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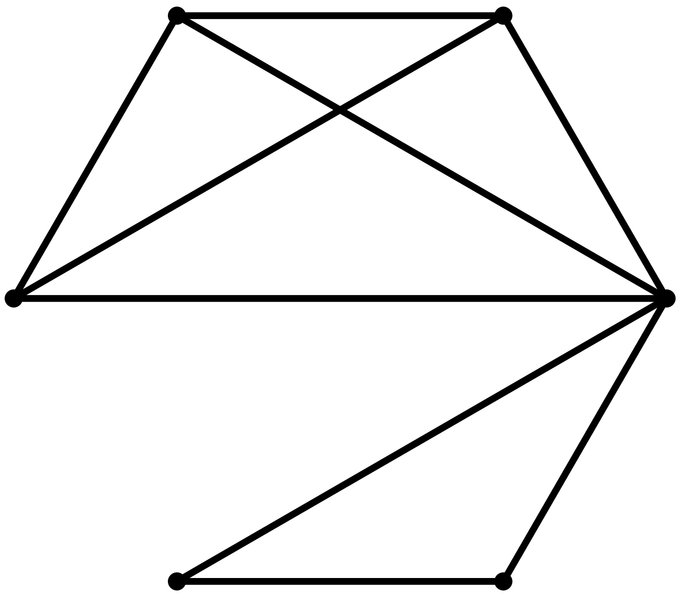
M500 278



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The Revision Weekend is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's web site.

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M500 Mathematics Revision Weekend 2018

The M500 Revision Weekend 2018 will be held at

Yarnfield Park Training and Conference Centre, Yarnfield, Staffordshire ST15 0NL
from Friday 18th to Sunday 20th May 2018.

We expect to offer tutorials for most undergraduate and postgraduate mathematics Open University modules, subject to the availability of tutors and sufficient applications. Application forms will be sent via email to all members who included an email address with their membership application or renewal form. Contact the Revision Weekend Organizer, **Judith Furner** if you have any queries about this event.

Inverted triangles

Tommy Moorhouse

Inversion in the plane Consider the unit circle centred on the origin in the plane $\mathbb{R}^2$. Inversion (denoted by the function $i : \mathbb{R}^2 \to \mathbb{R}^2$) maps the inside of the circle to the outside as follows. Given any point (x, y) draw a line from the origin through the point, and draw the point $(x/(x^2 + y^2), y/(x^2+y^2))$, which lies on this line. The circle maps to itself, while the map is not (yet) defined at the origin (we will look at this briefly later).

Inverting lines The inversion map has some fascinating properties, of which we will consider just one: under inversion straight lines map to circles (or circular arcs). To see this set up the x and y axes as usual, and consider the line $t \to (x_0, t)$ for some range of t. Then

$$i((x_0, t)) = \left(\frac{x_0}{x_0^2 + t^2}, \frac{t}{x_0^2 + t^2} \right).$$

Show that the image of the line lies on a circle of radius $1/(2x_0)$ that (asymptotically) passes through the origin. More general lines can be considered simply by changing x_0 and rotating the plane. Lines that do not intersect the circle are mapped to circles inside the unit circle. Those that intersect the unit circle map to circles that pass through the origin and intersect the unit circle in the same points as the straight line (a line tangent to the unit circle furnishes an obvious special case).

Inverted triangle Now consider two equilateral triangles, one (‘I’) inscribed in the unit circle, the other (‘O’) circumscribed, with the vertices of I touching the midpoints of the sides of O. Pick any of the outer three small triangles (‘T’) defined by one vertex of O and two of I. Show that the three vertices of the corresponding inverted triangle (i.e. the ‘triangle’ with circular arcs for sides obtained by inverting the sides of T) are collinear.

Explore! The interested reader could explore the properties of the inverted triangles (areas, perimeters) and other inverted polygons, with interesting visual representations. More generally, inversion preserves angles, and it is an interesting exercise to prove this. Inversion as defined here has the property that $i^2 = 1$ (the doubly inverted object is the same as the original): what about inversion in one circle followed by inversion in a different circle?

Loose ends The ‘problem’ of the origin can be resolved in projective space $\mathbb{RP}^2$ (which looks like $\mathbb{R}^2$ locally) by ‘blowing it up’ so that the origin is replaced by the set of directions defined by (unoriented) lines passing through it, and each point at infinity (defined by the class of parallel straight lines) maps to a distinct point of this 'origin', making inversion well defined. In this model any line approaching the origin passes through the point defined by its direction and emerges on the other side as one would intuitively expect. Alternatively, staying in 'ordinary' $\mathbb{R}^2$, adding a single point at infinity allows a well-defined inversion map. All straight lines meet the point at infinity, and all the circular images under i pass through the origin.

