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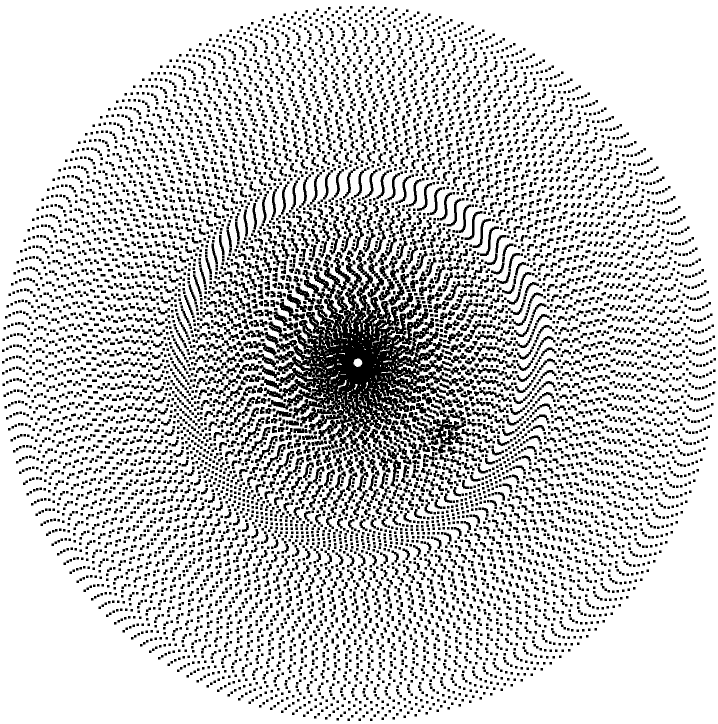
ISSN 1350-8539

M500 201



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The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: www.m500.org.uk.

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The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send s.a.e. to Jeremy Humphries, below.

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Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail.

On the average

John Bull

This article is a contribution to the recent run of interest in formulae that derive from the arithmetic–geometric mean inequality. Actually there are a lot more examples of means, all having fixed ordering between them for all values of variables. The most familiar are the harmonic (H), geometric (G), power (P) and arithmetic (A) means, where it is always true that $H \le G \le P \le A$. Proofs of these relationships can readily be found in books. Other means and their relationships are rather more elusive. In fact, there is still plenty of scope for interesting results and innovation, particularly involving the logarithmic and identric means, which are very curious beasts.

We are all familiar with the idea of a mean, although we usually have in mind the arithmetic mean: sum all the samples and take the average. To a lesser extent most of us will have experienced the harmonic and geometric means, usually in school by way of problems involving average speed and area respectively. To a lesser extent still, some will have come across the root mean square by way of average amplitude. This isn't the end of it. There are many more examples and some are given below. First let us introduce some means. For convenience and simplicity they are first given in terms of just two samples, a and b, but below they are generalized to any number of samples.

1. Arithmetic mean $\quad A(a,b) = \dfrac{a+b}{2}$
2. Harmonic mean $\quad H(a,b) = \dfrac{2ab}{a+b}$
3. Geometric mean $\quad G(a,b) = \sqrt{ab}$
4. Root mean square $\quad R(a,b) = \sqrt{\dfrac{a^2+b^2}{2}}$
5. Power mean $\quad H(a,b) = \left(\dfrac{\sqrt{a}+\sqrt{b}}{2}\right)^2$
6. Contraharmonic mean $\quad C(a,b) = \dfrac{a^2+b^2}{a+b}$
7. Centroidal mean $\quad T(a,b) = \dfrac{2(a^2+ab+b^2)}{3(a+b)}$

8.	Heronian mean	$N(a,b)$	$=$	$\dfrac{a+\sqrt{ab}+b}{3}$
9.	Logarithmic mean	$L(a,b)$	$=$	$\dfrac{b-a}{\log b-\log a}$
10.	Identric mean	$I(a,b)$	$=$	$\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$

All means have common properties.

If all the samples are scaled by a common factor, the final result will also be scaled by the same factor. In other words, if each sample is a unit of measure—it doesn't matter what the units are (feet, metres, miles/sec, etc.)—the mean will also be expressed in the same units. They are all *homogeneous.*

If all sample values are equal, the mean will also take this common value. 'Equal' is interpreted in the case of the logarithmic and identric means as the limit a as b tends to a (or b as a tends to b).

Otherwise a mean will always be less than the largest sample and greater than the smallest sample. In other words, a mean will always lie somewhere within this range of largest and smallest samples but different means will be biased towards one end of the range or the other.

Means are all symmetric: the values of a and b may be interchanged. If means are extended to any number of samples the result will be the same regardless of the order of the samples.

There are two ways to generalize: allow any number of samples; add parameters. In most cases extension to any number of samples is obvious; the exceptions being the logarithmic and identric means. In many cases it may also be seen how to generalize and consolidate through additional parameters in the function definition. Some examples follow.

1.	Arithmetic mean	$A(a_i)$	$=$	$\dfrac{\sum a_i}{n}$
2.	Harmonic mean	$H(a_i)$	$=$	$\dfrac{n}{\sum 1/a_i}$
3.	Geometric mean	$G(a_i)$	$=$	$\left(\prod a_i\right)^{1/n}$
4.	Root mean square	$R(r,a_i)$	$=$	$\left(\dfrac{\sum a_i^r}{n}\right)^{1/r}$

5. Power mean $\quad P(r, a_i) = \left(\dfrac{\sum a_i^{1/r}}{n} \right)^r$

6. Contraharmonic mean $\quad C(r, a_i) = \dfrac{\sum a_i^r}{\sum a_i^{r-1}}$

7. Centroidal mean $\quad T(a_i) = \dfrac{2}{n+1} \dfrac{\sum a_i a_j}{\sum a_i} \qquad (j \geq i)$

8. Heronian mean $\quad N(a_i) = \dfrac{2}{n(n+1)} \sum (a_i a_j)^{1/2} \qquad (j \geq i)$

Given the symmetry and homogeneous properties, it will be of little surprise that for any given set of samples the different means will be ordered. For the cases above where $r = 2$, it will always be true that $H \leq G \leq P \leq N \leq A \leq T \leq R \leq C$, equalities occurring when all the samples are equal.

For less obvious cases, such as for the logarithmic and identric means, there have been various suggestions for generalizations (e.g. [1]) but, while strong contenders have been proposed, none of these are definitive. Ideas for means and their generalizations have emerged from all branches of mathematics: algebra, calculus, geometry, ... (e.g. [2, 3]).

Means and their inequality relationships are fascinating because they are true regardless of the sample set taken; in other words they are true for all a_i. Also the relationships hold for any number n of a_i, even, in some instances, as n tends to infinity.

