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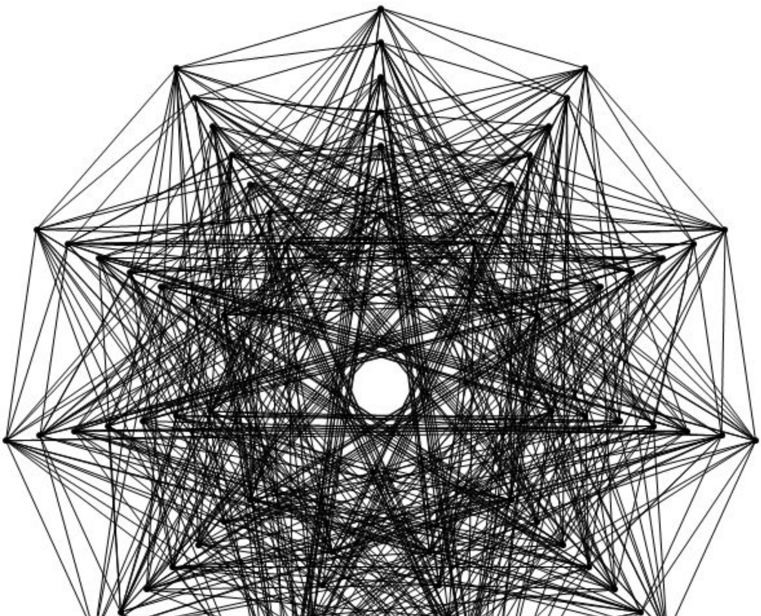
# M500 221



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**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

**The magazine** M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**MOUTHS** is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

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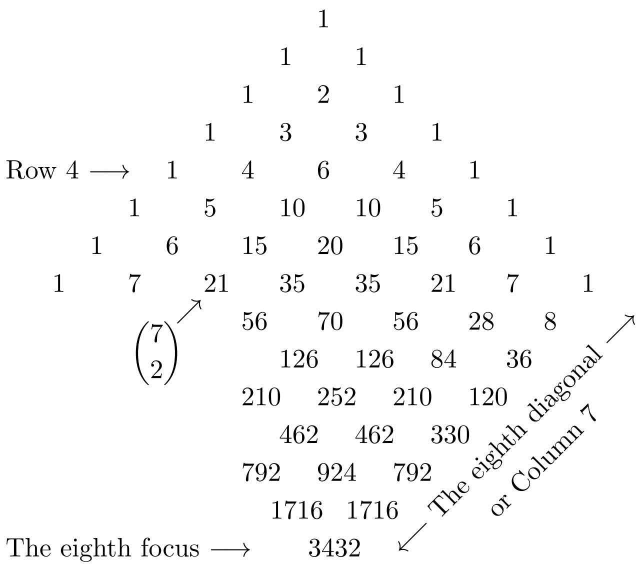
**Advice to authors**. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file to tony@m500.org.uk.

# Revisiting Pascal's triangle


## Martin Hansen


In the *Back To The Future* trilogy of time-travel films, the principal characters repeatedly find themselves back, both in space and time, at their small town's clock tower. Every return adds a layer of complexity to the moment at 10.04pm on the 12th of November 1955 when the tower is struck by lightning. The film's heroes, teenager Marty McFly and (mad) Dr Emmett Brown, repeatedly add to what they know from previous visits to navigate an increasingly intricate region of space and time without committing the ultimate time traveller's crime: meeting an earlier version of themselves.

The mathematical equivalent of the clock tower in Hill Valley's town square is, for me, Pascal's triangle of binomial coefficients. Recently, I had cause to rethink what I knew from previous visits. Past entanglements have seen me buy books on the triangle [1], print out material from websites devoted to it [2] and read accounts of how computers are being used to find and prove ever more obscure relationships between its entries [3]. Alas, there is too much information. The key ideas are buried amid the inconsequential. Having grappled with the triangle once again, I have a good understanding of what is worth knowing. I thought it worth sharing with the M500 readership.



A tour of the basics is a good place to start, for there are irritations that a newcomer needs to get to grips with and which occasionally trip up older hands, myself included.

Confusion can be minimized by careful use of words when talking about the rows and columns. In the diagram notice row 4 is the horizontal line '1 4 6 4 1'. It's actually the fifth row of the triangle. Worse, column 7 is the diagonal that begins '1 8 36 120 ...'. It's the eighth diagonal. To avoid muddle, never talk about the fourth row or the seventh column. Instead say row 4 and column 7. The 1 at the apex is at row 0, column 0.

Illogically, the letter $n$ represents row and $r$ column. The binomial coefficient at row $n$, column $r$, can be written ${}^nC_r$. Note carefully that $r$ does not represent 'row' nor $C$ 'column'. Unfortunately, this is too established to be changed. the symbol ${}^7C_2$ can better be written with the 7 above the 2 all inside vertically stretched curved brackets. There is a logic to this notation being the same as that for a vector as the 7 and the 2 provide a position vector of sorts to the entry, 21; ${}^7C_2$ is often said '7 choose 2'.

An obvious feature of the array is 'any internal element is the sum of the two terms directly above it, one on a previous column (or diagonal) to the left, the other, on the same column (or diagonal), to the right'. Translated into mathematics this becomes *Pascal's Rule*:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

I remember Pascal's Rule in words and translate it each time I require it. The words subtly keep me thinking correctly about the columns which, like the diagonals, slope downward from right to left.

