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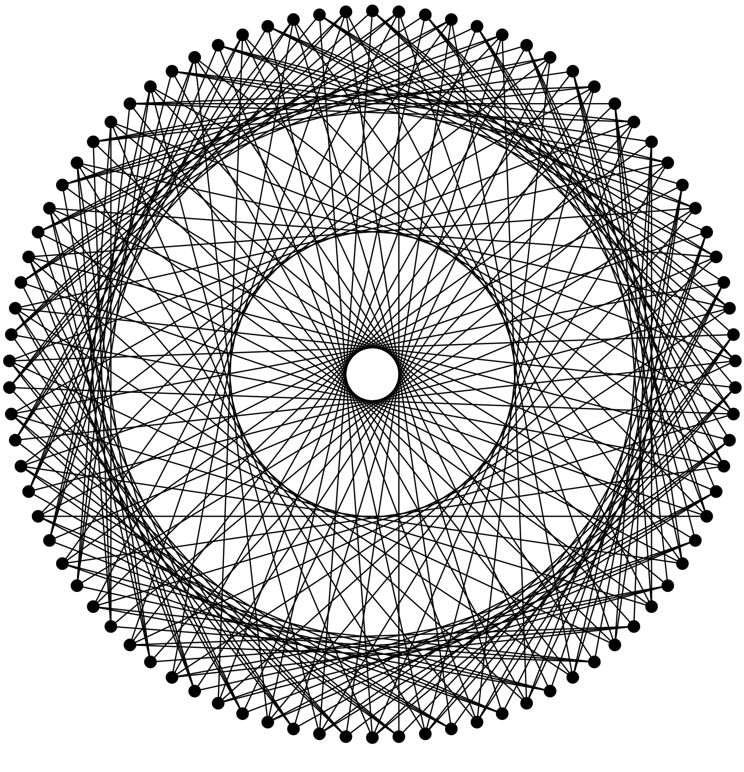

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M500 Winter Weekend 2018

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Florence Boot Hall, **Nottingham University**

Friday 5$^{\text{th}}$ – Sunday 7$^{\text{th}}$ January 2018.

Details, pricing and a booking form will be available nearer the time. Please refer to the M500 Web site.

http://www.m500.org.uk/winter.htm

Sylvester's Catalecticant

Jon Selig

1 Introduction

This work was prompted by a problem which was introduced by Tony Forbes in a Maths Study Group talk at London South Bank University and later resurfaced in M500 **266** as Problem 266.4 – Determinants. It involved a determinant of the form

$$\det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}. \qquad (1)$$

After searching on Wikipedia, [1], I found that this is an invariant discovered by Sylvester in 1852, [2]. The paper that introduces this determinant does a lot more, producing a sequence of invariants for binary forms of even degree. Here, I just want to look at one result, mentioned in the Wikipedia article.

One problem with reading old papers is that terminology has changed over the years. So, for example, a binary form of degree n is a homogeneous polynomial in two variables with degree n. A general binary form would be written as

$$\begin{aligned} f(x,y) &= a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{2}a_2x^{n-2}y^2 + \cdots + a_ny^n \\ &= \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i}y^i. \end{aligned}$$

The *catalecticant* of this form is (1), the determinant of its *catalecticant matrix*. The coefficients a_i are arbitrary constants, as usual. The inclusion of the binomial coefficients here was standard in the 19th century but has fallen out of use now. However, it is essential for the simplicity of the result.

Finally, note that the ground field will be taken to be the complex numbers $\mathbb{C}$. This is not usually specified in older work but is implicit since the fundamental theorem of algebra, that the complex numbers are complete, will be used.

2 The Theorem

A binary form of degree $2n$ can be split into a sum of powers of n linear forms if and only if its catalecticant is zero.

Consider a binary form of even degree,

$$f(x,y) \;=\; \sum_{i=0}^{2n} a_i \binom{2n}{i} x^{2n-i} y^i.$$

The theorem gives a condition for this to be equal to an expression of the form

$$\phi(x,y) \;=\; \sum_{j=1}^{n} (p_j x + q_j y)^{2n}. \tag{2}$$

Notice that for a form of odd degree, $2n+1$, there will be $2n+2$ constants $a_0, \ldots, a_{2n+1}$. Hence when we equate this to a sum of powers of $n+1$ linear factors, there will be the same number of constants to be determined. So we might expect that an odd degree binary form can always be decomposed into a sum of powers of $n+1$ linear factors. And this is indeed (almost always) the case, as shown by Sylvester in an earlier paper. In the even case considered here there is one fewer constant to be determined than there are coefficients in the binary form. So in this case we would expect there to be a single condition on the coefficients which ensures the decomposition can be performed.

As an example, consider the binary quartic

$$f_e(x,y) \;=\; 2x^4 + 12x^3y + 30x^2y^2 + 36xy^3 + 17y^4.$$

This can be written as

$$f_e(x,y) \;=\; 2\binom{4}{0}x^4 + 3\binom{4}{1}x^3y + 5\binom{4}{2}x^2y^2 + 9\binom{4}{3}xy^3 + 17\binom{4}{4}y^4;$$

that is, $a_0 = 2$, $a_1 = 3$, $a_2 = 5$, $a_3 = 9$ and $a_4 = 17$. The catalecticant is thus

$$\det\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} \;=\; \det\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{pmatrix} \;=\; 0.$$

Hence by the theorem the form can be written as the sum of two quartics of linear factors:

$$f_e(x,y) \;=\; (x+y)^4 + (x+2y)^4.$$

(Of course the computations here were performed in the reverse order to the presentation above.)

