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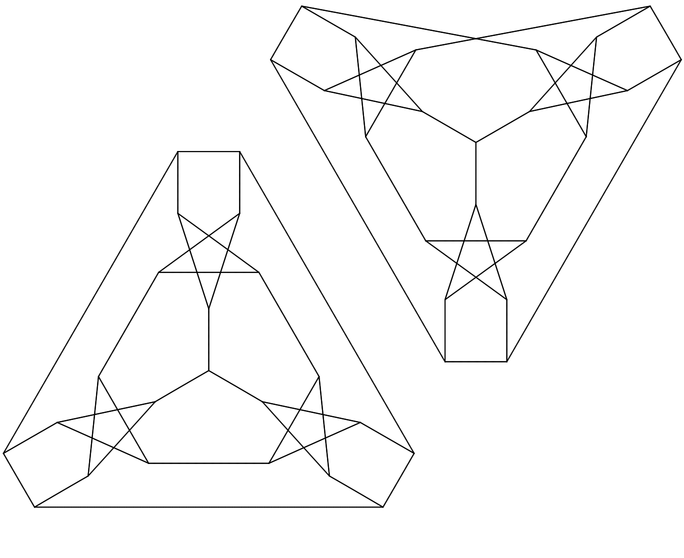

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Prime partitions: an asymptotic result

Tommy Moorhouse

Introduction This article is a brief summary of an investigation into the asymptotic behaviour of the number of partitions $P(n)$ of an integer into prime parts. This essentially means finding a good approximation for $P(n)$ when n is very large. Hardy and Ramanujan found a beautiful asymptotic formula for $p(n)$, the number of partitions of an integer n into positive integers. The formula was refined by Rademacher, and a general theory of asymptotics for a range of partitions was developed [Apostol, Andrews]. In this article we will look at a simple asymptotic bound on the logarithm of $P(n)$, the number of partitions of n into primes.

Background to the problem The classic result of Hardy and Ramanujan uses an identity between modular functions and a rather ingenious integration around a circle. The starting point is the expression for the generating function for unrestricted partitions

$$F(x) \;=\; \prod_{n=1}^{\infty} \frac{1}{1-x^n} \;=\; \sum_{n=0}^{\infty} p(n)x^n.$$

Taking x to be a complex variable we can extract the coefficient of x^n, namely $p(n)$, by integration:

$$p(n) \;=\; \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}dx$$

where C is any smooth contour lying inside the unit circle and circling the origin once in the positive direction. A clever choice of contour, close to the unit circle, gives a manageable series of integrals. The choice of contour was refined by Rademacher to give a better behaved asymptotic series. Indeed, $F(e^{2\pi iz})$ is closely related to a function for which a striking identity is known, and use of this identity allows the integration to be carried out. The details are in [Apostol 2] and [Andrews]. Unfortunately it is not a simple matter to apply this method directly to prime partitions because the modular function identity is not available, at least not in a form I have been able to deduce.

Asymptotics For large values of n we can obtain asymptotic conditions on many partition functions. For example we have [see e.g. Apostol]

$$p(n) \;<\; \frac{e^{\pi\sqrt{(\frac{2}{3}n)}}}{4n\sqrt{\pi}}.$$

Sometimes we can find asymptotic formulae that allow us to quickly compute partition functions to the nearest integer for large n. However, other techniques such as those in [Andrews] do not appear to be directly applicable to the problem of prime partitions.

The case of prime partitions We may wonder whether $\log P(n)$ has the same asymptotic dependence on n as $\log p(n)$. Given that a prime partition can only be made up from the $\pi(N)$ primes smaller than or equal to N we might postulate that

$$\log P(N) \;\sim\; K\sqrt{\pi(N)} \;\sim\; K\sqrt{\frac{N}{\log N}}.$$

This 'hand waving' argument can be strengthened, and we will consider this further below.