The following formula, usually proved by a counting argument [4], allows any given binomial coefficient, at row $n$, column $r$, to be evaluated:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

From this formula one can easily prove, using algebra, Pascal's Rule. Also, it confirms the value of ${}^7C_2$:

$$\binom{7}{2} = \frac{7!}{2!(7-2)!} = \frac{7!}{2!5!} = 21.$$

I enjoy rethinking old results in new ways. A fresh perspective can be gained by considering any given entry in Pascal's array as the $t$th term along the

$d$th diagonal. Note the change in the way language is being used with counting from a first diagonal and a first term rather than row zero and column zero. The 21 of the recent example is on the third diagonal where it's the sixth term. Thought yields a rewrite of the factorial formula:

$$\left\langle\!\!\left\langle \begin{matrix} d \\ t \end{matrix} \right\rangle\!\!\right\rangle \;=\; \frac{[(d-1)+(t-1)]!}{(d-1)!(t-1)!}.$$

This is written to be natural and memorable. When thinking rows and columns, readers hunt for the row first then the column. The existing notation places row $n$ above column $r$. When thinking diagonals and terms along a diagonal, it's the diagonal, $d$, that ones eye tracks for first, then the count along to the term, $t$. So, the $d$ is above the $t$ in the notation. This helps differentiate between the two representations of the same thing. In the one, the lower number, $r$, is less than or equal to the upper, $n$, whereas in the other it's the upper number, $d$, that is less than or equal to the lower, $t$. The double angled brackets notation I made up. It's not standard but it seemed appropriate to use vertically stretched brackets of some sort. To be absolutely clear, the 21 of the earlier example found via ${}^7C_2$ can be also found from

$$\left\langle\!\!\left\langle \begin{matrix} 3 \\ 6 \end{matrix} \right\rangle\!\!\right\rangle \;=\; \frac{[(3-1)+(6-1)]!}{(3-1)!(6-1)!} \;=\; \frac{7!}{2!5!} \;=\; 21.$$

Incidentally, before moving on, Pascal's Rule can be expressed as

$$\left\langle\!\!\left\langle \begin{matrix} d \\ t \end{matrix} \right\rangle\!\!\right\rangle \;=\; \left\langle\!\!\left\langle \begin{matrix} d \\ t-1 \end{matrix} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \begin{matrix} d-1 \\ t \end{matrix} \right\rangle\!\!\right\rangle.$$

The triangle is riddled with well-known number sequences. Along the third diagonal, for example, are the triangular numbers:

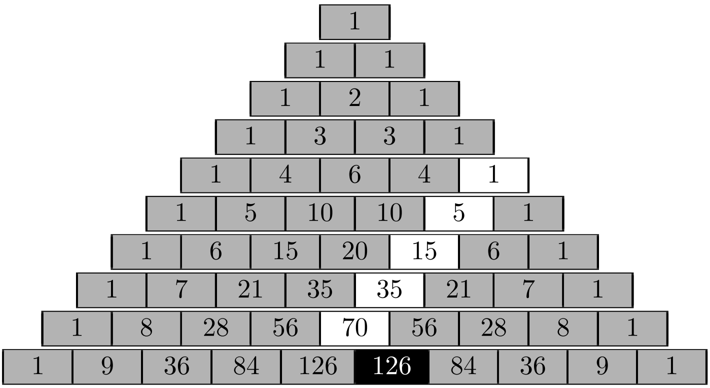
$$1, 3, 6, 10, 15, 21, \ldots, \frac{t(t+1)}{2}.$$

This is not a dead end observation. Pietro Mengoli discovered in 1650, and proved in his book *Novae Quadraturae Arithmetica*, that the sum of the reciprocals of the triangular numbers is 2:

$$\sum_{t=1}^{\infty} \frac{2}{t(t+1)} \;=\; 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \ldots \;=\; 2.$$

At the time, Mengoli's results, such as this, were ground breaking. Stopple in [5] looks at various ways of proving this result and the similar one for the tetrahedral numbers which lie along the fourth diagonal. In our very own M500, Nick Hobson recently [6] presented a method with proof that summed to infinity the reciprocal of any diagonal you care to pick—except the first two, which do not converge. Working the diagonals of Pascal's triangle can result in significant mathematics.

A key observation from the desire to work the diagonals is to notice what Tony Colledge in his poster on Pascal's Triangle [7] refers to as an *edge worm.*




The numbers in the diagonally descending edge worm's tail sum to its head which, if the body is a zig, is a zag. Formally, this result is known as *Fermat's combinatorial identity*:

$$\binom{n}{s} = \binom{f+1}{s+1}.$$

This identity says 'the sum of the edge worm's tail from row, $s$, (start) down to row, $f$, (finish) is in the head, which, if the tail is a zig, is a zag'. Again, this is phrased to be memorable and easy to translate into mathematics.

The highlighted edge worm runs from row 4 to row 8 along column 4:

$$\binom{n}{4} = \binom{9}{5} = 126.$$

In the literature there are convoluted explanations of why Fermat's combinatorial identity works. Michael Hirschhorn, alongside a campaign urging mathematicians to add this identity to their armoury, points out that its proof need only be a one line affair [8]:

$$\sum_{n=s}^{f}\binom{n}{s} = 1+\sum_{n=s+1}^{f}\left\{\binom{n+1}{s+1}-\binom{n}{s+1}\right\} = \binom{f+1}{s+1}.$$

This is a splendid example of telescopic cancelling. To appreciate it, write out the linking step in the proof for the example edge worm along column 4:

$$\begin{aligned}\sum_{n=4}^{8}\binom{n}{4} &= 1+\sum_{n=5}^{8}\left\{\binom{n+1}{5}-\binom{n}{5}\right\}\\ &= 1+\left\{\binom{6}{5}-\binom{5}{5}\right\}+\left\{\binom{7}{5}-\binom{6}{5}\right\}\\ &\quad+\left\{\binom{8}{5}-\binom{7}{5}\right\}+\left\{\binom{9}{5}-\binom{8}{5}\right\}\\ &= 1-\binom{5}{5}+\binom{9}{5} = \binom{9}{5} = 126.\end{aligned}$$