3 Proof

Before expanding the sum of the powers in (2) let $q_j = \lambda_j p_j$ for $j = 1, \ldots, n$. So now we can write

$$\phi(x, y) \;=\; \sum_{j=1}^{n} p_j^{2n} (x + \lambda_j y)^{2n}.$$

Expanding the powers of the linear factors gives

$$\phi(x, y) \;=\; \sum_{i=0}^{2n} \left(\sum_{j=1}^{n} p_j^{2n} \lambda_j^i \right) \binom{2n}{i} x^{2n-i} y^i.$$

Comparing the coefficients between the above expansion and the definition of the form $f(x, y)$ gives $2n + 1$ equations,

$$\begin{array}{ccccccccc}
p_1^{2n} & + & p_2^{2n} & + & \cdots & + & p_n^{2n} & = & a_0, \\
p_1^{2n}\lambda_1 & + & p_2^{2n}\lambda_2 & + & \cdots & + & p_n^{2n}\lambda_n & = & a_1, \\
p_1^{2n}\lambda_1^2 & + & p_2^{2n}\lambda_2^2 & + & \cdots & + & p_n^{2n}\lambda_n^2 & = & a_2, \\
 & & & & & & & \vdots & \\
p_1^{2n}\lambda_1^{2n} & + & p_2^{2n}\lambda_2^{2n} & + & \cdots & + & p_n^{2n}\lambda_n^{2n} & = & a_{2n}.
\end{array}$$

Notice the cancellation of the binomial coefficients. The equations can be written in the matrix-vector form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{2n} & \lambda_2^{2n} & \cdots & \lambda_n^{2n} \end{pmatrix} \begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix}. \tag{3}$$

Now take the first $n + 1$ rows of this system,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n} & \lambda_2^{n} & \cdots & \lambda_n^{n} \end{pmatrix} \begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{4}$$

The matrix on the left-hand side of this equation has order $(n+1)\times n$; hence there will be an $(n+1)$-vector, $(\Lambda_0, \Lambda_1, \ldots, \Lambda_n)$, which annihilates it:

$$(\Lambda_0\ \Lambda_1\ \ldots\ \Lambda_n)\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{pmatrix} \;=\; 0.$$

In particular, the elements Λ_i can be identified with the cofactors of the matrix,

$$\Lambda_0 \;=\; \det\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{pmatrix}, \quad \Lambda_1 \;=\; -\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{pmatrix}, \quad \ldots$$

up to

$$\Lambda_n \;=\; (-1)^n\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}.$$

The last determinant here is the Vandermonde determinant and in fact the others are multiples of the Vandermonde determinant by a symmetric polynomial in the λ_i. However, for our purposes this is not important.

Multiplying equation (4) by the vector of cofactors gives a linear equation,

$$a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2 + \cdots + a_n\Lambda_n \;=\; 0.$$

Next we take another $n+1$ rows from equation (3), this time starting from the second row,

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n+1} & \lambda_2^{n+1} & \cdots & \lambda_n^{n+1} \end{pmatrix}\begin{pmatrix} p_1^{2n} \\ p_2^{2n} \\ \vdots \\ p_n^{2n} \end{pmatrix} \;=\; \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}.$$

Notice that for any λ_i we have that

$$\Lambda_0\lambda_i + \Lambda_1\lambda_i^2 + \cdots + \Lambda_n\lambda_i^{n+1} \;=\; \lambda_i(\Lambda_0 + \Lambda_1\lambda_i + \cdots + \Lambda_n\lambda_i^n) \;=\; 0,$$

and hence we get another linear homogeneous equation,

$$a_1\Lambda_0 + a_2\Lambda_1 + a_3\Lambda_2 + \cdots + a_{n+1}\Lambda_n \;=\; 0.$$

Clearly we can repeat this procedure until we get $n+1$ equations,

$$\begin{array}{ccccccccccc}
a_0\Lambda_0 & + & a_1\Lambda_1 & + & a_2\Lambda_2 & + & \cdots & + & a_n\Lambda_n & = & 0, \\
a_1\Lambda_0 & + & a_2\Lambda_1 & + & a_3\Lambda_2 & + & \cdots & + & a_{n+1}\Lambda_n & = & 0, \\
 & & & & & & & & & \vdots & \\
a_n\Lambda_0 & + & a_{n+1}\Lambda_1 & + & a_{n+2}\Lambda_2 & + & \cdots & + & a_{2n}\Lambda_n & = & 0.
\end{array}$$

In matrix-vector form this is

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix} \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_n \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5}$$

Now clearly the vanishing of the catalecticant,

$$\det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix} \;=\; 0,$$

gives a necessary condition for the decomposition to be possible, otherwise only the trivial solution for the Λ_i would be possible.

To show that this condition is also sufficient, note that, if it holds, then equation (5) has a nontrivial solution. With the solutions for the Λ_i we can find the λ_i as the n solutions to the polynomial equation

$$\Lambda_0 + \Lambda_1\lambda + \cdots + \Lambda_n\lambda^n \;=\; 0.$$

Finally the p_i^{2n}, and hence the p_i, can be found by linear algebra, that is, from equation (3).

4 An Application

I found this problem in an old textbook, [3]: Show that the secant variety to the rational normal quartic curve is a cubic hypersurface. A hypersurface is an algebraic variety with dimension one less than that of the projective space it lies in; a primal in older language.

The rational normal quartic curve can be thought of as a mapping from the projective line $\mathbb{P}^1$ to $\mathbb{P}^4$. In particular, if the line has homogeneous coordinates $(s : t)$ then the mapping is given by

$$(s : t) \longrightarrow (s^4 : s^3t : s^2t^2 : st^3 : t^4).$$

This can be seen as a parametrization of the curve, with homogeneous parameters s and t. It is an example of a Veronese embedding, a general way to map one projective space into another of higher dimension.