Interesting results emerge when particular values are substituted for the a_i, such as the set of integers from 1 to n. An example was shown in M500 **196**, p. 21:

$$A = \frac{1 + 2 + \cdots + n}{n} = \frac{n+1}{2},$$
$$G = (1 \cdot 2 \cdot \ldots \cdot n)^{1/n} = (n!)^{1/n},$$
$$G \leq A \Rightarrow n! \leq \left(\frac{n+1}{2} \right)^n .$$

Other results that could be obtained by a similar method are

$$(n!)^{2/n} < \frac{(n+1)(2n+1)}{6} \quad \text{and} \quad (n!)^{3/n} < \frac{n(n+1)^2}{4}.$$

Further results could be obtained by substituting say $1/1^2$, $1/2^2$, $1/3^2$, ..., $1/n^2$, or perhaps the sequence $2/1$, $3/2$, $4/3$, ..., $(n+1)/n$, or the reciprocals $1/2$, $2/3$, $3/4$, $\ldots$, $n/(n+1)$. One might then consider what happens as n tends to infinity. These happen to be series that produce familiar results. Many other substitutions could be explored, perhaps involving functions.

The arithmetic and geometric means can also be extended by adding weights to the samples. These weighted means become

$$A^* = \frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{w_1 + w_2 + \cdots + w_n}$$

and

$$G^* = (a_1^{w_1} \cdot a_2^{w_2} \cdot \ldots \cdot a_n^{w_n})^{1/(w_1+w_2+\cdots+w_n)}.$$

It is still always true that $G^* \leq A^*$ for any chosen set of values of a_i and w_i. This leads to some other interesting results. For example, by assigning weights 1 and n to the numbers 1 and $1+x/n$ respectively, it can be shown that

$$\left(1 + \frac{x}{n+1}\right)^{n+1} \geq \left(1 + \frac{x}{n}\right)^n \qquad \text{if } x > -n.$$

In this case we use the weighted arithmetic–geometric mean inequality with just two samples, each weighted. Of course with more weighted samples, carefully chosen, this result could be extended.

Notice that when handling inequalities is often possible to obtain a simplification in the form of the inequality by relaxing its strength. Also, it may not be necessary to go to elaborate means to demonstrate an inequality; it may simply be constructed from an axiom. For example, consider the factorial inequality of M500 Problem 193.4, where we have already shown above that $n! \leq ((n+1)/2)^n$. Now consider the second part of the problem. We are given that n is a positive integer. So $0 < n$, and hence $0 \leq n-1$. By adding $n+1$ to each side it must therefore be true that $n+1 \leq 2n$ and hence that $(n+1)/2 \leq n$. Multiply both sides by $((n+1)/2)^3$, which, of course, can't be negative, and we have $((n+1)/2)^4 \leq n((n+1)/2)^3$, and hence

$$\left(\frac{n+1}{2}\right)^n \leq \left(n\left(\frac{n+1}{2}\right)^3\right)^{n/4}.$$

So the inequality required, which follows from

$$n! \leq \left(\frac{n+1}{2}\right)^n \leq \left(\frac{n(n+1)^3}{8}\right)^{n/4} \qquad \text{for all } n > 0,$$

is a first relaxation of strength. A second relaxation of the strength would be to take $0 \le n-2$ and see what follows, or generalize by taking $0 \le n-r$, where $1 \le r \le n$.

Given the host of results from the various types of mean, and given the greater freedom to manipulate inequalities by relaxing their strength, it is surprising that not more elegant and simple results have been published. Perhaps more will now appear in M500.

References

[1] K. B. Stolarsky, Generalisations of the logarithmic mean, *MAA Mathematics Magazine* **48** (1975), 87–92.

[2] Howard Eves, Means appearing in geometric figures, *MAA Mathematics Magazine* **76** (2003), 292–294.

[3] Brian C. Dietel & Russell A. Gordon, Using tangent lines to define means, *MAA Mathematics Magazine* **76** (2003), 52–61.

Problem 201.1 – Continued fraction

Prove that

$$\tan\theta \;=\; \cfrac{2}{\cot\dfrac{\theta}{2} - \cfrac{2}{\cot\dfrac{\theta}{2^2} - \cfrac{2}{\cot\dfrac{\theta}{2^3} - \cfrac{2}{\cot\dfrac{\theta}{2^4} - \dots}}}}.$$

Problem 201.2 – Sine series

Prove that

$$\theta \;=\; (\sin\theta)(\cos\theta) + \sum_{n=1}^{\infty} 2^n \sin\frac{\theta}{2^{n-1}} \sin^2\frac{\theta}{2^n}.$$

Problem 201.3 – 25 objects

Tony Forbes

Let V be a set of 25 objects. Let A_i, $i = 1, 2, ..., n$ be n different 12-element subsets of V. For $1 \le i < j \le n$, the number of elements in $A_i \cap A_j$ is 5 or 6. How large can n be?

More arctangent identities

Tony Forbes

The last thing Bryan Orman did in 'A class of arctangent identities', M500 **199**, was to ask about formulae of the kind

$$N \arctan \frac{1}{B} + M \arctan \frac{1}{C} \;=\; \arctan \frac{1}{A}, \tag{1}$$

where N and M are positive integers.

First some trigonometry. We need a couple of identities:

$$\tan(x+y) \;=\; \frac{\tan x + \tan y}{1 - (\tan x)(\tan y)} \tag{2}$$

and

$$\tan nx \;=\; \frac{\displaystyle\sum_{i=0}^{[(n-1)/2]} (-1)^i \binom{n}{2i+1} \tan^{2i+1} x}{\displaystyle\sum_{i=0}^{[n/2]} (-1)^i \binom{n}{2i} \tan^{2i} x}, \tag{3}$$

where n is a positive integer.

To prove (2) and (3), the best method by far is to ignore the usual text books and instead start with the complex exponential formula

$$e^{ix} \;=\; \cos x + i \sin x. \tag{4}$$

Thus

$$\cos(x+y) + i\sin(x+y) \;=\; e^{i(x+y)} \;=\; (\cos x + i \sin x)(\cos y + i \sin y).$$

Equating real and imaginary parts yields the familiar identities

$$\cos(x+y) \;=\; \cos x \cos y - \sin x \sin y,$$

$$\sin(x+y) \;=\; \cos x \sin y + \cos y \sin x$$

from which (2) follows on dividing one by the other. Also from (4) we have

$$\cos nx + i \sin nx \;=\; e^{inx} \;=\; (\cos x + i \sin x)^n.$$

When the right hand side is expanded the real terms sum to $\cos nx$ and the imaginary terms to $i \sin nx$. Dividing $\sin nx$ by $\cos nx$ gives (3).