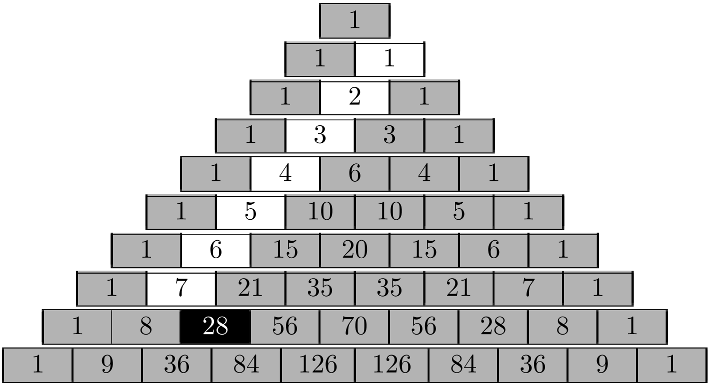
Here's the same result and proof expressed in diagonals and terms:

$$\sum_{t=s}^{f}\left\langle\left\langle {d \atop s}\right\rangle\right\rangle = 1+\sum_{t=s}^{f}\left\{\left\langle\left\langle {d+1 \atop t+1}\right\rangle\right\rangle - \left\langle\left\langle {d+1 \atop t}\right\rangle\right\rangle\right\} = \left\langle\left\langle {d+1 \atop f}\right\rangle\right\rangle.$$

Check:

$$\sum_{t=1}^{5}\left\langle\left\langle {5 \atop 1}\right\rangle\right\rangle = \left\langle\left\langle {6 \atop 5}\right\rangle\right\rangle = 126.$$

It was observed that the triangular numbers lie along the third diagonal of Pascal's triangle. They are intimately linked to the edge worm concept. For example, the illustration shows the seventh triangular number, 28, is the head of the worm along the second diagonal that contains seven terms, 1 + 2 + 3 + 4 + 5 + 6 + 7. This generalizes easily to the statement that 'the $t$th triangular number, is at the head of the worm along the second diagonal that contains $t$ terms'.




So, in general, the $t$th triangular number is formed from the tail given by

$$\sum_{n=1}^{t} \binom{n}{1}.$$

Applying Fermat's combinatorial identity to this is a lovely way of obtaining the formula for the $t$th triangular number:

$$\sum_{n=1}^{t} \binom{n}{1} = \binom{t+1}{2} = \frac{(t+1)!}{2!((t+1)-2)!} = \frac{t(t+1)}{2!}.$$

Care is needed when trying to obtain a similar formula for the tetrahedral numbers which lie along the fourth diagonal. The edge worm this time runs down the third diagonal from row 2 to the row that is one more than any given term, $t$, in the third diagonal that is being summed to. Thus

$$\sum_{n=2}^{t+1} \binom{n}{2} = \binom{t+2}{3} = \frac{(t+2)!}{3!((t+2)-3)!} = \frac{t(t+1)(t+2)}{3!}.$$

From these two illustrative steps, it's straightforward to generalize the argument and obtain a formula for the $t$th term on the $d$th diagonal:

$$\begin{aligned}\sum_{n=d-2}^{t+d-3}\binom{n}{d-2} &= \binom{t+d-2}{d-1} \\ &= \frac{(t+d-2)!}{(d-1)!((t+d-2)-(d-1))!} \\ &= \frac{(t+d-2)!}{(d-1)!(t-1)!} \\ &= \frac{1}{(d-1)!}t(t+1)(t+2)(t+3)\ldots(t+d-2).\end{aligned}$$

The manipulations to obtain this were long winded. More succinctly, recall

$$\left\langle\left\langle{d \atop t}\right\rangle\right\rangle = \frac{[(d-1)+(t-1)]!}{(d-1)!(t-1)!}.$$

From this, what has just been proven via delicate thinking is obvious:

$$\left\langle\left\langle{d \atop t}\right\rangle\right\rangle = \frac{1}{(d-1)!}t(t+1)(t+2)(t+3)\ldots(t+d-2).$$

I hope at this juncture, readers do not feel they've been lead a merry dance unprofitably. Both trains of thought are interesting. The longer comes from the established way of thinking about the triangle. Thinking differently gets the same result in fewer steps. As a bonus, being familiar with both paths places within reach a result which knowledge of the edge worm concept has made easy to grasp.

A formula has been found, in two ways, for the $t$th term on the $d$th diagonal. Pascal's combinatorial identity reveals this is also the sum of $t$ entries along the $(d-1)$th diagonal:

$$\sum_{t=1}^{T}\frac{1}{(d-2)!}t(t+1)\ldots(t+d-3) = \frac{1}{(d-1)!}T(T+1)\ldots(T+d-2).$$

Multiply both sides by $(d-2)!$:

$$\sum_{t=1}^{T}t(t+1)\ldots(t+d-3) = \frac{1}{(d-1)}T(T+1)\ldots(T+d-2).$$

Now let $r = d - 3$ and shuffle the letters. An abstract relationship results:

$$\sum_{i=1}^{n} \prod_{a=0}^{r} (i+a) \;=\; \frac{1}{r+2} \prod_{a=0}^{r+1} (n+a).$$

When I obtained this I was very pleased with myself. I've since found it in an old A-level mathematics formula book [9] where it is written in less abbreviated form:

$$\sum_{i=1}^{n} i(i+1)(i+2)\ldots(i+r) \;=\; \frac{1}{r+2} n(n+1)(n+2)\ldots(n+r+1).$$

This is a beautiful result, that we've proved fairly easily by thinking about Pascal's triangle in two different ways. It deserves to be far better known than would seem to be the case.

**References:**

[1] A. W. F. Edwards, *Pascal's Arithmetical Triangle*, Charles Griffin, 1987.

[2] *Pascal's triangle from top to bottom*, http://binomial.csuhayward.edu.

[3] Joe Gallian and Michael Pearson, An interview with Doron Zeilberger, *Focus: Mathematical Association of America Newsletter*, May/June 2007.

[4] David M. Burton, *Elementary Number Theory* (fourth edition), McGraw–Hill, 1998.

[5] Jeffrey Stopple, *A Primer of Analytic Number Theory*, CUP, 2003.

[6] Nick Hobson, Pascal triangle sums, M500 **216**, June 2007.

[7] Tony Colledge, *Pascal's triangle and the magic of its number patterns*, A wall poster published by Tarquin Publications.

[8] Michael D. Hirschhorn, An identity involving binomial coefficients, *The Mathematical Gazette*, July 2005.

[9] *Formulae for Advanced Mathematics with Statistics Tables* (second edition), CUP, 1989.