If the $\mathbb{P}^4$ has homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ then the curve is given by the intersection of six quadric (degree 2) hypersurfaces. These can be expressed as

$$\text{Rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} = 1.$$

That is the quadrics are given by the six degree 2 equations,

$$\begin{aligned} x_0x_2 - x_1^2 &= 0, \\ x_0x_3 - x_1x_2 &= 0, \\ x_0x_4 - x_1x_3 &= 0, \\ x_1x_3 - x_2^2 &= 0, \\ x_1x_4 - x_2x_3 &= 0, \\ x_2x_4 - x_3^2 &= 0. \end{aligned}$$

Now we can think of points in $\mathbb{P}^4$ as quartic binary forms; given a form

$$a_0y^4 + 4a_1y^3x + 6a_2y^2x^2 + 4a_3yx^3 + a_4x^4,$$

we will associate with it the point

$$(a_0 : a_1 : a_2 : a_3 : a_4) \in \mathbb{P}^4.$$

Note that multiplying the form by an overall non-zero constant doesn't change it, so these are points in a projective space.

Under this mapping, forms which can be decomposed as the fourth power of a linear factor describe a rational normal quartic curve. To see this consider the fourth power of an arbitrary linear factor,

$$(px + qy)^4 = q^4y^4 + 4pq^3xy^3 + 6p^2q^2x^2y^2 + 4p^3qx^3y + p^4x^4,$$

where p and q are arbitrary. Such forms will be mapped to the points $(q^4 : pq^3 : p^2q^2 : p^3q : p^4)$ in $\mathbb{P}^4$. That is, they lie on a rational normal quartic curve.

A secant line to a curve is a line which meets the curve in two points. The closure of the set of these lines will also include tangent lines to the curve, where the two points of intersection coalesce. The set of points on all possible secant lines to an algebraic curve will form a three dimensional variety; two dimensions given by varying the points along the curve and another dimension as the point can move along the line. This means that for our quartic curve, its secant variety will be a hypersurface in $\mathbb{P}^4$ and hence it will be given by a single equation. Since the points on the curve correspond to forms which are decomposable into single quartic factors, a point on a secant line to the curve will correspond to a linear combination of such quartics. So the condition for a quartic binary form to be decomposable into the fourth powers of a pair of linear terms will be the same as the condition for the point in $\mathbb{P}^4$ to lie on the secant variety to the rational normal quartic curve. That is,

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0,$$

clearly a homogenous cubic in the coordinates of $\mathbb{P}^4$.

References

[1] http://en.wikipedia.org/wiki/Catalecticant.

[2] Sylvester, J. J. (1852), On the principles of the calculus of forms, *Cambridge and Dublin Mathematical Journal* **VII**, 52–97 and also 179–217. These can be found in the *The Collected Mathematical Papers of James Joseph Sylvester* Volume 1, Cambridge University Press, 1904.

[3] Semple, J. G. and Roth, L., *Introduction to Algebraic Geometry*, Oxford University Press, 1985.

Problem 277.1 – Cooling towers

Find a function $f(x, y)$ that convincingly models the shape of a power station cooling tower. The picture on the right, taken at East Midlands Parkway station from inside the St Pancras to Nottingham train, shows these structures at the Ratcliffe-on-Soar Power Station. Why do they have that shape?

Solution 275.4 – Hidden die

> A special X-ray scanner can detect the results of a hidden die rolled inside a box. The results are 90 per cent accurate. The scanner shows that a six has been rolled. What is the probability that the die in the box actually shows a six?
>
> Several and varied answers were offered when the problem was aired at the 2017 M500 Winter Weekend in January. Thanks to Rob and Judith Rolfe for suggesting it.

Rob Rolfe

It is not totally impossible to get oneself in knots here.

(1) I've decided that I might have overlooked a possible credible alternative solution to the one that I delivered during the M500 Winter Weekend at Nottingham. The reasoning leading to 9/14 could be wrong, since if we say that 14/60 the output would be '6', we can say the same for each of the other five numbers, hence getting a total of 84/60.

The wrong output for the 50/60 actual outcomes 'not a 6', a total of five wrongs, are not all 6s, in fact only 1/5 of them, one, would be a 6. That is, in 60 throws, the scanner would output a 6 ten times, of which nine would be correct, hence the chances of the scanner giving the correct outcome is 90 per cent, as Diana suggested at the Weekend.

However ...

(2) Consider a die with five green faces and one red, and a scanner output which is 90 per cent accurate. In sixty throws, there will be on average ten reds, of which nine will be correctly reported. There will be fifty greens, of which 10 per cent, five, will be reported as red. Where is the reasoning false?

But ...

(3) Consider different probabilities. Take a hypothetical disease with an incidence of 10/1000, or 1 per cent. A 90 per cent accurate test of 1000 people would give: (a) of 10 people who had the disease on average, nine would be told so correctly (b) of 990 people who did not have the disease, 99 would be told they did, incorrectly. Thus 108 people would be told they had the disease, although only nine of them did. 892 would be told they did not have the disease, although only one of them did. Summarizing, taking the test and getting a positive outcome would increase the probability of you having the disease from 10/1000 to 9/108, while getting a negative outcome would decrease the probability of you having the disease from 10/1000 to

1/892. (If you were the worrying kind, would a negative outcome make any difference? A positive outcome certainly would for most people. This is an argument against scanning without other indicators, which is not our concern here.)

I think this problem could run and run, but I would like the Winter Weekend participants to know I've changed my mind.