With that out of the way we can now proceed. Let us apply tan to both sides of (1),

$$\frac{1}{A} \;=\; F_{N,M}(B,C) \;=\; \tan\left(N\arctan\frac{1}{B} + M\arctan\frac{1}{C}\right), \tag{5}$$

use (2) to get

$$F_{N,M}(B,C) \;=\; \frac{\tan(N\ \arctan 1/B) + \tan(M\ \arctan 1/C)}{1 - \tan(N\ \arctan 1/B)\ \tan(M\ \arctan 1/C)}$$

and then expand each of the tans with (3). The result is that we obtain a ratio $P(B,C)/Q(B,C)$, where $P(B,C)$ and $Q(B,C)$ are polynomials in B and C. For example,

$$F_{1,1}(B,C) \;=\; \frac{B+C}{B\,C-1},$$

$$F_{1,2}(B,C) \;=\; \frac{C^2+2\,B\,C-1}{B\,(C^2-1)-2\,C},$$

$$F_{1,3}(B,C) \;=\; \frac{C^3+3\,B\,C^2-3\,C-B}{-3\,C^2+B\,(C^2-3)\,C+1},$$

$$F_{1,4}(B,C) \;=\; \frac{C^4-6\,C^2+4\,B\,(C^2-1)\,C+1}{-4\,C^3+4\,C+B\,(C^4-6\,C^2+1)},$$

$$F_{2,2}(B,C) \;=\; \frac{2\,(B+C)\,(B\,C-1)}{(B\,(C-1)-C-1)\,(C\,B+B+C-1)},$$

$$F_{2,3}(B,C) \;=\; \frac{(3\,C^2-1)\,B^2+2\,C\,(C^2-3)\,B-3\,C^2+1}{(B^2-1)\,C^3-6\,B\,C^2-3\,(B^2-1)\,C+2\,B},$$

As you can imagine, if N and M are large, the polynomials become quite complicated.

Observe that $F_{1,2}(B,-2B) = -1/(4B^3+3B)$ thereby producing infinitely many solutions of (1) with

$$(N,M,B,C,A) \;=\; (1,2,B,-2B,-4B^3-3B). \tag{6}$$

And there is another infinite family of solutions for $N=1$, $M=2$ obtained by setting $B = 4p^3+3p$ and $C = -2(B-p)$, where p is a positive integer:

$$(N,M,B,C,A) \;=\; (1,2,4p^3+3p,-8p^3-4p,-p\,(16p^4+20p^2+5)).$$

In fact we can repeat this process indefinitely. We start with (6) and generate new solutions of (1) by the recursion

$$(1, 2, B, C, A) \;\to\; (1, 2, -A, 2A - C, A'), \tag{7}$$

where

$$A' \;=\; \frac{4\,A^3 - 2\,C - 4\,A^2\,C + A\,\left(3 + C^2\right)}{1 + 2\,A\,C - C^2}.$$

My proof that A' is an integer involves a fair amount of messy algebra the details of which are omitted. Let $D = C^2 + 2BC - 1$. From (5) and the definition of $F_{1,2}(B, C)$ we have

$$A \;=\; \frac{BC^2 - 2C - B}{D}.$$

Also

$$A' \;=\; \frac{4(B + C)(B^2 + 1)(C^2 + 1)}{D^2} - B.$$

Since A, B and C are integers, it follows that

$$(A - B)(A - C) \;=\; \frac{2C(B + C)(B^2 + 1)(C^2 + 1)}{D^2}$$

is also an integer. Furthermore, $\gcd(D, C) = 1$; therefore D^2 must divide $2(B + C)(B^2 + 1)(C^2 + 1)$. Hence A' is an integer.

However, (6) does not generate every $(1, 2)$ identity. If we start with $(1, 2, 1, -3, 7)$ and apply (7), we generate yet another infinite family of solutions.

Some examples are tabulated on the next page. An interesting phenomenon is the appearance of pairs involving the same B value which are generated from $(1, -2, -7)$ and $(1, -3, 7)$. For instance,

$$\arctan\frac{1}{8119} - 2\arctan\frac{1}{19601} \;=\; \arctan\frac{1}{47321},$$

$$\arctan\frac{1}{8119} - 2\arctan\frac{1}{13860} \;=\; -\arctan\frac{1}{47321}.$$

If you add them together, you get a $(1, 1)$ solution,

$$\arctan\frac{1}{19601} + \arctan\frac{1}{13860} \;=\; \arctan\frac{1}{8119},$$

but the process is not always reversible. You do not necessarily get two $(1, 2)$ identities from a $(1, 1)$ identity.

Initial (B, C, A)	Generated (B, C, A), $B < 55000$
$(1, -2, -7)$	$(7, -12, -41)$, $(41, -70, -239)$, $(239, -408, -1393)$, $(1393, -2378, -8119)$, $(8119, -13860, -47321)$, $(47321, -80782, -275807)$
$(2, -4, -38)$	$(38, -72, -682)$, $(682, -1292, -12238)$, $(12238, -23184, -219602)$
$(3, -6, -117)$	$(117, -228, -4443)$, $(4443, -8658, -168717)$
$(4, -8, -268)$	$(268, -528, -17684)$, $(17684, -34840, -1166876)$
$(5, -10, -515)$	$(515, -1020, -52525)$, $(52525, -104030, -5357035)$
$(6, -12, -882)$	$(882, -1752, -128766)$
$(7, -14, -1393)$	$(1393, -2772, -275807)$
$(8, -16, -2072)$	$(2072, -4128, -534568)$
$(9, -18, -2943)$	$(2943, -5868, -959409)$
$(10, -20, -4030)$	$(4030, -8040, -1620050)$
$(11, -22, -5357)$	$(5357, -10692, -2603491)$
$(12, -24, -6948)$	$(6948, -13872, -4015932)$
$(13, -26, -8827)$	$(8827, -17628, -5984693)$
$(14, -28, -11018)$	$(11018, -22008, -8660134)$
$(15, -30, -13545)$	$(13545, -27060, -12217575)$
$(16, -32, -16432)$	$(16432, -32832, -16859216)$
$(17, -34, -19703)$	$(19703, -39372, -22816057)$
$(18, -36, -23382)$	$(23382, -46728, -30349818)$
$(19, -38, -27493)$	$(27493, -54948, -39754859)$
$(20, -40, -32060)$	$(32060, -64080, -51360100)$
$(21, -42, -37107)$	$(37107, -74172, -65530941)$
$(22, -44, -42658)$	$(42658, -85272, -82671182)$
$(23, -46, -48737)$	$(48737, -97428, -103224943)$
$(1, -3, 7)$	$(7, -17, 41)$, $(41, -99, 239)$, $(239, -577, 1393)$, $(1393, -3363, 8119)$, $(8119, -19601, 47321)$, $(47321, -114243, 275807)$

One way, although perhaps not the best, of discovering further interesting arctangent identities is to perform a brute-force search for integral values of $1/F_{N,M}(B, C)$. Because of the tan function in (5), solutions found in this manner satisfy (1) modulo π. If we want proper solutions of (1), we must select those values of B and C for which $N \arctan 1/B + M \arctan 1/C$ lies in the interval $(-\pi/2, \pi/2]$. Having said that, apart from trivial solutions such as $(7, 5, 1, -1, 0)$, the only one I can find is given by $(1, 4, 1, -5, -239)$ and is usually known as Machin's formula.

Odds and ends

Probability, chance, odds, percentages, whatever. We are constantly affected by statistics in our daily lives—so you can imagine that it's a subject which all the popular media know about. Well, not quite all. Here for your amusement we have collected together some of the strange things people have said, trawled from a variety of sources over the past few years.

From Chapter 3 of John McEnroe's autobiography — 'The semifinal matchups that year were Bjorn Borg against Vitas Gerulaitis, and Jimmy Connors against ... me. Me! I remember walking into the Gloucester Hotel, the big players' hotel at the time, and seeing the odds posted on a chalkboard (everyone bets in London): "Borg, 2-to-1; Connors, 3-to-1; Gerulaitis, 7-to-1; McEnroe, 250-to-1."'