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*Q. What is the value of a contour integral around Western Europe?*

*A. Zero.*

*Q. Why?*

*A. Because all the Poles are in Eastern Europe!*

[Found on the Web. Presumably Western Europe does not include Britain.]

# Solution 216.3 – Reflection

> What is the function you get when you reflect graph of $y = e^x$ in the line $y = ax$, where $a$ is a constant?

## Steve Moon

Line $y = ax$ makes an angle $\theta = \tan^{-1} a$ with the positive $x$ axis at the origin. The matrix representation of reflection in this line is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Using standard trigonometric formulae

$$\cos 2\theta = \frac{1-a^2}{1+a^2}, \qquad \sin 2\theta = \frac{2a}{1+a^2},$$

let $(x', y')$ be the map of a point $(x, y)$ on reflection; then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \dfrac{1-a^2}{1+a^2} & \dfrac{2a}{1+a^2} \\ \dfrac{2a}{1+a^2} & \dfrac{a^2-1}{1+a^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence

$$\begin{aligned} (1+a^2)x' &= (1-a^2)x + 2ay, \\ (1+a^2)y' &= 2ax + (a^2-1)y. \end{aligned}$$

Solving yields

$$\begin{aligned} x &= \frac{1-a^2}{1+a^2}\, x' + \frac{2a}{1+a^2}\, y', \\ y &= \frac{2a}{1+a^2}\, x' - \frac{1-a^2}{1+a^2}\, y', \end{aligned}$$

and substituting for $x$ and $y$ in $y = e^x$ gives

$$\frac{2a}{1+a^2}\, x' - \frac{1-a^2}{1+a^2}\, y' = \exp\left(\frac{1-a^2}{1+a^2}\, x' + \frac{2a}{1+a^2}\, y'\right).$$

Using trigonometric formulae and replacing $x'$ by $x$ and $y'$ by $y$,

$$x \sin(2\tan^{-1} a) - y \cos(2\tan^{-1} a) = e^{x \cos(2\tan^{-1} a)} e^{y \sin(2\tan^{-1} a)}.$$

Check: Putting $a = 0$, $1$, $\infty$ produces $-e^x$, $\log x$, $e^{-x}$ respectively.

# The Golden Ratio pops out of 5-dimensional space

**Dennis Morris**

The 5-dimensional H-type natural algebra is

$$C_{5,1}L^1H^4 \;=\; \begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix}.$$

We have

$$\begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix} \begin{bmatrix} f & g & h & i & j \\ j & f & g & h & i \\ i & j & f & g & h \\ h & i & j & f & g \\ g & h & i & j & f \end{bmatrix}$$

$$= \begin{bmatrix} af+bj+ci+dh+eg & \sim & \sim & \sim & \sim \\ ef+aj+bi+ch+dg & \sim & \sim & \sim & \sim \\ df+ej+ai+bh+cg & \sim & \sim & \sim & \sim \\ cf+dj+ei+ah+bg & \sim & \sim & \sim & \sim \\ bf+cj+di+eh+ag & \sim & \sim & \sim & \sim \end{bmatrix}.$$

Thus, in non-matrix notation, the multiplication operation is

$$\begin{aligned} &(a+b\hat{m}+c\hat{n}+d\hat{o}+e\hat{p})\,(f+g\hat{m}+h\hat{n}+h\hat{o}+j\hat{p}) \\ &\quad= \;(af+bj+ci+dh+eg+(bf+cj+di+eh+ag)\hat{m} \\ &\qquad+(cf+dj+ei+ah+bg)\hat{n}+(df+ej+ai+bh+cg)\hat{o} \\ &\qquad+(ef+aj+bi+ch+dg)\hat{p}. \end{aligned}$$

From the adjoint matrix, the conjugate is

$$\begin{aligned} &\overline{(a+b\hat{m}+c\hat{n}+d\hat{o}+e\hat{p})} \\ &\quad= \;a^4-3a^2be-3a^2cd+2ab^2d+2abc^2+2ad^2e+ace^2 \\ &\qquad-b^3c-c^3e-bd^3-de^3+b^2e^2+c^2d^2-bcde \\ &\qquad+(e^4-3ade^2-3bce^2+2a^2ce+2ab^2e+2c^2de+2bd^2e \\ &\qquad\qquad-a^3b-b^3d-ac^3-cd^3+a^2d^2+b^2c^2-abcd)\,\hat{m} \\ &\qquad+(d^4-3abd^2-3cd^2e+2a^2de+2b^2cd+2ac^2d+2bde^2 \\ &\qquad\qquad-a^3c-b^3e-bc^3-ae^3+a^2b^2+c^2e^2-abce)\,\hat{n} \\ &\qquad+(c^4-3ac^2e-3bc^2d+2a^2bc+2b^2ce+2acd^2+2cde^2 \\ &\qquad\qquad-a^3d-ab^3-d^3e-be^3+a^2e^2+b^2d^2-abde)\,\hat{o} \end{aligned}$$

$$+ (b^4 - 3ab^2c - 3b^2de + 2a^2bd + 2bc^2e + 2bcd^2 + 2abe^2$$
$$- \; a^3e - c^3d - ad^3 - ce^3 + a^2c^2 + d^2e^2 - acde)\,\hat{p}.$$

Multiplying one of these matrices by its conjugate produces the determinant. This is an expression too horrendous for the gentle reader to view, but it leads to identities like $\hat{m}\hat{p} = 1$, $\hat{n}\hat{o} = 1$, $\hat{m} = \hat{o}\hat{o}$, $\hat{n} = \hat{o}\hat{p}$, $\hat{o} = \hat{m}\hat{n}$, $\hat{p} = \hat{m}\hat{o}$, $\hat{m}\hat{m} = \hat{o}\hat{p}$, $\hat{p}\hat{p} = \hat{m}\hat{n}$, $\hat{n}\hat{n} = \hat{m}\hat{o}$, $\hat{n} = \hat{m}\hat{m}$, $\hat{m}^5 = 1$, $\hat{n}^5 = 1$, $\hat{o}^5 = 1$, $\hat{p}^5 = 1$ and many more.