Finding an asymptotic bound For prime partitions the relevant partition function is given by a product over prime numbers:

$$Z(x) \;=\; \prod_p \frac{1}{1-x^p} \;=\; \sum_{n=0}^{\infty} P(n)x^n.$$

We will use the fact that, given the truncated product

$$\sum_{n=0}^{N} P(n)x^n + \sum_{n=N+1}^{\infty} r(n)x^n \;=\; \prod_{p\le N} \frac{1}{1-x^p},$$

we certainly have, for fixed N,

$$P(N) \;<\; \frac{1}{x^N}\prod_{p\le N}\frac{1}{1-x^p}.$$

This is simply because we have omitted all the other (positive) terms in the sum on the left. We can pick out a given $P(N)$ by choosing $x < 1$ appropriately, because the terms in the sum $P(n)x^n$ are products of the increasing function $P(n)$ and the decreasing x^n, so that the series of terms has a minimum for a value of n depending on x. Terms $r(n)$ ('remainder terms') are smaller than the terms $P(n)$ in the full product. Taking logs and setting $x = \exp(-s)$ where s is a positive real number, we have

$$\log P(N) \;<\; sN - \sum_{p\le N} \log(1-e^{-ps})$$

for all s. This means that if we substitute the value of s for which the expression on the right is smallest we should get a good upper bound for $P(N)$. Regarding the expression on the right as a function of s we can find the minimum in terms of N, and hence get an upper bound for $P(N)$. The sum of logarithms can be estimated using Abel's identity [Apostol, Theorem 4.2] to get the integral

$$\sum_{p\le N} \log(1-e^{-ps}) - \pi(N)\log(1-e^{-Ns}) \;\equiv\; I_N$$

$$= \;-s\int_2^N \pi(t)\frac{e^{-st}}{1-e^{-st}}dt \;=\; -s\int_2^N \pi(t)\sum_{n=1}^{\infty} e^{-snt}dt.$$

Using the prime number theorem we estimate the integrals by assuming that for large N we have $\pi(N) = N/\log N$. The lower integration limit can be ignored because $\pi(1) = 0$ so we will not pick up a contribution from this region. Now we integrate by parts (take $te^{-snt} = v', u = 1/\log t$) to find

$$I_N \;\sim\; \sum_{n=1}^{\infty}\left(\frac{N}{n}+\frac{1}{sn^2}\right)\frac{e^{-Nns}}{\log N} + \cdots$$

where the dots indicate terms that we neglect in the limit of large N (a fuller treatment would seek to justify this rigorously).

In the first sum $\sum e^{-snN}/n$ is forced to converge by the exponential factor, and the sum is found to be $-N\log(1-e^{-sN})/\log N$. For large N this is a slowly varying function of s with a value close to zero. The second sum is clearly bounded by $\zeta(2)/(s\log N)$.

Therefore we have

$$\log P(N) \;<\; Ns - \frac{N\log(1-e^{-sN})}{\log N} + \frac{\zeta(2)}{s\log N}.$$

To find the minimum value of this function of s we differentiate and solve for s. To do this we treat the second term on the right as constant (recall that it is slowly varying), and we find

$$s \;=\; \sqrt{\frac{\zeta(2)}{N\log N}}.$$

Substituting this into the inequality for $P(N)$ we find that the first and last terms add, and

$$\log P(N) \;<\; \pi\sqrt{\frac{2N}{3\log N}} - \frac{N\log(1-e^{-sN})}{\log N}.$$

We can make the final term (which is positive) as small as we like by choosing N large enough, and therefore we have the asymptotic bound

$$\log P(N) \ < \ \pi \sqrt{\frac{2N}{3 \log N}}.$$

References and useful books

[Apostol] T. Apostol, *Introduction to Analytic Number Theory* (5th printing), Springer, 1998.

[Apostol 2] T. Apostol, *Modular Forms and Dirichlet Series in Number Theory*, Springer, 1990.

[Andrews] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

Problem 242.1 – Interesting integrals

Tony Forbes

Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx \ = \ \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,dx \ = \ \frac{\pi}{e},$$

two interesting integrals both of which evaluate to that possibly rational number π/e.

Problem 242.2 – Quintic

Show that the real root of the cubic $x^3 - x - 1$ is also a root of the quintic $x^5 - x^4 - 1$. See if you can solve the problem without calculating the root in question, which is actually

$$A + B \ \approx \ 1.32472,$$

where

$$A = \sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}} \approx 0.337727 \quad \text{and} \quad B = \sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}} \approx 0.986991.$$

Observe that $AB = 1/3$.