Surely he cannot be serious. With odds like that the poor bookmakers would rapidly go out of business! I suppose 2-to-1 was a misprint for 2-to-1 on, or 0.5-to-1. That maintains the the bookmakers' advantage over the punters, just: $2/3 + 1/4 + 1/8 + 1/251 \approx 1.04565$.

British Heart Foundation poster — 'Nearly half of all people in Britain will, at some time, be struck by heart disease. If you believe it'll never happen to you, chances are you're wrong.'

Is it not the case that chances are you're right, rather than wrong? At least if the phrase 'nearly half' has its usual meaning. [Sent by Jeremy Humphries]

Robert Graves, *Claudius the God*, Chapter 18 — 'Sure enough, the three dice were lying in a neat equilateral triangle and each showed a six! The odds against Venus are 216 to one, so I can be pardoned for feeling great elation.'

Perhaps the dice were loaded. More likely, Graves was confusing the odds with the probability value, $1/6^3 = 1/216$; that's 215 to one against.

Letter to *The Times*, 15/11/2002 — 'Sir, You report that 98 per cent of convictions for dangerous driving were of men drivers. This means that men are 49 times more likely than women to be convicted of dangerous driving.' [Peter Fletcher]

BBC R4 News, 9.00, 2/9/03 — 'The chance the asteroid will hit the earth is one in 909,000.'

BBC R4 News, 10.00, 2/9/03 — 'There is just under a one in a million chance that the asteroid will hit the earth.'

One can imagine another interview:

"Good morning Professor. What is the chance that the sun will rise in the general direction of east tomorrow?"

"Well, taking all our calculations and observations into account, I'd say the chance of that happening is well under one in a million – something like one in one, in fact."

This is a strange usage of the adjective 'under'. In the usual ordering of the real numbers, '$1/a$ is under $1/b$' means '$1/a < 1/b$', not '$a < b$'. Recall that somewhere in M500 **192** we reported a local paper saying, 'Only one in two cameras are actually in operation, but this could increase to as many as one in three.' [Jeremy Humphries]

Advert at Euston Station — 'A rail route takes up to four times less land than a motorway.' [Peter Fletcher]

Herbal Health Newsletter, Issue 1 — ' Migraines affect approximately 14% of women and 7% of men; that's one fifth of the population.' [Peter Fletcher]

Man on BBC2 talking about the millennium bug — '82 per cent of big businesses are certain that they have a good chance of being relatively unaffected.'

Good chance? Relatively unaffected? How meaningful is such a precise figure as 82?

Macclesfield Express, 16/10/2002 — 'Legionnaires' disease is a rare form of pneumonia that is fatal in up to 15 per cent of cases. ... Most are treated with antibiotics and recover, but up to one in 15 dies.' [Peter Fletcher]

Breakfast Programme, BBC R5 — 'Nine out of ten people said that health was the most important issue in the election; four out of ten said Europe was the most important issue.' [Peter Fletcher]

The Fibonacci series and the golden section

Sebastian Hayes

I had always wondered how on earth it could be that the ratios of successive terms t_n/t_{n-1} of any *complex number* Fibonacci series, i.e. one verifying $t_{n+1} = t_n + t_{n-1}$, converge to $\phi = (1+\sqrt{5})/2$; so thanks to Dennis Morris for explaining this point in his fascinating article (M500 **198**, pp l–7). But why should there be any connection between the breeding of rabbits and the golden section in the first place?

The brilliant thirteenth century mathematician Fibonacci does not mention the golden section nor does he introduce the numbers forever associated with his name as a recursive series. He sets the reader a curious mathematical problem not unlike ones regularly appearing in M500 and which, at the other end of the time scale, could easily have figured in the Rhind Papyrus. (The conundrum *As I was going to St Ives, I passed a man with seven wives* ... probably does go back to the Egyptians.) Suppose a pair of rabbits is shut up in a certain place and they produce a pair every month, how many pairs will we have in a year? This is easy, at any rate to us—we have a geometric series with $t_1 = 1$ and constant ratio 2. But Fibonacci introduces the real-life constraint of an inevitable period of infertility during maturation setting this at one month by which we must understand a full month, beginning to end, passed in the enclosure. Thus, supposing p_n gives the number of pairs at the end of month n, $p_{n+1} = 2p_n -$ (infertile pairs). How do we know how many infertile pairs there are at the end of the month $n+1$? They will be precisely the pairs that have been born during the month n and so will be given by the discrepancy between the counts on successive months, i.e. $p_n - p_{n-1}$. Hence the formula,

$$p_{n+1} \;=\; 2p_n - (p_n - p_{n-1}) \;=\; p_n + p_{n-1}.$$

During month n, the ratio of the productive pairs (marked in black) to the unproductive pairs is very nearly equal to the ratio of the productive pairs to the total number of pairs.

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This already has a slightly more geometrical feel—in fact it is precisely the procedure of cutting a line segment in such a way that 'the ratio of the smaller part to the larger is equal to the ratio of the larger part to the whole,' i.e. what Euclid calls 'dividing a line in extreme and mean ratio.' Except that in the case of rabbit populations two successive ratios are never exactly equal.

Now suppose the non-productive period extends to two whole months. This time the unproductive newly born are given by the difference between population counts at two month intervals so (newly born) $= p_n - p_{n-2}$ and the formula becomes

$$p_{n+1} \;=\; 2p_n - (p_n - p_{n-2}) \;=\; p_n + p_{n-2}.$$

The first few terms are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, ... and the ratio of successive terms goes towards 1.4656.... Possibly ϕ is hidden here somewhere but I haven't yet dug it out.

In this model the unproductive pairs are those born in the month that has just ended and the preceding month, i.e. unproductive $= p_n - p_{n-2}$ while the productive pairs are those already in existence during month $n - 2$. So

$$\frac{\text{unproductive}}{\text{productive}} \;=\; \frac{p_n - p_{n-2}}{p_{n-2}} \;=\; \frac{p_n}{p_{n-2}} \;\approx\; 1.1478.$$

Extending the maturation period to $3, 4, \ldots, r$ years gives recursive series of the form $p_{n+1} = 2p_n - (p_n - p_{n-r}) = p_n + p_{n-r}$.

This is not the same set as the 'ϕ-type Fibonacci series' studied by Dennis Morris: the latter result from imposing a graded initial productivity. After a month's full stay a pair is, say, capable of producing a single pair and after two months two pairs. Now, if every pair had been capable of producing two new pairs while remaining alive itself we would have $p_{n+3} = 3p_n$ and so

$$p_{n+1} \;=\; 3p_n - (\text{not fully productive}).$$

The difference between p_n and p_{n-1} gives us the unproductive pairs born in the month that has just ended while $p_{n-1} - p_{n-2}$ tells us how many pairs will be producing a single pair only in the coming month instead of two. We end up with

$$p_{n+1} \;=\; 3p_n - 2(p_n - p_{n-1}) - (p_{n-1} - p_{n-2}) \;=\; p_n + p_{n-1} + p_{n-2}.$$

Extending the grading produces further series of the same type.