In the euclidean complex numbers the fifth roots of unity have the same relations with

$$\begin{aligned}
\hat{m} &\equiv \frac{\sqrt{5}-1}{4} + \hat{i}\,\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}, \\
\hat{n} &\equiv \frac{-\sqrt{5}-1}{4} + \hat{i}\,\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, \\
\hat{o} &\equiv \frac{-\sqrt{5}-1}{4} - \hat{i}\,\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}, \\
\hat{p} &\equiv \frac{\sqrt{5}-1}{4} - \hat{i}\,\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}.
\end{aligned}$$

We thus have the 2-dimensional shadow algebra of this 5-dimensional algebra:

$$\begin{aligned}
&\mathrm{Shad}\,(a + b\hat{m} + c\hat{n} + d\hat{o} + e\hat{p}) \\
&= a + b\left(\frac{\sqrt{5}-1}{4} + \hat{i}\,\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}\right) + c\left(\frac{-\sqrt{5}-1}{4} + \hat{i}\,\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right) \\
&\quad + d\left(\frac{-\sqrt{5}-1}{4} - \hat{i}\,\frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right) e\left(\frac{\sqrt{5}-1}{4} - \hat{i}\,\frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{4}\right) \\
&= a + \frac{(b-c-d+e)\sqrt{5} - (b+c+d+e)}{4} \\
&\quad + \hat{i}\sqrt{2}\left(\frac{\sqrt{5+\sqrt{5}}}{4}\,(b-e) + \frac{\sqrt{5-\sqrt{5}}}{4}\,(c-d)\right).
\end{aligned}$$

The determinant of this euclidean complex number is

$$a^2 + b^2 + c^2 + d^2e^2 - \phi(ac + ad + bd + be + ce) + \frac{ab + ae + bc + cd + de}{\phi},$$

where $\phi$ is the golden ratio. So distance in 5-dimensional space is connected to the golden ratio. The polar form of this complex number is

$$\exp\left(\frac{4a+(b-c-d+e)\sqrt{5}-(b+c+d+e)}{4}\right)$$
$$\cdot\big(\cos(0.95...b+0.59...c-0.59...d-0.95...e)$$
$$+\,\hat{\imath}\sin(0.95...b+0.59...c-0.59...d-0.95...e)\big),$$

where the numbers are more accurately 0.9510565161... and 0.5877852522.... We have

$$\frac{0.9510565161...}{0.5877852522...} \;=\; \phi.$$

Has anyone seen $\phi$ expressed as this ratio before?

# A word to the wise

## Eddie Kent

Oxford University Press has completed five years of the Oxford Corpus (part of the language research programme, where words are planted to see if they grow or die). To celebrate this some excerpts have been released, mostly I expect for entertainment. I'll mention some that relate to computing.

But first, one cannot ignore bling. This was first recorded in 1998, by the US rapper Baby Gangsta (B.G.), to refer to ostentatiously extravagant clothing and jewelery. By 2000 it was in wide use, having left the hip-hop world and even reached Britain. It has now given rise to a large family of offspring; currently more than 40. Bling-dripping arrived in 2006, three years after blingless.

The word scam had 347 instances in 2000, rising to 10409 this year; it is now associated with phishing, lottery, email, telephone and dialler. And here are half a dozen words that might be of interest to readers of this magazine: digilanti – volunteers dedicated to keeping scams (and spam) off the internet; bipodding – two people listening to the same mp3 player; crowdsourcing – outsourcing beyond the normal professional group; peerents – parents who want to be their children's friends; mojo – an amateur journalist using a mobile phone; and all the variants of cyber—(including the unlikely cyberloo). I can count 21 of these.

These words and many more have been collected by Countdown's Susie Dent from her latest Language Report recorded in the Oxford Corpus. She also lists those words that didn't make it: millennium bug hasn't been heard from this century, and the verb dyson had a sprited but doomed existence. The English language just keeps on evolving, with over two billion words now in the Corpus. If you are truly interested, visit www.askoxford.com.

# Problem 221.1 – Ten hats

## Tony Forbes

Ten people wear hats. On each there is a number 1, 2 or 3, chosen at random. As usual, each can see all the hat numbers except his/her own. At the appointed instant all persons simultaneously guess what number they have. If they are all correct, there is a Valuable Prize for everybody.

The probability of success, $3^{-10}$, is pretty poor, wouldn't you agree?

But the game is played again (with new numbers), and in the interval the participants have had time to discuss a strategy. Show that the probability of all correct can now be as high as 1/3.

And yet again. But this time the players are standing regularly spaced in a circular corridor; so each can see only the hat of his/her two immediate neighbours. Again, there is time for devising a strategy before the game commences. Now what probability of winning the prize can be achieved?

Thanks to **Søren Riis** of Queen Mary College, London for the idea behind this problem.

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# Problem 221.2 – Coefficients

Given $n$, find a simple formula for the coefficient of $x^k$ in the polynomial

$$(x+n-1)\,(2x+n-2)\;\ldots\;((n-1)x+1)\,(nx).$$

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# Problem 221.3 – Six tans

Show that

$$\frac{1}{\tan^2 \frac{1}{7}\pi + \tan^2 \frac{3}{7}\pi} + \frac{1}{\tan^2 \frac{3}{7}\pi + \tan^2 \frac{5}{7}\pi} + \frac{1}{\tan^2 \frac{5}{7}\pi + \tan^2 \frac{1}{7}\pi} = \frac{17}{26}.$$

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# Problem 221.4 – Eleven bottles

## Tony Forbes

Three people, A, B, C, are stuck in a lift over the weekend. They have 11 bottles of water, four supplied by A and seven by B, which are to be shared equitably. C donates £11 for the water. How is it to be divided between A and B?

The same thing happens the following weekend, but this time A has three bottles and B has eight. Again, £11 is to be split between A and B.

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# Solution 218.1 – Wands

> Having just returned from a night on the town Harry and his friends cast spells to ward off the effects of excessive drinking. But in the confusion on leaving the club they had each picked up a wand at random from the pile that was given back to them. Also the magic isn't perfect, and when it fails the caster will turn into a toad (albeit one without a hangover). The probability of the spell working properly is $p$ with one's own wand and zero with someone else's wand. Show that the probability of them all turning into toads is approximately $e^{-p}$. What is the expected number of toads created by this escapade?