On the other hand ...

(4) My thoughts at Nottingham were correct, but I did not generalize enough. In 60 throws of the die, the output would be '6' (which I shall call positive), 14 times. This exaggerates the number of sixes. Of those 14 outputs, only 9 would be correct. That is, the chance of the box actually containing a six, given the output '6', would be 9/14.

Consider if it were a fair coin and not a die. Same scanner, 90 per cent accuracy. Then in 100 tosses the scanner output 'heads' would occur $50 \times 0.9 + 50 \times 0.1$ times $= 50$, so no exaggeration.

Now consider if the actual incidence were 3/5. The positive output would be $0.6 \times 0.9 + 0.4 \times 0.1 = 0.58$, i.e. the number of positives is reduced.

To generalize: Any test of less than 100 per cent accuracy will generate false positives if the incidence is less than 0.5, be correct for the wrong reasons if the incidence is 0.5, and generate false negatives if the incidence is greater than 0.5.

As a further test, try incidences of 0 per cent and 100 per cent.

This is the big argument against medical screening without other indicators. Since practically all diseases have a very low incidence, the problems caused by false positives can far outweigh any benefit of screening. However, I do not in any way pretend any expertise in this field.

Jeremy Humphries

The probabilities the die is showing 1, 2, 3, 4, 5 or 6 are each 1/6. If the die is showing 1 and the 90 per cent X-ray says 6, the probability of that is

$$\frac{1}{6} \cdot \frac{1}{10} \cdot \frac{1}{5} \;=\; \frac{1}{300}.$$

Ditto if the die is showing 2, 3, 4 and 5. (Because the X-ray has 1/10 chance of going wrong, and 1/5 chance of giving 6 as its wrong answer. I'm assuming all wrong answers are equally likely.) So the total of those probabilities is 5/300, or 1/60.

If the die is showing 6 and the 90 per cent X-ray says it's a 6, the probability of that is

$$\frac{1}{6}\cdot\frac{9}{10} = \frac{9}{60}.$$

Therefore when the X-ray says it's a 6, it's right nine times out of ten. So the probability it really is a 6 is 9/10.

That's the same as the X-ray success rate. And that seems to be generally the case. If the X-ray is right with probability p, then if it says that the die is showing a particular number the probability it's really showing that number is p. Sounds obvious really.

Chris Pile

If the scanner shows a six and is 90 per cent accurate, then the probability that the die *is* a six is 90 per cent!! No problem.

To determine this by 'equally probable outcomes', consider rolling the die 600 times. Expected results would be 100 outcomes of each of the faces 1, 2, 3, 4, 5, 6. The scanner would show 540 results accurately. Thus it would show 90 sixes accurately. The remaining 60 outcomes (10 of each value) would not be shown accurately. Assuming the scanner is not biased in its inaccuracy, the 10 sixes would be shown as 2 of each value 1 to 5. The ten 1s would be shown as two each of the other values (including 6). Similarly for 2, 3, 4, 5. Therefore 10 sixes are shown inaccurately. Hence the scanner shows 90 accurate plus 10 inaccurate sixes. Therefore, if the scanner shows a six, the probability that the die is actually a six is 90 per cent as originally expected.

Tony Forbes

To deal with thorny probability questions which have presented themselves to me over the years, I always use a fool-proof method: computer simulation. Admittedly it can only give an approximate answer, but it is usually sufficient to provide the hint you need to obtain the correct solution analytically. Actually, when I say fool-proof I really mean that the method is guaranteed to work only if the program is correct. Otherwise it will solve a different problem. Anyway, I find that my computer simulations get the desired results often—about 90 per cent of the time.

Here is what happened with two typical runs of 1000000 trials. Let t be the number of times a 6 is reported and let s be the number of times a reported 6 is actually thrown.

(i) We assume that the X-ray scanner is acting as a 6-detector, getting it right with 90 per cent accuracy. A 6 is reported if it is a 6 (probability 1/6) and the machine is right (90%), or if it is not a 6 (5/6) and the machine gets it wrong (10%).

$$s = 149467, \quad t = 232877, \quad s/t \approx 0.641828 \approx 9/14.$$

(ii) Now suppose the X-ray scanner looks at the die and tries to decide what it is, getting it right 90 percent of the time with the remaining 10 per cent spread equitably over the five wrong numbers. A 6 is reported if a 6 is thrown (1/6) and the machine is right (90%), or an x is thrown (1/6) and the machine shows 6 (2%), $x = 1, 2, 3, 4, 5$.

$$s = 150271, \quad t = 166838, \quad s/t \approx 0.899681 \approx 90\%.$$

Reassuringly, $t/1000000$ is about 1/6, as it should be since this model of the scanner ought to show the numbers with approximately equal frequencies.

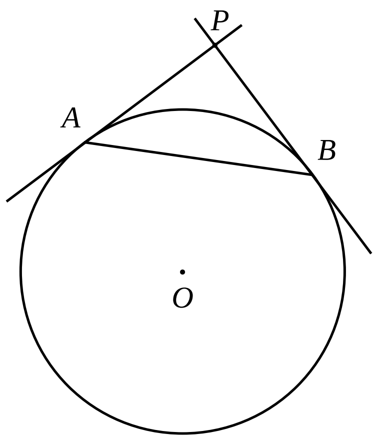
However, one can consider other models. For instance, a realistic possibility is that the device's spot-counting mechanism is only slightly faulty, with a maximum error of ± 1. So 6 gets reported as 6 (90%) or 5 (10%), 5 gets reported as 4 (5%) or 5 (90%) or 6 (5%), and so on.

$$s = 150783, \quad t = 158965, \quad s/t \approx 0.94853.$$

I leave it for someone else to analyse.