Problem 242.3 – Central trinomial coefficients

Tony Forbes

Let $T_n(b,c)$ denote the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. Thus the $T_n(b,c)$, which one might refer to as *central trinomial coefficients*, can be thought of as generalizing the more familiar central binomial coefficients $\binom{2n}{n}$. Indeed, putting $b=0$ gives $T_{2n}(0,c)=\binom{2n}{n}c^n$.

(i) Show that

$$T_n(b,c) \;=\; \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}\binom{2j}{j}b^{n-2j}c^j \;=\; \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{j}\binom{n-j}{j}b^{n-2j}c^j$$

and

$$(n+1)T_{n+1}(b,c) \;=\; (2n+1)bT_n(b,c)-n(b^2-4c)T_{n-1}(b,c), \qquad n\ge 1.$$

(ii) On the Internet forum NMBRTHRY, Zhi-Wei Sun of Nanjing University has made a number of interesting conjectures involving the central trinomial coefficients and π. For instance, he asserts that

$$\sum_{j=0}^{\infty}(-1)^j\,\frac{30j+7}{256^j}\binom{2j}{j}^2 T_j(1,16) \;=\; \frac{24}{\pi} \qquad (1)$$

and

$$\sum_{j=0}^{\infty}(-1)^j\,\frac{30j+7}{1024^j}\binom{2j}{j}^2 T_j(34,1) \;=\; \frac{12}{\pi}. \qquad (2)$$

Having no idea how to proceed, I would be delighted if anyone succeeds in actually finding proofs of (1) and (2). On the other hand, the obvious similarity between the two expressions suggests to me that it might be relatively easy to compute the ratio of the two left-hand sides. So for the second part to this problem, we merely ask you to show that

$$\sum_{j=0}^{\infty}(-1)^j\,\frac{30j+7}{256^j}\binom{2j}{j}^2 T_j(1,16) \;=\; 2\sum_{j=0}^{\infty}(-1)^j\,\frac{30j+7}{1024^j}\binom{2j}{j}^2 T_j(34,1).$$

If it helps, here are the first few of the relevant T_j coefficients.

j	0	1	2	3	4	5	6
$T_j(1,16)$	1	1	33	97	1729	8001	105441
$T_j(34,1)$	1	34	1158	39508	1350214	46222524	1584998556

Solution 238.4 – Wednesday's child

I have two children, one of whom is a boy born on a Wednesday. Assuming boys and girls as well as days of the week are equally likely, what's the probability that my other child is also a boy?

Dick Boardman

I would like to propose the following modification of the problem for consideration. Let there be a characteristic X which occurs in boys with a probability p, $0 < p \leq 1$. For the first child there are three possible events with probabilities

Boy with X	probability $p/2$	(event 1),
Boy without X	probability $(1-p)/2$	(event 2),
Girl	probability $1/2$	(event 3).

For the second child there are three similar events. These are independent events and hence the probability of any two happening is the product of the probabilities.

	First child					
Second child	(1)		(2)		(3)	
(1)	$\frac{1}{4}p^2$	$(*)$	$\frac{1}{4}p(1-p)$	$(*)$	$\frac{1}{4}p$	$(**)$
(2)	$\frac{1}{4}p(1-p)$	$(*)$	$\frac{1}{4}(1-p)^2$		$\frac{1}{4}(1-p)$	
(3)	$\frac{1}{4}p$	$(**)$	$\frac{1}{4}(1-p)$		$\frac{1}{4}$	

According to this 'standard' procedure the probability of having two children, at least one of which is a boy with X, is the sum of the boxes marked either $(*)$ or $(**)$; that is $(4p-p^2)/4$. The probability of having two boys, at least one of whom has characteristic X is the sum of the boxes marked $(*)$; that is $(2p-p^2)/4$. The ratio of these is

$$(2-p)/(4-p).$$

This ratio varies between 1/2 and 1/3. To answer the original problem, put $p = 1/7$. Then $(2-p)/(4-p) = 13/27$. In the limiting case, where $p = 1$, the problem simplifies to: 'I have two children, one of whom is a boy; what's the probability that my other child is also a boy?' Then the ratio agrees with the direct computation:

$$\frac{\mathbb{P}(\mathrm{BB})}{\mathbb{P}(\mathrm{BG}) + \mathbb{P}(\mathrm{GB}) + \mathbb{P}(\mathrm{GG})} = \frac{1}{3}.$$