## John Smith

A solution written in terms of $n$ friends can become a maze of notation. So first we consider the case of three friends, and later sketch the more general argument.

Suppose Harry goes clubbing with Albert and Bertie. Let $H$ be the event that, after the wand allocation and spell casting, Harry is not turned into a toad. Define $A$ and $B$ similarly for Albert and Bertie. Write $p(H)$, $p(A)$ and $p(B)$ for the probabilities that Harry, Albert and Bertie are not turned into toads. Let $P_3$ be the probability that all three are turned into toads. Using the inclusion–exclusion principle, we find that $P_3$ is given by

$$\begin{aligned} P_3 &= 1-(p(H)+p(A)+p(B))+(p(H\cap A)+p(A\cap B)+p(B\cap H)) \\ &\qquad -p(H\cap A\cap B). \end{aligned}$$

Since the problem has symmetry in the events $H$, $A$ and $B$,

$$P_3 = 1-3p(H)+3p(H\cap A)-p(H\cap A\cap B).$$

The probability $p(H)$ is given by the product of the probability of Harry getting his own wand, and the probability of Harry's spell working given that he has the right wand. There are six possible allocations of wands to friends. Of these six, Harry gets his own wand just twice. Thus $p(H)=\frac{1}{3}p$.

The probability $p(H\cap A)$ is given by the product of the probability of both Harry and Albert getting their own wands, and the probability of Harry's and Albert's spell working given that they have their own wands. In just one of the six possible allocations of wands to friends do both Harry and Albert get their own wands. Thus $p(H\cap A)=\frac{1}{6}p^2$.

Similarly, $p(H\cap A\cap B)=\frac{1}{6}p^3$.

Thus

$$P_3 = 1 - p + \frac{p^2}{2!} - \frac{p^3}{3!}.$$

The extension to $n$ friends is now practical. Suppose the friends are $A_1, A_2, \ldots, A_n$, and the probability that they all turn into toads is $P_n$. Then

$$\begin{aligned} P_n &= 1 - \sum_i p(A_i) + \sum_{i<j} p(A_i \cap A_j) - \sum_{i<j<k} p(A_i \cap A_j \cap A_k) + \cdots \\ &\quad + (-1)^n p(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) \\ &= 1 - {}^nC_1 p(A_1) + {}^nC_2 p(A_1 \cap A_2) - {}^nC_3 p(A_1 \cap A_2 \cap A_3) + \cdots \\ &\quad + (-1)^n\, {}^nC_n p(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n). \end{aligned}$$

The probability $p(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_r)$ is given by the product of $p^r$ and the probability that friends $1, 2, \ldots, r$ of the $n$ all receive their own wands. The allocation of wands to friends defines a permutation on the $n$ friends. There are $n!$ possible permutations, of which in $(n-r)!$ permutations the friends $1, 2, \ldots, r$ all get their own wand. Thus the probability that friends $1, 2, \ldots, r$ of the $n$ receive their own wand is $(n-r)!/n!$. Then

$$\begin{aligned} P_n &= 1 - p + \frac{p^2}{2!} - \cdots + (-1)^r \frac{n!}{(n-r)!r!} \times \frac{(n-r)!}{n!} p^r + \cdots + (-1)^n \frac{p^n}{n!} \\ &= 1 - p + \frac{p^2}{2!} + \cdots + (-1)^r \frac{p^r}{r!} + \cdots + (-1)^n \frac{p^n}{n!}. \end{aligned}$$

Thus $P_n$ is the first $n+1$ terms of the power series for $e^{-p}$; so $P_n$ is approximately $e^{-p}$.

The problem also asks for the expected number of toads. This requires less calculation. The expected number of toads into which Harry alone turns as a result of this escapade is $1 - p/n$. Thus the expected number of toads resulting from all $n$ friends is $n$ times $1 - p/n$, which is $n - p$.

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Use your calculator to show that

$$\left(1782^{12} + 1841^{12}\right)^{1/12} = 1922,$$

and then deduce that $1782^{12} + 1841^{12} = 1922^{12}$. See if you can discover further counterexamples to Fermat's Last Theorem.

# Solution 218.3 – Nearly an integer

Let

$$\alpha = \sqrt[3]{\frac{1}{2}\left(27 + 3\sqrt{69}\right)}, \qquad \beta = \left(\frac{\alpha}{3} + \frac{1}{\alpha}\right)^{2000}.$$

Show that $\beta$ is within $10^{-120}$ of an integer.

## Dick Boardman

Let

$$\gamma \;=\; \frac{\alpha}{3} + \frac{1}{\alpha},$$

so that $\beta = \gamma^{2000}$. To prove the stated property of $\beta$ we show first that the number $\gamma$ is the solution of a cubic equation, and secondly that $\gamma^n$ is part of a solution to a recurrence relation between integers and that the other parts are small when $n$ is large.

Consider the recurrence relation

$$f(n+3) \;=\; f(n+1) + f(n). \tag{R}$$

The values of the sequence depend on three initial values, $f(0)$, $f(1)$ and $f(2)$. Obviously, if these three values are integers, then all of the terms in this sequence are integers.

Let $x$, $y$ and $z$ be the roots of the cubic equation $X^3 = X + 1$, where $x$ is real and $y$ and $z$ are complex conjugates. Using the well-known relationships between the roots of equations and the coefficients of the equation we have

$$\begin{aligned} x + y + z &= 0, & (1)\\ xy + xz + yz &= 1, & (2)\\ xyz &= -1. & (3) \end{aligned}$$

Dividing equation (2) by equation (3) and simplifying, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \;=\; -1. \tag{4}$$

It is well known that the general solution to (R) is

$$f(n) \;=\; Ax^n + By^n + Cz^n,$$

where $A$, $B$ and $C$ are arbitrary constants. Choose $A = B = C = 1$. Then, using (1), $f(0) = 3$, $f(1) = 0$ and $f(2) = x^2 + y^2 + z^2$.