Problem 277.2 – Circle

William R. Bell



The points A and B lie on the circle with equation

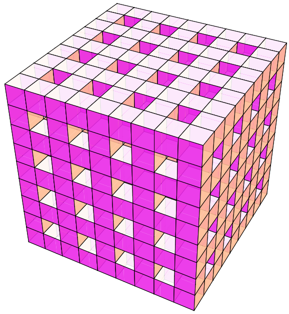
$$x^2 + y^2 = 25.$$

The tangents to the circle at A and B meet at the point $P = (1, 7)$. Show that the chord AB has equation

$$x + 7y = 25.$$

In what ratio does this chord divide the area of the circle?

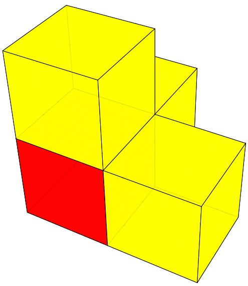
Solution 274.2 – Holey cube

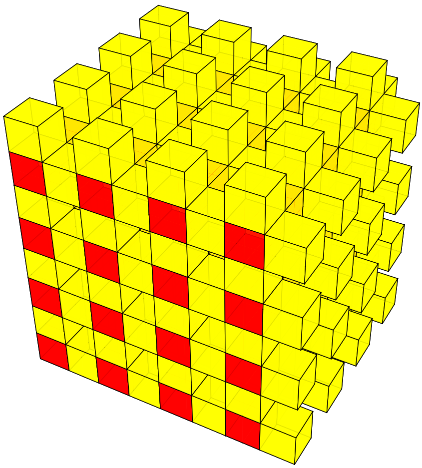


A $(2h+1) \times (2h+1) \times (2h+1)$ cube has three mutually orthogonal $h \times h$ arrays of 1×1 holes running through it. Find a formula for $c(h)$, the number of little cubes used in its construction, and $f(h)$, the number of exposed facelets. Hence or otherwise compute the limit of $f(h)/c(h)$ as h tends to infinity.

Robin Marks

Consider a substructure S consisting of a cube with three orthogonal cubes attached, as in the left-hand illustration, below.





We assemble $(h+1)^3$ copies of S to make a holey cube with $(h+1)^2$ extra cubes attached to three of its faces, as illustrated on the right, above. From the pictures it is plain that

$$c(h) \;=\; 4(h+1)^3 - 3(h+1)^2$$

and

$$f(h) \;=\; 12(h+1)^3 + 6(h+1)^2 - 3 \cdot 4(h+1)^2 \;=\; 12(h+1)^3 - 6(h+1)^2.$$

Hence $f(h)/c(h) = (12h^3 + O(h^2))/(4h^3 + O(h^2))$, which has limit 3 as h tends to infinity.

Problem 277.3 – Coasters

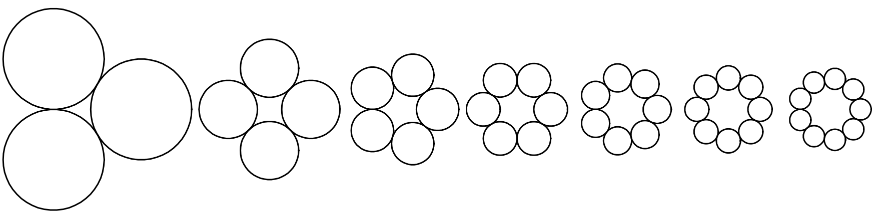
Tommy Moorhouse

Three identical circular coasters lie on a table. The coasters are in contact so that the shape of the 'hole' in the middle is a triangle with curved concave edges. If the distance from the centre of that 'triangle' (call it a 'c-triangle') to a vertex (where the coasters touch) is called ρ, what is the area of the c-triangle?

It is possible to generalize this and an interesting thing emerges. Suppose N circles are arranged so that each touches two neighbours and their centres lie on a larger circle so that a regular c-N-gon (an N-gon with curved edges) is formed. Again, let the distance from the centre of the large circle to a vertex of the c-N-gon be ρ. What is the area A_N of the c-N-gon? Try small values of N at first and then see if you can find a general formula depending on ρ and N.

Check that $A_N \to \pi\rho^2$ as $N \to \infty$. Now consider the length of the curve formed by the N small circular edges, and show that it tends to $\pi^2\rho$ and not $2\pi\rho$.

If this surprises you then another curiosity may throw some light on it. A straight line is drawn in the plane and two points A and B a distance 1 apart are chosen. A path between A and B is drawn by forming the two sides of an equilateral triangle with apex C above the midpoint of AB. This path is twice as long as the line AB. Now break the lines AC and CB at their midpoints and invert the peak, so that C now lies at the midpoint of AB, and the path is now an M shape of length 2 (as we have not altered the length of the path at all). The apexes are half the height of the original apex C. Repeat this process with each peak, so that the Nth iteration is a path with 2^n peaks of height $1/2^n$ and length 2. As $N \to \infty$ the path goes to zero height above AB but still has length 2. The limiting path is continuous but does not have a definite slope ('is not differentiable') at any point. A similar thing is happening with the coaster construction.



Solution 253.4 – Two colours

> Is it possible to colour each point of $\mathbb{R}^2$ red or blue in such a manner that no continuous curve containing more than one point is monochromatic?

Reinhardt Messerschmidt

Such a colouring exists if the axiom of choice is assumed. I found the idea behind the following solution at [1], after many of my own unsuccessful attempts.