Robin Marks

We show the possibilities below.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_7			X											
. . .			. . .											
G_1			X											
B_7			X											
B_6			X											
B_5			X											
B_4			X											
B_3	X	X	X	X	X	X	X	X	X	X	X	X	X	X
B_2			X											
B_1			X											

The labels along the bottom represent the first child, where B = boy, G = girl, and 1, 2, 3, ... represent Monday, Tuesday, Wednesday, ... The labels on the left represent the second child. Initially let us assume the children are statistically independent. Each pair which includes at least one B_3 is shown by the letter X; there are 13 cases of 2 boys, and 14 of a boy and a girl. Hence the probability that your other child is also a boy is 13/27.

The assumption of independence fails if the children are identical twins, when the probability that your other child is also a boy is 1. The rate of male identical twins in the world is about 3/2000. The probability that your other child is also a boy then becomes about

$$\frac{1997}{2000} \cdot \frac{13}{27} + \frac{3}{2000} \cdot 1 \;=\; \frac{13021}{27000}.$$

Re: Solution 233.1 – Hill

Ken Greatrix

Perhaps Norman Graham and other readers might be interested in the paper http://journals.tubitak.gov.tr/engineering/issues/muh-04-28-6/muh-28-6-3-0404-5.pdf, which shows a derivation for the trajectories of projectiles over long distances, and includes air resistance, the radius of the Earth and variable gravity. But it's only a cubic approximation—which doesn't work for the higher levels of drag which you would get with arrows (as in my hobby of archery).

Solution 240.1 Two tins of biscuits

> There are two tins, each containing $n > 0$ biscuits. Take a biscuit from a tin chosen at random. Keep doing this until one tin is empty. What is the expected number of biscuits that remain in the other tin?

Dick Boardman

I am an engineer and programmer, not a mathematician. I do computer assisted puzzle solving just as engineers in industry use computers to help them solve their problems. When attacking a problem like this, I find it helps to calculate the answer for small values of n, say up to 10. I use a programming language like C or MATHEMATICA although BASIC or PASCAL would do just as well. In this case, I used a random number generator to play 50000 versions of each game from $n = 3$ to $n = 10$. To estimate the expected value, I used the average over the 50000 games. Of course, these results cannot play any part in the proof but if my general method gives results very different from these, it is probably wrong. My first attempt at a general method failed this test so I had to re-work it.

Following a suggestion from Tony [Forbes], I noted that there are a maximum of 2^{2n-1} versions. I calculated each one for $n = 3$ to 10, summed and found the expected value in each case. This is called constructing the probability tree. This gives precise values and any general solution has to match these exactly. I looked into the cases where the number left was n or $n - 1$ or $n - 2$ and saw how many there were and how they arose, and this suggested that I looked at a number if different routes to a state times the number of ways the state was reached. The number of ways was always a power of 2, 2^n when n were left, 2^{n+1} where $n - 1$ were left, 2^{n+2} where $n - 2$ were left and so on.

I tried to work with the game as stated but found it very confusing and difficult to get right so I used a similar game, which emphasized the patterns in the numbers and made each stage clearer.

Consider a related problem. Start with empty tins and at each stage add a biscuit to either the left tin or the right tin with a probability of 1/2. Call the state where there are L biscuits in the left tin and R biscuits in the right tin $s(L, R)$. I shall also use $s(L, R)$ to denote the number of ways of getting there. In order to get to state $s(L, R)$ we must go through either $s(L - 1, R)$ or $s(L, R - 1)$.

If we draw a grid of squares which puts the number in the left tin along the bottom and the number in the right up the side, and in each square we put the number of routes to the square, we get a well-known pattern, Pascal's triangle. The number of routes to $s(L, R)$ is $\binom{L+R}{L} = \binom{L+R}{R}$.