Inversion preserves angles

Tommy Moorhouse

Inversion in the plane Previously we considered inversion i in the unit circle centred on the origin in the plane $\mathbb{R}^2$. We asserted that this map preserves angles. It is what is called a conformal map: the scalar product of two vectors is preserved up to a scale factor. In this guided problem we will consider a proof that inversion preserves angles and also find the scale function.

Step 1 An angle can be defined by two straight lines meeting at a point. Let the lines be parametrized by

$$\gamma_1(t) = (t, m_1 t + c_1); \quad \gamma_2(t) = (t, m_2 t + c_2).$$

Step 2 Show that the lines meet at

$$p = \frac{1}{m_2 - m_1}(c_1 - c_2,\ c_1 m_2 - c_2 m_1),$$

which we will write as

$$\frac{-1}{\delta m}(\delta c,\ c \times m),$$

where the minus sign reminds us that $\delta m = m_1 - m_2$.

Step 3 A vector V at a point p will be written V_p. Show that the tangent vectors to the above lines have the representation

$$(\dot{\gamma}_i)_p = (1,\ m_i)_p$$

and that the cosine of the angle between them is

$$\frac{1 + m_1 m_2}{\sqrt{(1 + m_1^2)(1 + m_2^2)}}.$$

Step 4 Now we need to map the vectors under i_*. The map i_* takes a vector at the point p and maps it to a vector at the point $i(p)$ as follows:

$$i_*(V_p) \;=\; \left[\frac{d}{dt}\, i(p+tV)\right]_{t=0},$$

where p and tV are added component-wise and the point $i(p)$ is understood. Intuitively, as we move along the vector away from p by a small amount, the image of the moving point defines the tangent vector to the image circle at $i(p)$. Show that

$$p + t\dot{\gamma}_1 \;=\; \frac{1}{\delta m}\left(t\delta m - \delta c,\; m_1 \delta m t - c \times m\right),$$

with an analogous expression for γ_2.

Step 5 Find the square of the length of $p + t\dot{\gamma}_1$ and write it in the form $At^2 - 2Bt + D$, where A, B and D are expressions you will find in $\delta m, \delta c$ and so on (for example, $D = \delta c^2 + (c \times m)^2$). Find the image of $p + t\dot{\gamma}_k$ and carry out the differentiation—remembering that taking $t = 0$ at the end allows some simplification. Conclude that

$$i_*(\dot{\gamma}_k)_p \;=\; \left(\frac{\delta m}{\delta c^2 + (c \times m)^2}\right)^2 \left(-E - 2m_k F,\; m_k E - 2F\right),$$

where $E = \delta c^2 - (c \times m)^2$ and $F = \delta c(c \times m)$. Notice that $E^2 + 4F^2 = D^2$.

Step 6 Finally, denoting the usual scalar product by $\langle v, w \rangle$, show that

$$\frac{\langle i_*(\dot{\gamma}_1)_p, i_*(\dot{\gamma}_2)_p \rangle}{|i_*(\dot{\gamma}_1)_p||i_*(\dot{\gamma}_2)_p|} \;=\; \frac{1 + m_1 m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}.$$

Thus the angle between the image vectors is the same as that between the original lines.

The scale factor To find the scale factor $\Omega((\dot{\gamma}_k)_p) = |i_*(\dot{\gamma}_k)_p| / |(\dot{\gamma}_k)_p|$ we rotate the axes so that $\gamma(t) = (x_0, t)$. Then a straightforward calculation gives