If A is a set, then A^{c} will denote its complement with respect to $\mathbb{R}$, i.e. $A^{\mathrm{c}} = \mathbb{R} - A$. All statements involving measurability or measure will be with respect to the Lebesgue σ-algebra and the Lebesgue measure λ on $\mathbb{R}$.

We will prove later that there exists a subset X of $\mathbb{R}$ with the following property:

$$\text{if } u < v \text{ then } X \cap [u, v] \text{ and } X^{\mathrm{c}} \cap [u, v] \text{ are not measurable.} \qquad (*)$$

Let

$$\begin{aligned} R &= \big(X \times \mathbb{Q}\big) \cup \big(X^{\mathrm{c}} \times \mathbb{Q}^{\mathrm{c}}\big), \\ B &= \mathbb{R}^2 - R = \big(X \times \mathbb{Q}^{\mathrm{c}}\big) \cup \big(X^{\mathrm{c}} \times \mathbb{Q}\big). \end{aligned}$$

Colour the set R red and the set B blue.

Let $I = [0, 1]$ and suppose f is a continuous function from I into R. We will show that f must be constant. Let g, h be the coordinates of f; i.e. g, h are functions from I into $\mathbb{R}$ and $f(t) = (g(t), h(t))$ for every t. The functions g, h are continuous because f is continuous; therefore $g(I) = [a, b]$ and $h(I) = [c, d]$ for some $a \le b$ and $c \le d$. Since $f(I)$ is contained in R,

$$g(t) \in X \text{ if and only if } h(t) \in \mathbb{Q};$$

therefore

$$X \cap [a, b] = g\big(h^{-1}(\mathbb{Q})\big) = \bigcup_{r \in \mathbb{Q}} g\big(h^{-1}(\{r\})\big).$$

For every $r \in \mathbb{Q}$, the set $h^{-1}(\{r\})$ is a closed subset of the compact set I; therefore it is compact. Continuous functions preserve compactness; therefore $g(h^{-1}(\{r\}))$ is compact; therefore it is closed. It follows that $X \cap [a, b]$ is a countable union of closed sets; therefore it is measurable. This implies that $a = b$, because if $a < b$ then $X \cap [a, b]$ is not measurable by $(*)$. In other words, g is constant. If the constant value of g is in X then $[c,d]$ is contained in $\mathbb{Q}$, otherwise $[c,d]$ is contained in $\mathbb{Q}^{\mathrm{c}}$. In either case, $c=d$; therefore h is constant. It follows that f is constant.

If $f(I)$ is contained in B, then

$$X^{\mathrm{c}} \cap [a,b] \;=\; \bigcup_{r\in\mathbb{Q}} g\bigl(h^{-1}(\{r\})\bigr),$$

and it follows by a similar argument that f is constant.

Existence of a set X satisfying $(*)$

If the axiom of choice is assumed, then there exists a subset X of $\mathbb{R}$ with the following property:

$$\text{every measurable subset of } X \text{ or } X^{\mathrm{c}} \text{ has a measure of zero.} \qquad (**)$$

For a proof of this claim, see Proposition 1.4.9 in [2]. We will show that X also satisfies $(*)$. Suppose, for a contradiction, that $u<v$ and $X\cap[u,v]$ or $X^{\mathrm{c}}\cap[u,v]$ is measurable. Both $X\cap[u,v]$ and $X^{\mathrm{c}}\cap[u,v]$ must then be measurable, because

$$X\cap[u,v] \;=\; [u,v]-\bigl(X^{\mathrm{c}}\cap[u,v]\bigr), \quad X^{\mathrm{c}}\cap[u,v] \;=\; [u,v]-\bigl(X\cap[u,v]\bigr).$$

It follows by $(**)$ that

$$\lambda\bigl(X\cap[u,v]\bigr) \;=\; \lambda\bigl(X^{\mathrm{c}}\cap[u,v]\bigr) \;=\; 0;$$

therefore

$$0 \;<\; v-u \;=\; \lambda\bigl([u,v]\bigr) \;=\; \lambda\bigl(X\cap[u,v]\bigr)+\lambda\bigl(X^{\mathrm{c}}\cap[u,v]\bigr) \;=\; 0,$$

which is a contradiction.

References

[1] `https://mathoverflow.net/questions/156` (accessed 29 April 2017).

[2] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.

Student: I am planning to take both *Real Analysis* and *Complex Analysis*. Which should I do first?

Tutor: It doesn't matter. Each is a prerequisite for the other.

— Sent by Eddie Kent.

Solution 274.6 – Tan integral

Show that

$$I = \int_0^{\pi} \frac{\tan(t/6)\tan(t/3)}{\tan(t/2)}\,dt = \log\frac{3^6}{2^9}.$$

Richard Gould

We first use the substitution $u = \tan(t/6)$ to give

$$\tan\frac{t}{3} = \frac{2u}{1-u^2}, \quad \tan\frac{t}{2} = \frac{u + \dfrac{2u}{1-u^2}}{1 - \dfrac{2u^2}{1-u^2}} = \frac{3u-u^3}{1-3u^2},$$

$$\frac{du}{dt} = \frac{1}{6}\sec^2\frac{t}{6} = \frac{1}{6}(1+u^2), \quad u(0) = 0, \quad u(\pi) = \frac{1}{\sqrt{3}}.$$

From which

$$\begin{aligned} I &= 6\int_0^{1/\sqrt{3}} u\cdot\frac{2u}{1-u^2}\cdot\frac{1-3u^2}{3u-u^3}\cdot\frac{1}{1+u^2}\,du \\ &= 12\int_0^{1/\sqrt{3}} \frac{u(1-3u^2)}{(1-u^2)(3-u^2)(1+u^2)}\,du. \end{aligned}$$

After a somewhat lengthy skirmish with partial fractions this becomes

$$\begin{aligned} I &= \int_0^{1/\sqrt{3}} \left(-\frac{6u}{1-u^2} + \frac{12u}{3-u^2} + \frac{6u}{1+u^2}\right)du \\ &= \left[3\ln|1-u^2| - 6\ln|3-u^2| + 3\ln|1+u^2|\right]_0^{1/\sqrt{3}} \\ &= \left[\ln\left|\frac{(1-u^4)^3}{(3-u^2)^6}\right|\right]_0^{1/\sqrt{3}} \\ &= \ln\left(\frac{8^3}{9^3}\times\frac{3^6}{8^6}\times\frac{3^6}{1^3}\right) \\ &= \ln\left(\frac{3^6}{2^9}\right), \end{aligned}$$

as required.