To mimic the original game, we decide the grid ends when the when the number in either tin reaches k. Then there is only one route to $s(k, R)$, that is from $s(k-1, R)$. The route from $s(k, R-1)$ is barred. There is no route to $s(k, k)$. Thus in the truncated grid $s(k+R, R) = s(k+R-1, R)$.

R

1	4	10	20	0
1	4	10	20	20
1	3	6	10	10
1	2	3	4	4
1	1	1	1	1

L

Example with $k = 5$

The probability of each route is $(1/2)^{L+R}$. In the truncated grid, the probability of getting to $s(k, R)$ is $\binom{k+R-1}{R}/2^{k+R}$, where R is in the range 0 to $k-1$. To return to the original problem, we would like to find the probability, prob$[n, r]$, where n is the starting number in each pile and r is the number left, where r goes from n to 1. In a truncated game it is evident that $n = k$ and $r = k - R$ so that

$$\begin{aligned} \text{prob}[n, r] &= s(k, R) + s(L, k) \;=\; 2s(k, R) \\ &= \frac{2}{2^{2n-r}}\binom{2n-r-1}{n-r} \;=\; \frac{1}{2^{2n-r-1}}\binom{2n-r-1}{n-r}. \end{aligned}$$

Checks. Since the answer must be in the range 1 to n, $\sum_{r=1}^{n} \text{prob}[n, r]$ must be 1. I use the symbolic algebra package MATHEMATICA, for which I had to pay, but there are similar free packages like MAXIMA, which any puzzler can have. MATHEMATICA shows that $\sum_{r=1}^{n} \text{prob}[n, r] = 1$. The function prob$[n, r]$ matches exactly the values given by constructing the probability tree and finding the expected value:

$$\frac{3}{2}, \frac{15}{8}, \frac{35}{16}, \frac{315}{128}, \frac{693}{256}, \frac{3003}{1024}, \frac{6435}{2048}, \frac{109395}{32768}, \frac{230945}{65536}. \tag{1}$$

To find the expected value $\sum_{r=1}^{n} r\, \text{prob}[n, r]$, MATHEMATICA gives

$$2^{2-2n} n \binom{2n-1}{n}. \tag{2}$$

Tony Forbes

Dick and I agree there is still work to do. We need proofs that the probabilities really do add up to 1 and that $\sum_{r=1}^{n} r \, \mathrm{prob}[n,r]$ really does give the stated value, (2), for the expected number of biscuits in the other tin. I shall set these tasks as Problem 242.4.

In parallel with Dick, I also was working on the problem. By developing the probability tree in the usual manner I arrived at the same answers for $n = 1$ to 10 as in (1), above, also the help of MATHEMATICA. Then I noticed to my amazement a connection with the hyperbolic cosine function. Indeed, the answer to the problem is actually

$$\frac{2}{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n} x}}.$$

But the integral is doable. Suppose $m \geq 3$. Integrating by parts with

$$u = \frac{1}{\cosh^{m-1} x}, \quad \frac{du}{dx} = \frac{-(m-1)(\sinh x)}{\cosh^m x}, \quad v = \sinh x, \quad \frac{dv}{dx} = \cosh x,$$

we have

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^{m-2} x} \;=\; \int_{-\infty}^{\infty} u \frac{dv}{dx} dx \;=\; uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \frac{du}{dx} dx$$

$$= \frac{\sinh x}{\cosh^{m-1} x}\Bigg|_{-\infty}^{\infty} + (m-1)\int_{-\infty}^{\infty} \frac{\sinh^2 x}{\cosh^m x} dx = (m-1)\int_{-\infty}^{\infty} \frac{\cosh^2 x - 1}{\cosh^m x} dx,$$

the uv part vanishing because $m \geq 3$. This gives the recursion formula

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^m x} \;=\; \frac{m-2}{m-1} \int_{-\infty}^{\infty} \frac{dx}{\cosh^{m-2} x} \tag{3}$$

from which we can compute

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n} x} \;=\; \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x}.$$

To do the integral on the right, make the substitution $x = \operatorname{arcsinh}(\tan z)$, $dx/dz = |\sec z|$ to get