The imposition of further constraints provides further series or variations on the basic ones. If we throw in four newly-born rabbits in the first month and an extra one in the second, we have $p_1 = 4$, $p_2 = 5$, and though this throws the numbers out, the ratios soon become barely distinguishable from the standard set. Further sophistications would be to have a proportion of the pairs remaining permanently infertile, then to make this proportion itself variable, &c. &c.

As far as we know, the Greeks were not aware of the Fibonacci series. An Alexandrian craftsman who actually wanted to employ the golden section in a building or artefact (because of its alleged aesthetic qualities) would not have got a lot out of Euclid's strictly non-numerical 'construction'. He would have wanted numbers to use in measurement but a mathematician at the University of Alexandria would have assured him there were none available. 'In effect you require a trio $1, m/n, (m/n)^2$, where m/n is in its lowest terms and $m^2/n^2 = 1 + m/n$, or $m^2 = n^2 + mn$. For reasons of parity there are no such numbers since if m is even, n odd, we have even = odd + even, which is impossible, while if m is odd, n even, we have odd = even + even, likewise. And if there is no unitary solution, there are no solutions. My regrets.'

It has in fact been plausibly conjectured by Tannery that the original 'irrational number' proof concerned not $\sqrt{2}$ but ϕ.

Our craftsman, working by trial and error, might have stumbled on the Fibonacci numbers by trial and error. A reasonable first attempt would be to divide a given length in the middle giving the ratios $1:1$ and $1:2$. The difference in areas is $1 \cdot 2 - 1^2 = 1$ square unit. Chopping the given length into three equal sections gives the ratios $1:2$ and $2:3$ while the difference in areas, rather pleasingly, remains one square unit. After some more experimentation our craftsman, if he were really sharp, might realize that using the second ratio in an attempted golden sectioning as the first ratio in the next attempt always brings the diagonals closer together and keeps the difference in the areas at 1 square unit. Since the unit is always decreasing in real value, the above procedure is sufficient to obtain pairs of numbers for measurement purposes to any required degree of accuracy. Tri-golden sectioning would have been more difficult—I do not know of anyone trying to build a golden parallelepiped—but the principle remains the same. This time we are looking for numbers a, b and c where $a/b = b/c = c/(a+b+c)$ and setting $a = 1$ we have $c = b^2$; hence $b^3 = 1+b+b^2$. The ϕ_3-Fibonacci numbers, as Dennis Morris calls them, i.e. $1, 1, 1, 3, 5, 9, 17 \ldots$, move steadily towards the desired situation without ever attaining it.

I originally assumed the remarkable property $F_n^2 = F_1 \cdot F_{n+1} \pm 1$ was true for all Fibonacci$_2$ style series and I was considerably put out that I couldn't prove this directly. In fact it isn't! With the starting points $p_1 = 4, p_2 = 5$

we obtain $5^2 = 4 \cdot 9 - 11$ and it is ± 11 that persists. What we have is as follows. Suppose $F_{n-1}F_{n+1} = F_n^2 + C$. Then

$$\begin{aligned} F_{n+2}F_n &= (F_n + F_{n+1})F_n \\ &= F_n^2 + F_nF_{n+1} \\ &= F_{n-1}F_{n+1} - C + F_nF_{n+1} \\ &= F_{n+1}(F_{n-1} + F_n) - C \\ &= F_{n+1}^2 - C. \end{aligned}$$

The spin-off is that the difference between the ratios of successive terms remains $b/a - (a+b)/b = \pm C/(ab)$ with ab increasing and so, irrespective of the starting points, we get closer and closer to the situation where $b^2 = a(a+b)$ *exactly*. If our starting points are arbitrary positive fractions we eventually end up in the same situation and this is even true of complex series—it is this same relation $F_{n+2}F_n = F_{n+1} \pm C$, which makes the numerator of the imaginary part go to zero incidentally (see bottom of page 2 of Dennis Morris's article). If we have two starting values of different signs, eventually successive terms are either both positive or both negative; so the ratio is positive. Thus all roads lead to Rome.

Solution 197.2 – Consecutive cubes

Which integers can be expressed as sums of three or more consecutive cubes?

Ted Gore

Using $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$, we have

$$\begin{aligned} \sum_{i=r}^{n} i^2 &= \left(\frac{n(n+1)}{2} - \frac{(r-1)r}{2}\right)\left(\frac{n(n+1)}{2} + \frac{(r-1)r}{2}\right), \\ &= mj\,(mj + r(r-1)), \end{aligned} \tag{1}$$

where m and j are as defined in my solution to Problem 197.1 (page 17).

Let

$$z = mj\,(mj + r(r-1)).$$

This has the form $z = a(a+2b)$, where $b \geq 0$, so that either z is odd or z is divisible by 4. This implies that no number of the form $4w+2$ can be represented as the sum of consecutive cubes.

It is easily shown that

$$r = m - \frac{j}{2} + \frac{1}{2}. \tag{2}$$

In order for z to be expressible as the sum of consecutive cubes we need a triple of numbers (j, m, r) that fulfils (1) and (2).

Taking the case where z is odd, we see that mj is odd, so that either m is odd and j is odd (both odd factors of z) or $m = s/2$ and $j = 2t$ (where s and t are odd factors of z).

As examples: (i) Consider the case $m = 5$, $j = 3$. From (2), $r = 4$, so that $z = 405 = 4^3 + 5^3 + 6^3$. (ii) Let $m = 2\frac{1}{2}$, $j = 6$. From (2), $r = 0$, so that $z = 225 = 0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3$.

When z is even, mj is even, so that either m is even and j is odd (both factors of z), or $m = s/2$ and $j = 2t$ (where s is an odd factor of z and t is an even factor).

As examples, consider: (i) $m = 4$, $j = 3$. From (2), $r = 3$, so that $z = 216 = 3^3 + 4^3 + 5^3$. (ii) $m = 3\frac{1}{2}$, $j = 4$. From (2), $r = 2$, so that $z = 224 = 2^3 + 3^3 + 4^3 + 5^3$.

The following procedure can be used to decide whether a number, z, is expressible as a sum of consecutive cubes.

(1) The number z is of the form $a^2 + 2ab$, where $b \geq 0$ and $a = mj$, so a has the same parity as z and $a \leq \sqrt{z}$.

(2) For each value of a, assign values to m and j.

(3) For each m, j pair, calculate r.

(4) Check that the (m, j, r) triple gives rise to z.

A slightly different approach is possible. From (1) and (2) we can derive

$$\frac{z}{j} = m^3 + m\left(\frac{j^2 - 1}{4}\right)$$

and this with $p = (j^2 - 1)/4$ and $q = z/j$ yields

$$m = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

So m and r can be calculated for each possible value of j and the triple (m, j, r) tested to see whether it gives rise to z, bearing in mind the constraints on the possible values of j and m for a given z.

Solution 197.1 – Consecutive integers

Which integers can be expressed as sums of three or more consecutive integers?