Tony Forbes

As with Problem 272.1 and its solution in M500 **274**, one is tempted to consider a generalization

$$T(a,b,c) \;=\; \int_0^{\pi} \frac{\tan(t/a)\tan(t/b)}{\tan(t/c)}\,dt,$$

and again we see some of the various kinds of behaviour that were observed in **274**. Using the multiple-angle formula,

$$\tan n\theta \;=\; \frac{\operatorname{Im}(1+i\,\tan\theta)^n}{\operatorname{Re}(1+i\,\tan\theta)^n},$$

we can put

$$m \;=\; \operatorname{lcm}(a,b,c), \quad \alpha \;=\; \frac{m}{a}, \quad \beta \;=\; \frac{m}{b}, \quad \gamma \;=\; \frac{m}{c}, \quad u \;=\; \tan\frac{t}{m},$$

and follow Richard Gould's argument to obtain

$$T(a,b,c) \;=\; m\int_0^{\tan(\pi/m)} \frac{\operatorname{Im}(1+iu)^{\alpha}}{\operatorname{Re}(1+iu)^{\alpha}}\cdot\frac{\operatorname{Im}(1+iu)^{\beta}}{\operatorname{Re}(1+iu)^{\beta}}\cdot\frac{\operatorname{Re}(1+iu)^{\gamma}}{\operatorname{Im}(1+iu)^{\gamma}}\cdot\frac{du}{1+u^2}.$$

Now we are stuck unless we choose a, b and c with some care. Here are a few examples:

$$\begin{aligned}
T(3,3,2) &= 3\log(32/9) - 3,\\
T(3,3,6) &= 3 + \log 8,\\
T(4,6,3) &= 2\sqrt{3} - 3 + \log\frac{2^{12}}{3^{9/2}} + \log\frac{(-1+\sqrt{3})^4}{(1+\sqrt{3})^2},\\
T(6,6,2) &= \log(2^{24}/3^{15}),\\
T(6,6,3) &= \log(64/27) - 1/2,\\
T(6,6,4) &= 6 - 4\sqrt{3} + \log(2^{18}/3^{21}) + 12\log(1+\sqrt{3}),\\
T(6,6,12) &= 4\sqrt{3} - 6 + \log(64/27),\\
T(6,12,3) &= 3 - 2\sqrt{3} + \log(2^9/9) + \log(-45 + 26\sqrt{3}).
\end{aligned}$$

The original problem, $T(3,6,2)$, was also solved in a similar manner by **Bruce Roth**.

Solution 274.8 – Binomial coefficients

Let n_1, n_2, …, n_r be $r \geq 0$ numbers. Show that

$$\sum_{i=1}^{r} \binom{n_i}{2} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_i n_j \;=\; \binom{n_1 + n_2 + \cdots + n_r}{2}.$$

Tommy Moorhouse

The starting point is the relation

$$(1+x)^n \;=\; \sum_{i=0}^{n} \binom{n}{k} x^k.$$

We choose any partition of n, say $n = n_1 + n_2 + \cdots + n_r$ and write

$$\begin{aligned} (1+x)^{n_1+n_2+\cdots+n_r} &= (1+x)^n, \\ \prod_{i=1}^{r} \left(\sum_{k_i=0}^{n_i} \binom{n_i}{k_i} x^{k_i} \right) &= \sum_{k=0}^{n} \binom{n}{k} x^k. \end{aligned}$$

The coefficient of x^2 can be read off on each side:

$$\sum_{k_1+k_2+\cdots+k_r=2} \; \prod_{i=0}^{r} \binom{n_i}{k_i} \;=\; \binom{n}{2}.$$

Using

$$\binom{n}{1} \;=\; n$$

it is clear that the products on the left are either of the form

$$\binom{n_j}{2}\binom{n_k}{0}\cdots \;=\; \binom{n_j}{2}$$

or

$$\binom{n_j}{1}\binom{n_m}{1}\binom{n_k}{0}\cdots \;=\; n_j n_m.$$

Each pair $(n_i n_j)$ occurs just once, so we can specify that $i < j$. Summing, we conclude that

$$\sum_{i=1}^{r} \binom{n_i}{2} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_i n_j \;=\; \binom{n_1 + n_2 + \cdots + n_r}{2}.$$

Solution 248.6 – Bus stop

> Buses arrive at a bus stop according to the Poisson process with arrival rate β. People arrive at the same bus stop also according to the Poisson process but with arrival rate α. You arrive at the same bus stop and see n people (other than yourself) waiting. How long would you expect to wait for the bus. Assume for simplicity that only one bus route is served by the stop.

TF writes. I am pleased to inform readers of M500 that the problem has been solved by **Rafael Prieto Curiel**. See his article, which is available on the website of the magazine *Chalkdust*: 'Why I hate an empty bus stop', http://chalkdustmagazine.com/blog/empty-bus-stop/, 11 August 2016.