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x} \;=\; \int_{-\infty}^{\infty} \frac{dx}{1 + \sinh^2 x} \;=\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec z \, dz}{1 + \tan^2 z} \;=\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos z \, dz \;=\; 2.$$

Thus

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n} x} \;=\; \frac{\bigl((2n-2)(2n-4)\ldots(2)\bigr)^2 \cdot 2}{(2n-1)!} \;=\; \frac{2^{2n-2}\bigl((n-1)!\bigr)^2 \cdot 2}{(2n-1)!}$$

and therefore

$$\frac{2}{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n} x}} \;=\; \frac{2^{2-2n}(2n-1)!}{\bigl((n-1)!\bigr)^2} \;=\; 2^{2-2n} n \binom{2n-1}{n},$$

in agreement with (2).

The recursion formula (3) also works for odd m:

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n+1} x} \;=\; \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \cdots \cdot \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{\cosh x}.$$

As before, the integral on the right is evaluated by the same substitution, $x = \operatorname{arcsinh}(\tan z)$, $dx/dz = |\sec z|$, but with a slightly different outcome:

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh x} \;=\; \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1+\sinh^2 x}} \;=\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec z\, dz}{\sqrt{1+\tan^2 z}} \;=\; \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \;=\; \pi.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n+1} x} \;=\; \frac{(2n)!\,\pi}{(2^n n!)^2} \;=\; \frac{\pi}{2^{2n}} \binom{2n}{n}$$

and hence there is yet another solution to Problem 240.1:

$$\frac{2n}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\cosh^{2n+1} x}.$$

Problem 242.4 – Two sums

Prove that

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r \;=\; 2^{2n-1}$$

and

$$\sum_{r=1}^{n} \binom{2n-r-1}{n-r} 2^r r \;=\; 2n \binom{2n-1}{n}.$$

Solution 240.3 – Double sum

Show that

$$\sum_{r=1}^{\infty} \sum_{s=r+1}^{\infty} \frac{1}{r^2 s^2} = \frac{\pi^4}{120}.$$

Tony Forbes

It is no coincidence that I chose to put this problem in the same issue as Steve Moon's solution of Problem 237.5. Recall that you were to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2 - 9}{3}.$$

The key step in Steve's solution of this last problem was the use of the Taylor series as well as the product formula for the function $(\sin x)/x$. Indeed, we have

$$\begin{aligned}\frac{\sin x}{x} &= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \\ &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots.\end{aligned}$$

Now equate the coefficients of x^{2n} to obtain (after cancelling $(-1)^n$)

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \cdots \sum_{r_n=r_{n-1}+1}^{\infty} \frac{1}{r_1^2 r_2^2 r_3^2 \ldots r_n^2 \pi^{2n}} = \frac{1}{(2n+1)!}.$$

Putting $n = 1$ yields the familiar equality

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

But of course we now have an infinite number of similar identities, including the one stated in the problem, the case $n = 2$, in which the sequence of denominators in numerical order is 4, 9, 16, 25, 36, 36, 49, 64, 64, 81, 100, 100, 121, 144, 144, 144, 169, 196, 196, 225, 225, 256, 256, 289, 324, 324, 324, 361, 400, 400, 400, 441, 441, 484, 484, 529, 576, 576, 576, 576, 625, 676, 676, 729, 729, 784, 784, 784, 841, 900, 900, 900, 900, 961, ...

Five hats

Eddie Kent

Do you know what hat you are wearing? Many problems depend on people being able to deduce this without looking. Sometimes it gets them out of prison, or even helps them avoid the gallows. Here is a problem someone gave me many years ago. I wasn't sure it was possible when it turned up recently, but I managed it and so can you.

Five people including you, Anabel, are wearing hats. Three hats are square and two are round. You all know this but not what yours is.

shape	name
	Anabel
square	Bert
round	Clarence
square	Dick

Ernestine

Who can deduce and call out the shape of your (that is, Anabel's) hat. No one can look back. Only Dick can see Ernie. Clearly Bert can tell his hat is square because if it was round he knows Anabel would know her own was square.

Tracks

Tony Forbes

Next time you see a tank in action, answer these simple questions.

(i) At what speeds are various bits of the tracks moving?