$$\Omega(V_p) \;=\; \frac{1}{|p|^2}.$$

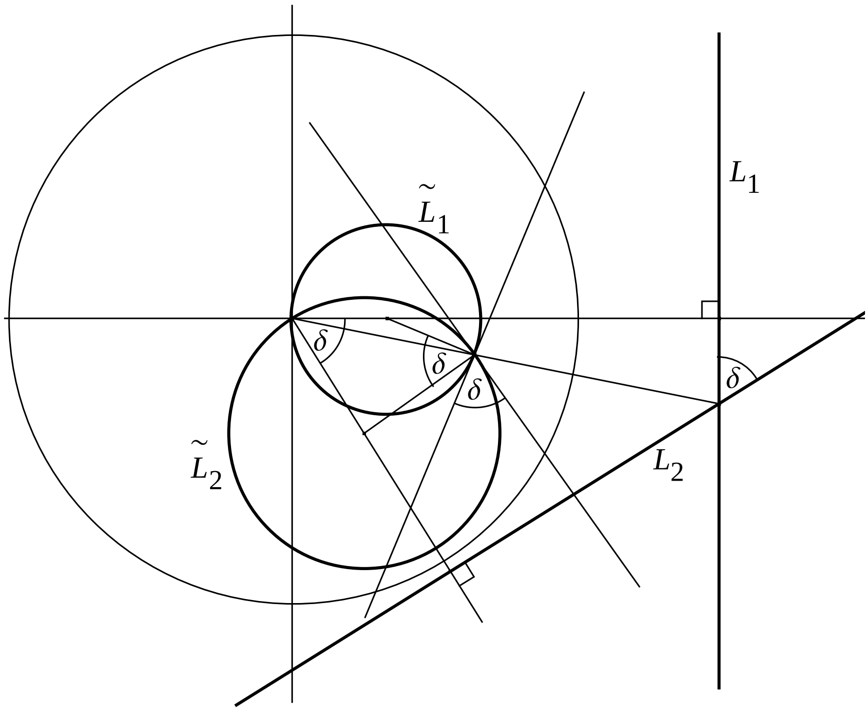
The scale depends only on the position of the vector, not its direction or size.

Conclusion We have shown that i is a conformal map and found the scale factor.

Inversion preserves angles: a geometric proof

Tommy Moorhouse

Inversion visualized Inversion i in the unit circle centred on the origin in the plane $\mathbb{R}^2$ preserves angles, but the differential geometric proof using tangent vectors may seem obscure. In fact there is a simple geometric proof. Anyone who enjoys compass and ruler constructions can have a go at it. There are three cases to consider—the first is outlined here.



Step 1 Draw the unit circle (choosing a 'unit' that makes the rest of the construction easy).

Step 2 Construct two intersecting lines L_1 and L_2 outside the circle by drawing radial lines from the centre of the circle, measuring a distance along the line and constructing the perpendiculars. The lines should intersect in a convenient place.

Step 3 Construct the image circle of each line (say $\tilde{L}_1$ and $\tilde{L}_2$), using the property that the images are circles centred on the radial line with radius $1/(2x)$.

Step 4 Construct the intersecting tangent lines to the image circles $\tilde{L}_1$ and $\tilde{L}_2$. The centre of the unit circle is collinear with the intersection point of L_1 and L_2 and that of $\tilde{L}_1$ and $\tilde{L}_2$.

Step 5 Denote by δ the smaller angle of intersection between L_1 and L_2 (perpendicular lines are a special case). The perpendicular to one of the tangent lines at the point of intersection passes through the centre of the corresponding circle, and plane geometry tells us that the angle formed by this perpendicular and the line joining the centre of the unit circle to the intersection points is δ.

Step 6 By 'chasing' the angles around the image intersection point deduce that the tangent lines intersect at an angle of δ.

The other cases are: lines intersecting inside the unit circle, and lines intersecting on the unit circle. These are left to the reader.

Conclusion We have shown using plane geometry that i preserves angles.

Problem 278.1 – Pistachio nuts

Tony Forbes

There is a bowl containing n pistachio nuts. How many times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?

(i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or just half of a pistachio nut shell.

(ii) If it is a half-shell, you discard it.

(iii) Otherwise you split the shell into two halves, remove the kernel, which you eat, and return the two shell fragments to the bowl.

This is like Problem 275.6 (see Tommy Moorhouse's solution in this issue) except that now we have adopted a more efficient nut-eating strategy.

Problem 278.2 – Symbols

There are q symbols. How many unordered n-tuples of symbols are there? For instance, when $q = 3$ and $n = 2$ the answer is 6, AA, AB, AC, BB, BC, CC.