As an afterthought, perhaps one can suggest a possible refinement. As you can imagine, the situation nowadays is different due to the widespread availability of smart phones. It seems to me, therefore, that it might now be appropriate to count persons at the bus stop with some weight $w > 1$ if they are staring at their telephones, assuming those who are not have no information regarding bus activity. Of course, this problem is soon to be irrelevant because by 2019, so I am told, every bus stop will have a 'Next Bus' indicator, and hence the arrival time will be determinable exactly—well, perhaps with a smallish plus-or-minus error.

Chalkdust is a mathematics magazine for undergraduates, edited by Rafael Prieto Curiel and published by The Department of Mathematics, University College, London. It is *free*. The current and all past issues are available online at http://chalkdustmagazine.com/, and you can usually obtain a paper copy from UCL. The title presumably derives from the waste product that is the inevitable result of mathematical activity involving the writing with chalk on a blackboard. However, one cannot help being reminded of the time when the white lines on grass tennis courts were painted with a slurry consisting of powdered chalk and water together with some kind of binding agent. A consequence of using such material was that the impact of a tennis ball hitting a line at high speed would throw up a cloud of white powder the presence of which could provide assistance to the umpire in reaching his or her decision to award the point one way or the other. This was especially important in the days before HAWKEYE, when player and umpire had no alternative but to resolve questionable line-judge calls by evidence-based discussion and debate, as, for example, "I saw chalkdust, you moron!"

Problem 277.4 – Characteristic polynomial

For $n \geq 3$, let M be an $n \times n$ symmetric $\{0, 1\}$ matrix with 0 all along the diagonal, and let

$$m_0 + m_1 x + \cdots + m_n x^n \;=\; \det(xI - M)$$

be its characteristic polynomial. Show that

$$\operatorname{trace}(M^j) \;=\; -\,j\,m_{n-j}, \quad j = 1,\ 2,\ 3.$$

For example,

$$M \;=\; \begin{bmatrix} 0\,1\,1\,1 \\ 1\,0\,1\,0 \\ 1\,1\,0\,0 \\ 1\,0\,0\,0 \end{bmatrix}, \quad M^2 \;=\; \begin{bmatrix} 3\,1\,1\,0 \\ 1\,2\,1\,1 \\ 1\,1\,2\,1 \\ 0\,1\,1\,1 \end{bmatrix}, \quad M^3 \;=\; \begin{bmatrix} 2\,4\,4\,3 \\ 4\,2\,3\,1 \\ 4\,3\,2\,1 \\ 3\,1\,1\,0 \end{bmatrix},$$

the characteristic polynomial of M is $x^4 - 4x^2 - 2x + 1$, $\operatorname{trace}(M) = 0$, $\operatorname{trace}(M^2) = 8$ and $\operatorname{trace}(M^3) = 6$.

Problem 277.5 – Closure and complement

Show that if X is any set of real numbers, then, with the usual topology, the closure of the complement of the closure of the complement of the closure of the complement of the closure of X is equal to the closure of the complement of the closure of X.

For example, if X is the set of rational numbers in the interval $(0, 1)$, you should end up with $(-\infty, 0] \cup [1, \infty)$ in both cases. This problem is actually a special case of Theorem of the Day, number 239, Kuratowski's 14-set Theorem, http://www.theoremoftheday.org.

Problem 277.6 – Rational sum

Tony Forbes

Suppose a, b and c are rational numbers and let

$$\begin{aligned} d &= 1 + 20a - 10b + 4c, \\ e &= 3 + 38a - 17b + 5c, \\ f &= 3 + 20a - 8b + 2c. \end{aligned}$$

Show that

$$\sum_{n=1}^{\infty} \frac{n^6 + fn^5 + en^4 + dn^3 + cn^2 + bn + a}{n^4(n+1)^4}$$

is rational.

The origin of our dates and degrees

Peter L. Griffiths

Ancient astronomers observing the sky at night and at daytime will have concluded that there was a yearly cycle of about 360 days, and that equal night and day occurred twice during this period separated by about 180 days. Furthermore these equinoxes were separated by 90 days from the two solstices, the summer solstice and the winter solstice.

This is the origin of 360 degrees to a circle, and 90 degrees for a right angle. Later astronomers were able to calculate the number of days to the year more accurately as $365\,^1/_4$. It was felt that the phases of the Moon should be fitted into this, and it was recognised that one third of the 90 degrees would approximate to the Moon's approximate cycle of 30 days. The days of the week seem to originate with the Romans, but the planetary names of the days of the week seem to be Anglo-French as follows.

	French	English
Sunday	dimanche	Sunday
Monday	lundi	Monday
Tuesday	mardi	Mars
Wednesday	mercredi	Mercury
Thursday	jeudi	Jupiter
Friday	vendredi	Venus
Saturday	samedi	Saturday

Host (of *The Chase*): "On an electricity bill, what does kWh stand for?"

Contestant: "Kilowatts per hour."

Host: "Correct."

— sent by Jeremy Humphries

TF: For my enlightenment, I looked up *The Chase* to see what it was about and in passing I noticed that the *Sun* had spent a lot of columns per inch writing about the daft answers offered by various contestants. However there was no indication of any such answers actually getting accepted.

Jeremy Humphries: When I was a lot younger I used to read a motoring magazine. The magazine always gave the units of torque as lb/ft. Did anybody ever take it seriously? 'OK, it says do up the cylinder head bolts to 70 lb/ft. We've got a 2 foot spanner, so stick 140 pounds on the end of it.'

TF again: Sorry, I meant column-inches.

Contents

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Front cover A 5-regular graph with girth 5.