Since $x$, $y$ and $z$ are the roots of the cubic equation it follows that

$$x^2 = 1 + \frac{1}{x}, \quad y^2 = 1 + \frac{1}{y} \quad \text{and} \quad z^2 = 1 + \frac{1}{z}.$$

Adding these three equations and using (4), we get $f(2) = x^2 + y^2 + z^2 = 2$.

The three solutions of $X^3 - X - 1 = 0$ are

$$\begin{aligned}
x &= \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/3}}{3^{2/3}}, \\
y &= -\frac{1+i\sqrt{3}}{6}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} - \frac{(1-i\sqrt{3})\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/3}}{2\cdot 3^{2/3}}, \\
z &= -\frac{1-i\sqrt{3}}{6}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} - \frac{(1+i\sqrt{3})\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/3}}{2\cdot 3^{2/3}}.
\end{aligned}$$

Then

$$\gamma = \left(\frac{2}{27+3\sqrt{69}}\right)^{1/3} + \frac{1}{3}\left(\frac{27+3\sqrt{69}}{2}\right)^{1/3}.$$

Moreover, $\gamma^3 - \gamma - 1 = 0$; so $\gamma$ is my $x$.

A simple calculation shows that

$$x \approx 1.3247, \quad y \approx -0.66236 + 0.56228i, \quad z \approx -0.66236 - 0.56228i$$

$$\text{and} \quad |y| \approx |z| \approx 0.86884.$$

Thus $y^n$ and $z^n$ become small when $n$ is large. Indeed, we have

$$|y^{2000} + z^{2000}| \leq |y^{2000}| + |z^{2000}| \leq 10^{-121}.$$

However, the 2000th term of the recurrence relation (R) is an integer and is given by $x^{2000} + y^{2000} + z^{2000}$. Hence $\beta = x^{2000}$ is within $10^{-120}$ of an integer.

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Stone, 14 lb, £14; Plácido Domingo, tenor, £10; poorly cuttlefish, sick squid, £6; ladies underwear, pair of knickers, £2; type of pig, guinea, £1/1/-; sixteen ounces, 1 lb, £1; royal headdress, crown, 5/-; boy's name, Bob, 1/-; leather worker, tanner, 6d; cockney breast, thrupenny bit (tit), 3d; bicycle, penny farthing, $1\frac{1}{4}$d; girl's name, Penny, 1d; primate's leg joint, ape knee, $\frac{1}{2}$d.

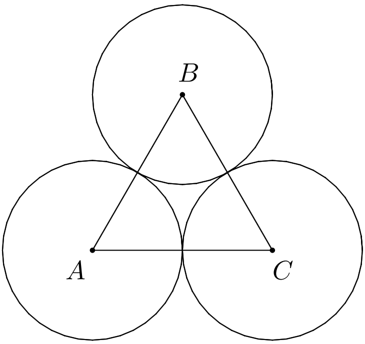
# Solution 202.2 – Five spheres

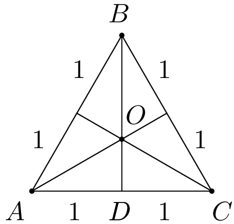
> Four spheres of radius $a$ are arranged so that their centres are at the vertices of a regular tetrahedron. Each sphere touches the other three. A fifth sphere of radius 1 is in the middle of the structure and it touches each of the other four spheres. What is $a$? Can you generalize to $n$ spheres of equal radius surrounding a sphere of radius 1?

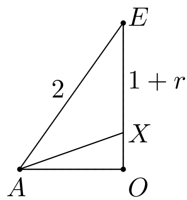
## Steve Moon

I found the algebra easier if I determined the radius of the included small sphere surrounded by four spheres of unit radius, then scaled up. Let this be $r$.

First, consider the plane in which the centres of the three base spheres lie.




The centres of the small sphere and the fourth sphere in the tetrahedron lie on the perpendicular through the point of intersection of the lines joining the vertices and opposite side midpoints in the middle triangle, above. By Pythagoras and $|OD| = \frac{1}{3}|BD|$, we have $|OD| = \sqrt{3}/3$ and $|OA| = 2/\sqrt{3}$.

Now consider the vertical plane passing through the centres of one sphere in the base, the upper sphere in the tetrahedron ($E$) and the small sphere. We have (right-hand diagram, above) $|OX|$ from $|OE|^2 = 4 - 4/3$; $|OE| = 2\sqrt{2}/\sqrt{3}$; Hence $|OX| = 2\sqrt{2}/\sqrt{3} - (1 + r)$. Then by Pythagoras on $\Delta AXO$,

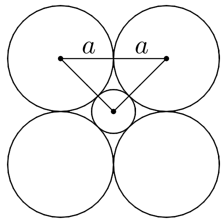
$$(1+r)^2 \;=\; \frac{4}{3} + \frac{8}{3} + (1+r)^2 - \frac{4\sqrt{2}(1+r)}{\sqrt{3}}.$$

Hence $r = \sqrt{3/2} - 1 \approx 0.225$; $a = 1/r = \sqrt{2}/(\sqrt{3} - \sqrt{2}) \approx 4.449$.

I don't think you can generalize this for arrangements of spheres of radius $a$ where not all the spheres are equivalent. For example, with five spheres you could try either a triangular bipyramid (but no central hole—just two tetrahedral ones), or a square based pyramid (but the apical sphere not equivalent to the four comprising the base).

You can solve for six spheres in an octahedral hole as they are all equivalent geometrically. In the plane of four spheres, by Pythagoras, $4a^2 = 2(a+1)^2$. Hence

$$a \;=\; 1+\sqrt{2} \;\approx\; 2.414.$$




Next, the cube. For eight spheres, consider the plane through opposite edges and centre. The length of the main diagonal of a $2a \times 2a \times 2a$ cube is

$$\sqrt{4a^2+4a^2+4a^2} \;=\; 2\sqrt{3}a.$$

But this is equal to $2a+2$. Hence

$$a \;=\; \frac{2}{2\sqrt{3}-2} \;=\; \frac{\sqrt{3}+1}{2} \;\approx\; 1.366.$$

I postulate that the only other arrangements that might work are the remaining Platonic solids, the icosahedron (20 spheres) and the dodecahedron (12 spheres), where all spheres are equivalent. However, whilst I would expect the values of $a$ to continue to decrease, their calculation I happily leave for others.