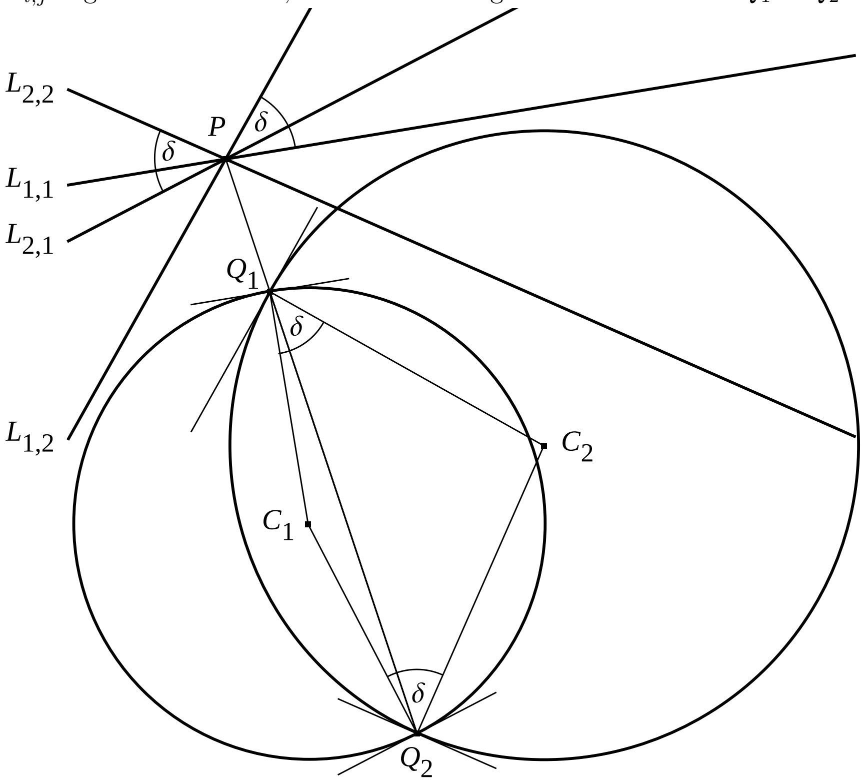
Problem 278.3 – Two circles and four lines

Tony Forbes

Motivated by the diagram on page 4, which, perhaps without the big circle, must surely have been known to Archimedes, I offer the following interesting exercise involving no more than high-school geometry.

Suppose there are two distinct circles, with centres C_1 and C_2, that intersect at points Q_1 and Q_2. Let $\delta = \angle C_1 Q_1 C_2 = \angle C_1 Q_2 C_2$. Let P be a point on the line defined by Q_1 and Q_2. For $i, j = 1, 2$, let $L_{i,j}$ be the line that passes through P and is perpendicular to the line $Q_i C_j$. Show that for $i = 1, 2$, the lines $L_{i,1}$ and $L_{i,2}$ intersect at angle δ.

Note that if Q_1 and Q_2 are not distinct, then $\delta = 0$ and the four lines $L_{i,j}$ degenerate into one, the common tangent to the circles at $Q_1 = Q_2$.



Equilateral triangles

Chris Pile

We answer the questions posed by Tommy Moorhouse in 'Equilateral triangles', M500 **275**, p. 22. In each of the three diagrams, below, the bordering square has side 1.

(i) Figure 1 clearly shows that the area of an equilateral triangle is equal to the shaded area and therefore is less than half the area of a square with the same side length.

(ii) If an equilateral triangle has area 1/2, height p and side length $2b$, then $p = b\sqrt{3}$, $pb = 1/2$ and therefore $2b = \sqrt{2/\sqrt{3}}$. If the triangle is positioned as in Figure 2, then $4b^2 = a^2 + 1$, and hence $a^2 = 2/\sqrt{3} - 1$. The part of the triangle in the $a \times 1$ rectangle is less than half the area. Therefore the part of the triangle in the $(1-a)\times 1$ rectangle must be greater than half the area. So from the diagram (Figure 2) it is clear that the third vertex is outside the square.

(iii) With the equilateral triangle positioned as in Figure 3, $p = \sqrt{2} - b = b\sqrt{3}$. Therefore $b = \sqrt{2}/(1+\sqrt{3})$ and $p = \sqrt{6}/(1+\sqrt{3})$. Hence the area of the triangle is $pb = 2\sqrt{3} - 3 \approx 0.4641$.

(iv) The smallest equilateral triangle with all three vertices touching the square must have side long enough to touch opposite sides; therefore the smallest is a unit equilateral triangle. From Figure 3, the area of the triangle is $pb = b^2\sqrt{3} = \sqrt{3}(1+a^2)/4$, which reaches its minimum when $a = 0$ and $b = 1/2$, as in Figure 1.

