 9 possible observations. Hence the problem is impossible with this first weighing.

Now suppose the first weighing balances five against five, with three tarts off the scales. Suppose this weighing does not balance. Then the odd one out is one of the 10 on the scales, that is 20 possible answers. There are only 18 possible observations (9 for left side heavy and 9 for right side heavy), so no solution is possible for this weighing. Any other first weighing is worse and so no solution is possible for the problem as stated.

However, suppose a fourteenth tart, known to be of the correct weight, is available for the first weighing. A solution is now possible. All you have to do is alter Tony Forbes's weighings of M500 **191**, page 23, as follows.

1st weighing:	`A B C D M`	against	`E F G H N`
2nd weighing:	`A B C E N`	against	`D I J K M`
3rd weighing:	`A D F I N`	against	`B G J L M`

Here, `M` is the 13th tart and `N` is the known good tart. (If you want to be pedantic, replace `N` in the 2nd and 3rd weighings by another good tart.)

If `M` is bad, the left pan will go either (up, down, down) or (down, up, up). But these two combinations were previously impossible; hence they now unambiguously indicate `M` light or `M` heavy, respectively.

If `M` is good then the `M` and the `N` have no effect; we treat the situation as if we had just tarts `A`–`L`, and we can read off the result from the twelve tarts table in M500 **191**.

This also solves the slightly more difficult problem where you are given only that at most one tart is not of the correct weight. For then 'all correct' will be detected by (balance, balance, balance), which is the third unused entry in Forbes's table.

Solution 193.7 – Binomial coefficients

Which ${}^{2n}C_n$ are not divisible by the square of an odd prime?

Sebastian Hayes

The first ${}^{2n}C_n$ divisible by the square of an odd prime is

$$ {}^{10}C_5 \;=\; \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5} \;=\; 10\cdot 9\cdot 8\cdot 7\cdot 6 \;=\; 252 \;=\; 28\cdot 3^2. $$

Why does this work? Because 3 occurs once below and three times above so we gain an extra two 3s.

The key then is to gain (at least) an extra two occurrences in the numerator. Now if there are r multiples of p, our prime, in the denominator, there will be at most $r+1$ multiples in the numerator and the latter will only occur if $n = rp + m$, where $m > p/2$. This results in a gain of only one occurrence of p but a square will provide a second. We require then the double condition $n = rp + m$ with $p > m > p/2$ and $p < n < p^2 < 2n$, where $p > 2$.

Thus ${}^{26}C_{13}$ will work, also ${}^{28}C_{14}$ since both the conditions are met. But ${}^{30}C_{15}$ will not work because $m < 5/2$. Solutions will in fact come in batches $(p-1)/2$ long.

The smallest ${}^{2n}C_n$ will occur when p^2 is at one end of the numerator, and the largest when it is at the other:

$$ \frac{p^2(p^2+1)\ldots 2(p^2-1)}{1\cdot 2\ldots(p^2-1)} \;=\; {}^{2(p^2-1)}C_{p^2-1}, $$

$$ \frac{(p^2+3)/2\ldots(p^2+1)}{1\cdot 2\ldots(p^2+1)/2} \;=\; {}^{p^2+1}C_{p^2+1}/2. $$

In the sequence $n = 1$, 2, 3, 4, **5**, 6, 7, **8**, 9, 10, 11, 12, **13**, **14**, 15, 16, 17, **18**, **19**, 20, 21, 22, **23**, **24**, **25**, **26**, **27**, the bold n indicate what I shall call *prime-square* coefficients, i.e. n for which ${}^{2n}C_n$ is a multiple of the square of an odd prime. The pattern is that for each prime p we have $(p+1)/2$ clusters of prime-square coefficients between $n = (p^2+1)/2$ and $n = p^2 - 1$ and each of them is $(p-1)/2$ digits long.

Thus for 3 we have two occurrences at 5 and 8, each a single term long. Then 5 homes in at 13 with three two-term clusters $(13, 14)$, $(18, 19)$ and $(23, 24)$, and 7 commences at 25. As it happens, 7 takes over when 5 ends, so we get a cluster of $2 + 3 = 5$.

The same sort of thing will work for all higher powers of p. Certainly, since $(p^r + 1)/2 > p^{r-1}$, all powers lower than p^r, the highest, will be squeezed into the denominator and the numerator will not contain any fewer occurrences of multiples of a particular power than the denominator. Thus 3 enters the picture again at $n = 14$ with further occurrences at $n = 17, 20, 23, 26$, and we can fill in one or two gaps in the list to produce $n =$ 1, 2, 3, 4, **5**, 6, 7, **8**, 9, 10, 11, 12, **13**, **14**, 15, 16, **17**, **18**, **19**, **20**, 21, 22, **23**, **24**, **25**, **26**, **27**, 28, 29, 30, 31, **32**, **33**, **34**.