Thus we have the radii of the holes in which the small sphere fits, $r$, in an array of $n$ larger spheres of radius $R$.

| $n$ | 4 | 6 | 8 |
|---|---|---|---|
| $r/R$ | 0.225 | 0.414 | 0.732 |

This is used in solid-state chemistry in analysing crystal lattice structures for ionic solids, where one atom ion packs in an infinite hexagonal or cubic array and its oppositely charged ion can be thought of as fitting in the holes created—the bigger the ion, the larger the hole it needs to pack well.

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Mathematics is not a deductive science—that's a cliche. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork. [Paul Halmos]

# Solution 209.3 – $x^y + y^x$

Show that if $x$ is an odd power of 4, then $x^y + y^x$ is composite for all integers $y \geq 2$. What if $x$ is an even power of 4?

## Steve Moon

Let $x = 4^{2k+1}$ and $y = 2m + 1$, the problem being trivial if $y$ is even. Then

$$x^y + y^x \;=\; 4^{(2k+1)y} + y^{4^{2k+1}} \;=\; 2^{(4k+2)y} + y^{2^{4k+2}},$$

which is positive for $k \geq 0$ and $y \geq 1$. Completing the square, we have

$$\begin{aligned} x^y + y^x &= \left(2^{(2k+1)y} + y^{2^{4k+1}}\right)^2 - 2 \cdot 2^{(2k+1)y} y^{2^{4k+1}} \\ &= \left(2^{(2k+1)y} + y^{2^{4k+1}}\right)^2 - 2^{2ky+y+1} y^{2^{4k+1}}. \end{aligned}$$

Now substitute $y = 2m+1$, in which case $2^{2ky+y+1} y^{2^{4k+1}}$ is a square. Hence

$$\begin{aligned} x^y + y^x &= \left(2^{(2k+1)(2m+1)} + (2m+1)^{2^{4k+1}}\right)^2 - \left(2^{2km+k+m+1}(2m+1)^{2^{4k}}\right)^2 \\ &= \left(2^{(2k+1)(2m+1)} + (2m+1)^{2^{4k+1}} + 2^{2km+k+m+1}(2m+1)^{2^{4k}}\right) \\ &\quad \times \left(2^{(2k+1)(2m+1)} + (2m+1)^{2^{4k+1}} - 2^{2km+k+m+1}(2m+1)^{2^{4k}}\right). \end{aligned}$$

Hence $x^y + y^x$ is composite if the second factor is greater than 1. Write $K = 2^{2km+k+m}$ and $M = (2m+1)^{2^{4k}}$. Then the second factor is

$$2K^2 - 2KM + M^2 \;=\; K^2 + (K - M)^2,$$

which is greater than 1 unless $k = m = 0$, in which case $x = 4$, $y = 1$ and $x^y + y^x = 5$, a prime.

So $x^y + y^x$ is composite for $x$ an odd power of 4 and $y \geq 2$.

Now consider $x^y + y^x$ for $x = 4^{2k}$. If $k = 0$, then $x^y + y^x = 1 + y$, which clearly takes both prime and composite values. So we can assume that $k \geq 1$, and obviously we can assume that $y = 2m + 1$ is odd. Unfortunately the method involving the difference of squares fails, since

$$x^y + y^x \;=\; 2^{4ky} + y^{2^{4k}} \;=\; \left(2^{2ky} + y^{2^{4k-1}}\right)^2 - 2 \cdot 2^{2ky} y^{2^{4k-1}}$$

and the last term is not a square.

So for $x \neq 1$ an even power of 4, $x^y + y^x$ is composite for even $y$ but, in the absence of any evidence to the contrary, I can only conjecture that $x^y + y^x$ is composite for odd $y \geq 3$.

# A century of theorems

## Tony Forbes

Congratulations to **Robin Whitty**, who reached 100 on February 20th this year.

Well, no; Robin has a long way to go for his 100th birthday. What I am referring to is his website at www.theoremoftheday.org, which attained its one hundredth theorem on that date.

In this excellent site, Robin has collected together a cornucopia of beautiful mathematical results. Each theorem is presented in a masterful way to appeal as far as possible to the enthusiastic mathematician in the street. Here you can find theorems on graphs, including the famous *Four Colour Theorem* (which is actually first in the list), numbers (such as *The Fifteen Theorem* and *Fermat's Last Theorem*), and many if not all of the other diverse branches of our subject.

He also has results in geometry, such as *Miquel's Triangle Theorem*, which are illustrated by clever JAVA programs using David Joyce's GEOMETRY APPLET software showing how various elements change their shapes as you move points around. And of course the human side of mathematics has not been forgotten. Two of his theorems feature weddings—*The Happy Ending Problem* and *The Marriage Theorem*.

But why am I particularly excited about *Theorem of the Day*? How appropriate it is that Theorem 100 should be *The Design of the Century*, which first appeared on the front cover of M500 **205**!

# Old money

## Gareth Harries

This is more suitable for older readers of M500.

From the cryptic clues, fill in the amounts in pre-decimal currency to reach the given total. To make it easier the amounts are in descending value.

Answers somewhere in this issue.

| | £ | s | d |
|---|---|---|---|
| stone | | | |
| Plácido Domingo | | | |
| poorly cuttlefish | | | |
| ladies underwear | | | |
| type of pig | | | |
| sixteen ounces | | | |
| royal headdress | | | |
| boy's name | | | |
| leather worker | | | |
| cockney breast | | | |
| bicycle | | | |
| girl's name | | | |
| primate's leg joint | | | |
| | 34 | 7 | $11\frac{3}{4}$ |

## Contents M500 221 – April 2008



04.02

**Cover**: The graph $M_{22}$. The vertices are the 77 blocks of the Steiner system $S(3, 6, 22)$, and two vertices are joined when the blocks they represent are disjoint [http://mathworld.wolfram.com/M22Graph.html].