(ii) Where is the best place to put the driving wheels? I ask because I notice, for instance, that the Pz Kpfw IV had them at the front (so that the tracks are dragged from the back of the vehicle and pushed towards the ground) whereas on the T-34 they were placed at the rear.

Teacher: "We need to work out seven times nine. I'm hopeless at arithmetic. Can someone do it for me, please."

Student: "Forty-two."

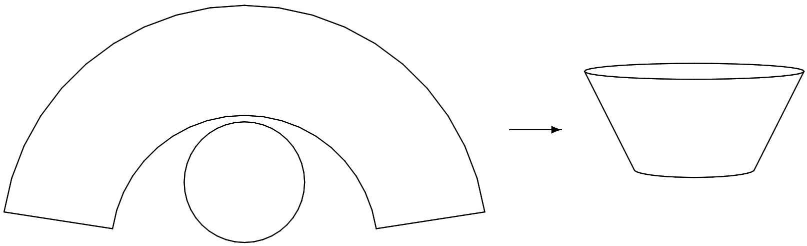
Another student: "I make it fifty-six."

Teacher: "Come on, guys. You can't both be right. It has to be one or the other."

Problem 242.5 – Coffee cup

Tony Forbes

A coffee cup in the form of a truncated cone closed at its thin end is made from plastic sheeting. There are two parts. A section of an annulus of radii r and R, $R > r$, subtending an angle of θ, and a disc of radius $r\theta/(2\pi)$. Assuming that the total surface area is 1 unit, choose the parameters to maximize the volume of the cup.



The diagram actually represents what I believe to be the optimum volume. So you could just make some careful measurements with a ruler and a protractor if you only want the answer without all that tedious mathematical reasoning.

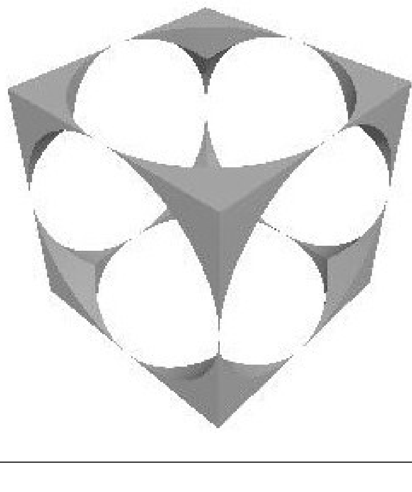
This is like 'Problem 230.5 – Cup-cake holder' except that the receptacle is constructed in a different manner.

Problem 242.6 – Three cylinders

Start with a $1\,\mathrm{m}^3$ cube. Take out three mutually orthogonal cylinders of length 1 m and diameter 1 m. What is the volume that remains?

The cylinders should of course fit snugly inside the cube along its main axes, as suggested by the picture on the right.

Thanks to John Faben and Andy Drizen for communicating this problem to me (TF).



Problem 242.7 – Numbers in a row

Arrange two ones, two twos, two threes and two fours in a row such that exactly k numbers occur between two ks.

Here is one way to do it with two ones, two twos and two threes.

$$3\ \ 1\ \ 2\ \ 1\ \ 3\ \ 2$$

Can you do it with two ones, two twos, two threes, two fours and two fives?

Things that do not exist

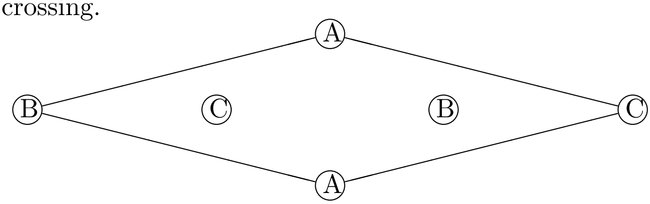
Tony Forbes

A considerable part of mathematical activity is concerned with the investigation of things that do not exist. Sometimes they even have famous names associated with them. For instance, Siegel zeros (real zeros in the range $(0, 1)$ of the Dirichlet L functions) have been the subject of intensive investigation in past years and, although the general opinion is that there aren't any, the question of their existence is still open. Anyway, suitably inspired, we offer the following examples of non-existent items taken from outside mathematics.