Ted Gore

Using $\sum_{i=1}^{n} i = n(n+1)/2$, we have

$$\sum_{i=r}^{n} i \;=\; \frac{n(n+1)}{2} - \frac{(r-1)r}{2} \;=\; \frac{n+r}{2}\cdot(n-r+1) \;=\; mj,$$

where m is the mean of j consecutive numbers.

Now if j is odd then m is the middle value in the sequence of integers, k, say, so that

$$z \;=\; \sum_{i=r}^{k} i \;=\; jk, \tag{1}$$

while if j is even then m is the mean of the two central values, k and $k+1$;

$$z \;=\; j\left(k+\tfrac{1}{2}\right). \tag{2}$$

If z is any non-prime number that has an odd factor, we apply (1). For example, $15 = 3\cdot 5$; let $j=3$, $k=5$ so that $15 = 4+5+6$. Clearly we can always choose $j \ge 3$.

An interesting development occurs when some of the integers are allowed to be negative. For example, 33 can be represented as $10+11+12$ ($j=3$, $k=11$), or $-2-1+0+1+2+3+4+5+6+7+8 = 3+4+5+6+7+8$ ($j=11$, $k=3$). It can be shown that once cancellations have been carried out between corresponding positive and negative integers, the number of consecutive integers left, $j^* = 2k$, is bound to be greater than 3 for non-primes with odd factors.

If j is even then $z = j\left(k+\frac{1}{2}\right)$. For any odd prime p we have $p = 2\cdot(p/2)$ ($j=2$, $m = k+\frac{1}{2}$), so that $k = (p-1)/2$, $k+1 = (p+1)/2$ and

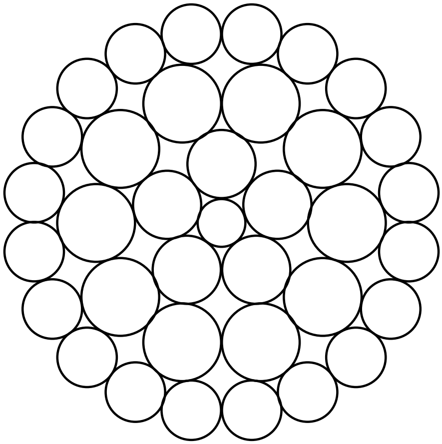
$$p \;=\; \frac{p-1}{2} + \frac{p+1}{2}.$$

For any power of two we have $2^s = 2^t\cdot 2^{s-t}$, so that $2^{s-t} = k+\frac{1}{2}$, which can only be true if $k=0$ and $j^* = 2\left(k+\frac{1}{2}\right) = 1$. Thus 2^s can only be expressed (trivially) as one consecutive integer.

In summary: (a) Any non-prime with an odd factor (not less than 3) can be expressed as the sum of three or more consecutive integers. (b) Odd primes can be expressed as the sum of two consecutive integers. (c) Powers of 2 cannot be expressed as sums of two or more consecutive integers.

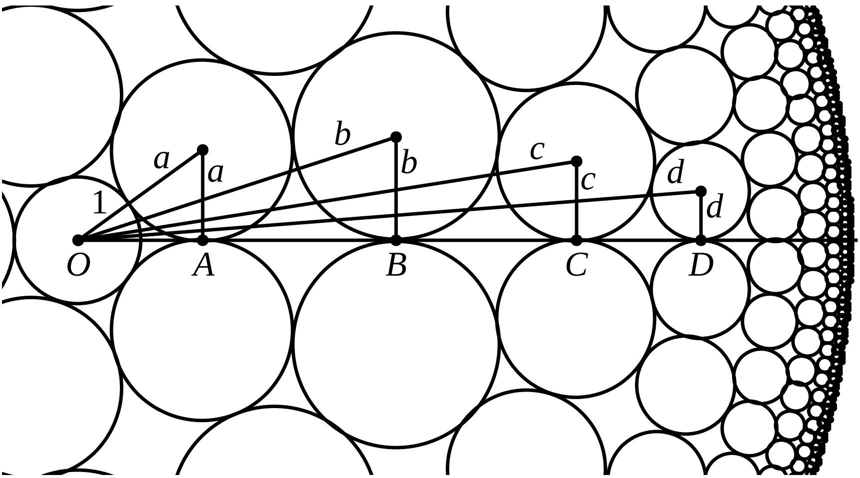
Solution 197.6 – 36 circles

Look at the diagram on the right. The innermost circle has radius 1, the circles in the inner ring have radius a, the circles in the middle ring have radius b and the circles in the outer ring have radius c. Determine a, b and c.



Basil Thompson

Inspired by the front cover of issue **198** I decided to have a go. I am unable to find any simple sequence or connection between the radii, apart from, of course, the tangent formula used for a, b, c, etc. Instead I worked them all out to five decimal places hoping for inspiration.



Referring to the diagram, we have $OA = \sqrt{2a+1}$, $AB = 2\sqrt{ab}$, $BC = 2\sqrt{bc}$, $CD = 2\sqrt{cd}$, and so on. Also $\sin 36^\circ = a/(1+a)$; hence

$$a \;=\; \frac{\sin 36^\circ}{1-\sin 36^\circ} \;\approx\; 1.42592.$$

For b, we have

$$\tan 18^\circ \;=\; \frac{b}{\sqrt{2a+1}+2\sqrt{ab}}$$

and on substituting the values for a and $\tan 18^\circ$ we obtain this equation,

$$0.63769 + 0.77600\sqrt{b} \;=\; b, \tag{1}$$

which on rearranging and squaring becomes

$$b^2 - 1.87754b + 0.40665 \;=\; 0.$$

This equation has two solutions:

$$b \;\approx\; 1.62771 \qquad \text{and} \qquad b \;\approx\; 0.24983.$$

The correct value to use is $b \approx 1.62771$; the other one, generated by the squaring process, is not a valid solution of (1).

To calculate c, we have

$$\tan 9^\circ \;=\; \frac{b}{\sqrt{2a+1}+2\sqrt{ab}+2\sqrt{bc}}$$

and this time we get

$$0.79344 + 0.40414\sqrt{c} \;=\; c,$$

which has the solution $c \approx 1.24423$. Similarly,

$$\tan 4\tfrac{1}{2}^\circ \;=\; \frac{b}{\sqrt{2a+1}+2\sqrt{ab}+2\sqrt{bc}+2\sqrt{cd}},$$

$$0.61826 + 0.17558\sqrt{d} \;=\; d,$$

$d \approx 0.77259$. The formula used to find b, c and d can be extended. From the diagram it can be seen that the circles get smaller after B.

Tony Forbes

I am inclined to agree with Basil's observation that there is no simple formula for the radii as the circles get further from the centre—in spite of a promising start. Using the formula

$$\sin 36^\circ \;=\; \tfrac{1}{2}\sqrt{\tfrac{1}{2}\left(5-\sqrt{5}\right)}$$

we obtain this expression:

$$a \;=\; 5-2\sqrt{5}+\sqrt{50-22\sqrt{5}}.$$

Curiously, $a-1$ is very nearly equal to $11^3/5^5$. A coincidence, surely.