To sum up, the conditions for ${}^{2n}C_n$ being a prime-square coefficient are that for some p and $r \geq 2$,

$$\frac{p^r + 1}{2} \leq n \leq p^r - 1 \quad \text{and} \quad n \equiv \frac{p+1}{2}, \ldots, p - 1 \pmod{p}.$$

A point that needs to be investigated is whether the intervals $[(p^r + 1)/2, p^r - 1]$ overlap in order to provide complete coverage for any n. This seems to be the case from 13 onwards but I wouldn't care to prove it.

David Porter

The answer is possibly 1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 21, 22, 28, 29, 30, 31, 36, 37, 54, 55, 57, 58, 110, 171, 784 and 786.

Why do I say this? Well to tell the truth I cheated and wrote a computer program (in MAPLE) that checked on all the n up to one million and those are the only square free ones it found. To allow the program to perform without having to handle ridiculously large numbers I did first have to indulge in the following bit of mathematics.

Let $T_n = {}^{2n}C_n = (2n)!/(n!)^2$. So $T_{n+1} = (2n + 2)(2n + 1)T_n/(n + 1)^2$ which on back stepping one stage gives us the recurrence relationship

$$T_n = 2T_{n-1}(2n - 1)/n.$$

This relationship allows the calculation to progress from one factorization to the next by just adding in the odd prime factorization of $2n - 1$ and subtracting that of n.

A few facts. For $n = 786$ T_n had 169 odd prime factors ranging from 3 to 1571. For none of the T_n that contained an odd prime square was it necessary to check primes above 43, and this was first necessary for $n =$ 3250.

For $n = 1000000$ the odd prime factorization of T_n contained one 8th power (3), one 4th power (17), six cubes (5, 19, 43, 47, 53 and 73) and 61 squares (from 31 up to 1409). Based on this very small amount of evidence I would guess that there are no more to be found.

Letters to the Editor

Three friends

Dear Tony,

You asked why there were apparently two solutions to Problem 189.6. [I have three friends, Alan, Bert and Curt. I write a different positive integer on the forehead of each of them . . .] As the source of the problem I thought I'd better have a look at it and, to my horror, I realized that I'd made a mistake in the wording. The second sentence should read: 'I write an integer greater than zero on the forehead . . . ', i.e. the numbers are not necessarily different. According to my reasoning the problem now has a unique answer, which is neither (50, 40, 10) nor (50, 10, 40). Hopefully this is now OK.

David Kerr

OK. We present the corrected version as Problem 196.5. See page 24.

LEDs

Dick Boardman's solution to Problem 190.5 (Eight switches) for LEDs is, in fact, the way that an LED display screen is controlled, using transistors in place of the switches. The fact that you have to use one transistor for each row and column of LEDs, with an unholy amount of associated wiring (and have to do this three times over for a colour screen) is the reason why flat screens are still so expensive. So far no one has found a better way of doing it. If anyone were to come up with an answer, they should promptly build a small working model, patent it, and sell the rights to Matsushita or somebody for an immense sum.

Best wishes,

Ralph Hancock

Inequalities

Dear Tony,

It is curious that not more inequality problems are proposed as there must be infinitely many more inequality problems than equality ones. One might have thought this would lead to an even greater number of novel and innovative solutions than we actually see.

John Bull

M500 194

Tony,

Problem 194.1. A friend of mine wrote a paper asking us to start by considering a pencil of matrices with particular properties. So presumably a pencil is a collective noun.

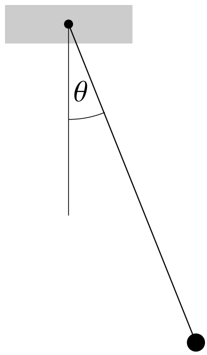
Fermat numbers. I did not realize that 645815 was such a good approximation to $\lceil \log_{10} \log_{10}(2^{2^{2145351}} + 1) \rceil$. Even so, my answer is the same as yours within my estimation of error.

Getting dressed. I foresee problems with assuming the usual conventions. A model I once knew said that, when dressing for the catwalk, she usually put her hat on first, and that as the topology of a skirt allowed for certain items of underwear to be put on last, she herself had no usual conventions for the order of dressing.

Colin Davies

Problem 196.6 – Pendulum

ADF



Show that

$$\theta(t) \;=\; 4 \arctan e^{\sqrt{g/L}\; t} - \pi$$

is a solution of

$$\frac{d^2\theta}{dt^2} \;=\; -\frac{g}{L} \sin\theta,$$

the equation of motion of a pendulum of length L under gravitational acceleration g.

This appeared on Stan Wagon's *Problem of the Week* (Internet mailing list) but when the answer arrived it was just quoted without explanation.

Notice that when $t = -\infty$ the pendulum points vertically upwards, is stationary and has zero acceleration—so it should stay there forever. But when $t = 0$ the pendulum is pointing downwards and moving with speed $2\sqrt{gL}$. It's amazing what can happen after an infinite amount of time!

Crossnumber 195 solution

ACROSS 3. 162^2, 5. 288^2, 6. 219^2, 7. 3^{17}, 11. 224^2, 12. 164^2, 13. 5^6

DOWN 1. 241^2, 2. 3^{10}, 3. 5^{12}, 4. 6^6, 8. 145^2, 9. 3^9, 10. 192^2

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