- A poor banker
- Chinese food that does not benefit from the addition of a considerable quantity of chilli sauce [Vaughan's conjecture]
- A tin of paint that does not have in the instructions, 'The surface to be painted must be clean, dry and free from dust and grease'
- A box of matches for children
- A packet of peanuts that may not contain nuts
- A restaurant with a dessert menu that includes raspberry ice-cream

(Any more?)

Connect A to A, B to B and C to C with no lines (including the ones in the picture) crossing.



Now do it again but this time with the A–A path inside the diamond.

Solution 231.5 – Four cos and four sins

Prove that

$$\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1 \quad \Rightarrow \quad \frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1.$$

David Wild

An alternative method to that printed in M500 **238** is as follows. Multiply the equation $\dfrac{\cos^4 A}{\cos^2 B} + \dfrac{\sin^4 A}{\sin^2 B} = 1$ by $\cos^2 B \sin^2 B$, and replace $\sin^2$ by $1 - \cos^2$. This gives

$$\cos^4 A(1 - \cos^2 B) + (1 - \cos^2 A)^2 \cos^2 B = \cos^2 B(1 - \cos^2 B).$$

Simplifying gives

$$\cos^4 A - 2\cos^2 A \cos^2 B + \cos^4 B = (\cos^2 A - \cos^2 B)^2 = 0.$$

Therefore $\cos^2 A = \cos^2 B$ and $\sin^2 A = \sin^2 B$. So

$$\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1.$$

Letter

Re: The Four Card Problem

Dear M500,

There is a stupid error in my article on the Four Card Problem [M500 **241**, 1–8]. The last paragraph should start: 'Now, suppose the card PARIS does have TRAIN in the back ....' The article reads 'Now, suppose the card PARIS does have AIR on the back ....'

I am a suspect for a murder committed on a Eurostar train bound for Paris and I make a statement to the effect that it cannot be me, "Because I never travel to a French city possessing an airport by train." The police find a train ticket to Paris and the Prosecutor flourishes it: this information is presented as a 'card' with PARIS on one side and TRAIN on the other. I am dumbfounded until I remember that on the date mentioned there was an air strike (not a train strike) and so I had to take the train. The point is that a single counter-example does not in real life falsify a general statement especially since there may be special circumstances.

I have made different versions of this train/air Four Card Problem and got mixed up. My apologies,

Sebastian Hayes

Gender in mathematics

Eddie Kent

Grace Chisholm was an English mathematician and as such is one of the subjects of a book by Claire G. Jones: *Femininity, Mathematics and Science*, 1880–1914. Grace married William Henry Young, adding his surname to her own, and collaborated with him in his research. However, for reasons connected with male attitudes (and not only her husband's) and financial expediency, they often submitted under his name, thus promoting for him a distinguished career.

The Youngs were friends of A. S. Besicovitch, and Patricia Rothman tells an anecdote in 'Grace Chisholm Young and the division of the laurels' (*Notes and Records of the Royal Society* **50** (1996) 97) that hints at feelings of insecurity.

'William Henry Young was out swimming one day with Besicovitch and he got into difficulties. Besicovitch swam over to help him. With Besicovitch's assistance W. H. Young came up for a "third time" coughing, his long beard bobbing in the waves. He spluttered out as he gasped for breath, "Are you one of those people who think my wife is a better mathematician than I am?"'

M500 Winter Weekend 2012

Join with fellow mathematical enthusiasts for a weekend of mathematical fun. If you are interested in mathematics and want a great weekend, then this is for you, accessible to anyone who has studied mathematics. The **thirty-first M500 Society Winter Weekend** will be held at

Florence Boot Hall, Nottingham University, 6–8 January 2012.

Cost: £193 to M500 members, £198 to non-members. You can obtain a booking form from the M500 site.

http://www.m500.org.uk/winter/booking.pdf

If you have no access to the internet, send a stamped addressed envelope to

Diana Maxwell.

See the inside of the front cover for her address.

We will have the usual extras. On Friday we will be running a pub quiz with Valuable Prizes, and for the sing-song on Saturday night we urge you to bring your favourite musical instrument (and your voice). Hope to see you there.

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Front cover: The two Loupekine snarks.