Thereafter things get horribly messy. With a certain amount of effort it is possible to get exact formulae for b and c. Having done the work I feel obliged to pass on the results of my labours. It is not sensible to include all the intermediate details, so I just quote the final answer:

$$b \;=\; 16-7\sqrt{5}+\sqrt{554-\frac{1238}{\sqrt{5}}+2\sqrt{259-\frac{579}{\sqrt{5}}+\frac{4}{5}\sqrt{209050-93490\sqrt{5}}}}$$

and

$$\begin{aligned}
c \;=\; & 98-47\sqrt{5}+(17+6\sqrt{5})j_2+\left(12+\frac{34}{\sqrt{5}}\right)j_3 \\
&+j\left(57-27\sqrt{5}-6\left(1+\frac{1}{\sqrt{5}}\right)j_3\right) \\
&+\frac{2}{\sqrt{5}}\sqrt{\Bigg(97710-43614\sqrt{5}+32(103+45\sqrt{5})\,j_1} \\
&\quad +20(73-\sqrt{5})j_2+8\big(73\sqrt{5}-5+(225+103\sqrt{5})j_2\big)\,j_3 \\
&\quad +j\Big(123(535-241\sqrt{5})-4\big(275-77\sqrt{5}+(155+63\sqrt{5})j_2\big)j_3\Big)\Bigg),
\end{aligned}$$

where

$$j \;=\; \sqrt{5+2\sqrt{5}}, \qquad j_1 \;=\; \sqrt{209050-93490\sqrt{5}},$$

$$j_2 \;=\; \sqrt{554-1238/\sqrt{5}} \quad \text{and} \quad j_3 \;=\; \sqrt{1295-579\sqrt{5}+4j_1}.$$

By the way, in case you are worried about your eyesight, staring at the 36 circles diagram does have a dizzying effect. All those circular tangents seem to confuse the brain, and the eyes are unable to focus correctly.

Highest common factor

John Byrne

In M500 **198**, page 21, the problem you pose is to find the prime factors of a and b, where

$$a = 3\ 94020\ 06196\ 39447\ 92330\ 46804\ 46811\ 60078\ 93398\ 29458\ 92387\ 22475\ 04307\ 08947\ 55507\ 64181\ 30273\ 08398\ 62936\ 98852\ 77753\ 11113\ 49458\ 02497$$

and

$$b = 3\ 94020\ 06196\ 39447\ 92330\ 46804\ 48005\ 17655\ 67233\ 17400\ 56861\ 65122\ 75841\ 35826\ 02387\ 09158\ 68013\ 02250\ 74049\ 82336\ 01099\ 13554\ 78053\ 53497.$$

The common factors are 7922 81625 14264 33760 39837 22123 and 7922 81625 14264 33760 39837 22127. The additional factors for a are 7922 81625 14264 33760 39837 22117 and 7922 81625 14264 33760 39837 22121. The additional factors for b are 7922 81625 14264 33760 39837 22129 and 7922 81625 14264 33760 39837 22133. Such is the wonder of computers.

ADF writes—You may have noticed that there is not much difference between the prime factors. In fact we have here a modest example of a *prime sextuplet*, six primes in a range of 16. (You can't get closer than 16 except for a few special cases at the beginning of the prime number sequence.) Thus

$$\begin{aligned} a &= P(P+4)(P+6)(P+10), \\ b &= (P+6)(P+10)(P+12)(P+16) \end{aligned}$$

and

$$g = \gcd(a,b) = (P+6)(P+10),$$

where $P = 79228162514264337603983722117$.

I realized after M500 **198** went to press that it is possible (though somewhat laborious) to factorize g, a/g and b/g by hand—provided you attack them in the right manner. The trick is to use Fermat's method. Take g, for instance. Trying $y = 1, 2, 3, \ldots$ in turn, we test $g+1$, $g+4$, $g+9, \ldots$ until we find a perfect square, $g + y^2 = x^2$, say. Then $g = (x+y)(x-y)$. The method succeeds because the factors of g are so close together. Indeed, you only need to go as far as $y = 2$, at which point you discover that $g+4$ is a square, $(P+8)^2$.

Letter to the Editor

Dear Tony,

For a moment I thought that our amazing M500 magazine had answered my question before I had asked it. It involved a test for division by 7. We had just been on holiday with a friend, who said that she was taught a method at school. Sadly, she cannot remember it and I couldn't work out anything quicker than an essentially modular approach. Even then, I think children would find it easier to work through the two digits at a time knocking out multiples of 7. Dennis Morris's article looked as if it would solve the problem but 7 is sadly missing. Since $10 \equiv 3 \pmod 7$ there is still a lot of work to do. Does any M500 reader have a simpler approach?

He doesn't point out that since $10 \equiv 1 \pmod 3$, the same test applies for division by 3, a fact that people who know the test for 9 don't always realize. As 7 is so important in our calendar system, you would expect some simple test to have emerged.

The other intriguing feature of 7 is the decimal version of the basic fractions all consisting of the digits 142857 recurring with the initial digit varying. Can anyone explain why this strange pattern appears in the decimal system and why 0, 3, 6 and 9 do not appear, although all ten digits from 0 to 9 appear in multiples of 7?

I also enjoyed Dennis's article on ϕ in the previous edition **198**. He calls it the golden ratio without actually describing it in terms of dividing a line in the ratio $\phi : 1$ or the 'golden rectangle' with sides in the same ratio such that $1/\phi = \phi/(\phi+1)$, which immediately gives the equation $\phi^2 = \phi+1$ and so the values $\frac{1}{2}(1 \pm \sqrt{5})$.

Its continued fraction form $\phi = [1; 1, 1, 1, \ldots]$ demonstrates its claim to be the most irrational number since it converges most slowly to its ultimate value, where $\phi = 1 + 1/\phi$. Its successive values give precisely the successive Fibonacci ratios, $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \ldots$. An alternative approach using the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, setting up the Fibonacci numbers from any initial pair of numbers, explains why it is so amazing that the ratios converge towards ϕ. The ultimate value depends on the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. (Can someone fill in the details as I can no longer remember them or work them out.) By the usual calculation, they come from $\begin{bmatrix} 1-\lambda & 1 \\ 1 & \lambda \end{bmatrix} = 0$, giving $\lambda^2 - \lambda - 1 = 0$; so $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$, as we had before.

Finally, looking back a long time to the 12 pennies problem and Ian Stewart's laboured approach [*Telegraph*, 8 February 2003] for which you gave a much more systematic method [M500 **191**], can you tell me how you reached the array you proposed? I was working on a similar array by eliminating redundant arrangements from all possible ones but I ran into problems. Can you put me out of my misery?

Many thanks for all you do to keep the magazine going—I do enjoy it.

Yours,

John V. Budd

Tony Forbes writes — In all probability there does not exist a divisibility test for 7 of the same simplicity and elegance as that for 3, say, or even 11. Once one notices that $1001 \equiv 0 \pmod 7$, one can devise the following, which is similar to the test for 11:

> Split the number into groups of three digits, starting from the units. Alternately add and subtract the resulting three-digit numbers. If the sum is divisible by 7, so is the original number.

In case this description is too obscure, let us work through a typical example. Is 314 159 265 divisible by 7? Yes, because $265 - 159 + 314 = 420$ and 420 is a multiple of 7. The main problem is the committing to memory of the seven times table all the way up to $142 \times 7 = 994$. However, you don't have to go any further. For if the number is large, you can perform the test iteratively. Thus $314\,159\,265\,358\,979\,323\,846\,264\,338 \to 338 - 264 + 846 - 323 + 979 - 358 + 265 - 159 + 314 = 1638$. Then $1\,638 \to 638 - 1 = 637$, and 637 is divisible by 7.

The same test works for 13 as well. Also, since $1001 = 7 \cdot 11 \cdot 13$, it works for 11—but of course we already have a simpler test for that number. Similarly, if you form groups of four digits instead of three, you have divisibility tests for 73 and 137 should you ever need them.

With regard to the twelve tarts solution in M500 **191**, I have to admit that there is no subtlety here. I arrived at the weighing instructions by brute force—simply trying out possible combinations until I found one which worked. That is why I gave no explanation. It's rather like the way a professional mathematician solves a quadratic equation which factorizes.

(i) Use the formula $(-b \pm \sqrt{b^2 - 4ac})/(2a)$.
(ii) Notice that the solutions are rational.
(iii) Solve by factorizing the equation.
(iv) Erase all evidence that you did (i).

An exponential sum

Tony Forbes

Look at the front cover of this issue. Yes, I'm afraid that like 36 circles on page 18, this picture also seems to make the brain go dizzy. I am sure we would be interested if there any psychologists out there who can offer an explanation of this phenomenon. Anyway, the cover picture is a plot of the first 20000 partial sums of

$$S_N \;=\; \sum_{n=1}^{N} n^{200\,\pi i}. \tag{1}$$

That is, for each N from 1 to 20000, you plot a point at S_N in the complex plane.

As you can see from the blow-up of the hole in the middle of the dark area (upper plot on the next page), the centre of the thing, z_0, say, is at approximately $z_0 = -7.07 + 4.80\,i$. I have no idea why this point is significant. Also I notice that the diameter of the hole is 1, at least very nearly. Therefore it seems that the first 20000 partial sums of (1) avoid the region $|z - z_0| < \frac{1}{2}$. I suggest two problems.

(i) Find the correct exact expression for z_0, the centre of the hole.

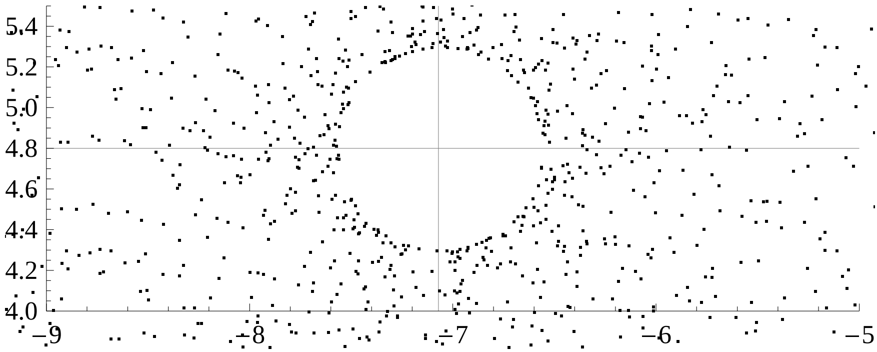
(ii) Either prove that S_N lies outside the region $|z - z_0| < \frac{1}{2}$ for all positive integers N, or find a counter-example.

The smudgy bit to the south-east of the centre represents the first few partial sums of (1) before it settles down to produce the more regular part of the pattern. This is illustrated by lower plot on the next page, which shows $S_1, S_2, \ldots, S_{1000}$.

Reflecting on the form of (1), I feel that I ought to explain my choice of the exponent $200\pi i$. All I can suggest is that it is a special case of a general type of exponential sum,

$$\sum_{n=1}^{N} \exp(2\pi i f(n)) \;=\; \sum_{n=1}^{N} (\cos 2\pi f(n) + i \sin 2\pi f(n)),$$

where $f(n)$ is a given real function. The presence of the factor $i = \sqrt{-1}$ ensures that each term of the sum has absolute value 1. In fact, what we are really doing is joining together matchsticks where $f(n)$ determines the angle (as a fraction of 360 degrees) between the nth matchstick and the positive real axis. Here I have chosen $f(n) = 100 \log n$.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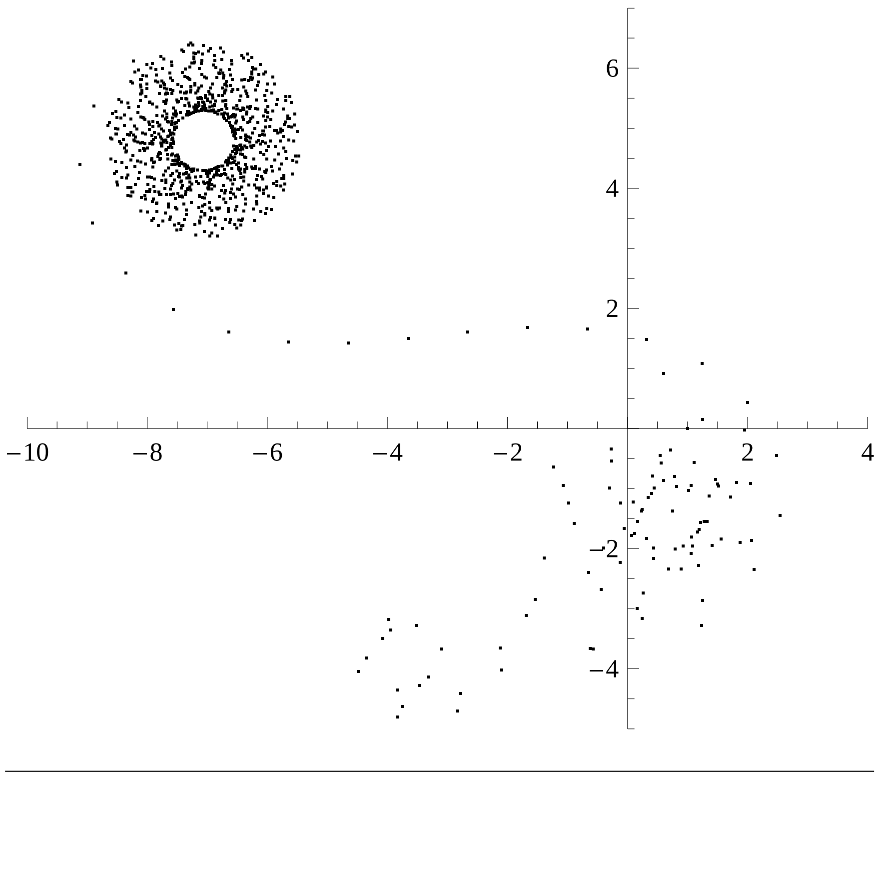
6
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Contents — M500 201 – December 2004