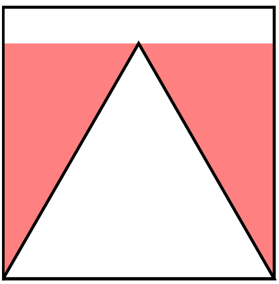


Figure 1

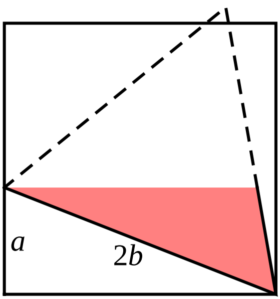


Figure 2

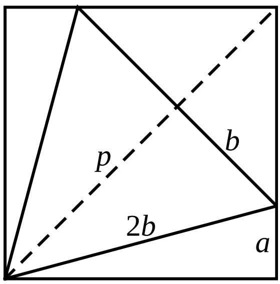


Figure 3

An interesting series

Abdul Ahad

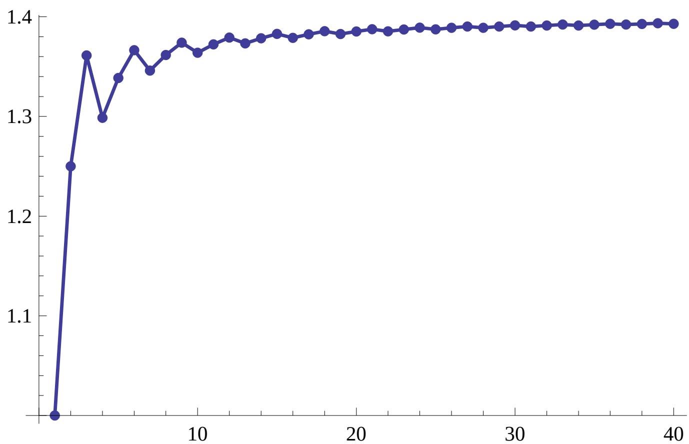
The series is a sum of reciprocal squares all the way to infinity that is dealt with in an alternating fashion: adding and subtracting consecutive terms in the ratio 2:1, respectively. The series was first stated by the author on the Math forum, [1], as follows: For $n = 1$ to infinity, starting with 1 the next two consecutive terms of $1/n^2$ are added, then the next term of $1/n^2$ is subtracted, the following two consecutive terms of $1/n^2$ are added, the subsequent term of $1/n^2$ is subtracted, ..., repeating like that all the way to infinity. Thus

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \cdots. \tag{1}$$

The sum of reciprocal squares, as first stated and solved by Euler, [2], adds all terms and converges exactly to $\pi^2/6$. Hence, by algebraic manipulation, the sum of series (1) can be rewritten as a difference of the original sum of reciprocal squares and the new sum expressed as an exact result:

$$S = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}. \tag{2}$$

By calculating the first 40 consecutive terms and plotting their cumulative sums on a graph it is clear that the series is convergent and that the limit exists.



We note that the sum appears to be tending to a limiting value of around 1.4. Convergence can also be proved analytically by using an appropriate test. The comparison test in particular is used as follows.

The first term in (2) is a constant, so all we need to prove is that the sum in the second term converges. This can be compared with a similar series:

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2} \;<\; \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{3}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely the new series also converges absolutely.

Inputting the series into Wolfram Alpha, [3], the sum value (taken to infinity) is computed to be approximately 1.4014680389755.... This appears to be an irrational number by comparison to sums of other similar series. We note that the construct of the sum on the left of (3) bears a close resemblance to each of the following series that have been proved to have irrational sums related to π.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}, \quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

References

[1] A. Ahad, Proofs of my new infinite alternating series, Google Groups, sci.math (April 10, 2017).

[2] A. Eremenko, How Euler found the sum of reciprocal squares (2013).

[3] Wolfram Alpha, https://www.wolframalpha.com/examples/Math.html.

Problem 278.4 – Polynomial integration

(i) Find a polynomial $Q(x,y)$ of degree 2 in x and y such that for any quadratic $P(x)$,

$$\int_{-1}^{1} P(x)Q(x,y)\,dx \;=\; P(y). \tag{1}$$

(ii) Find a polynomial $Q(x,y)$ of degree 1 in x and y such that (1) holds for any linear function $P(x)$.

Hint: Do (ii) first.

Solution 276.7 – Three primes

If p and $p^2 + 8$ are both prime, prove that $p^3 + 4$ is also prime.

Stuart Walmsley

The initial statement: p and $p^2 + 8$ are both primes.

If $p = 2$, then $p^2 + 8$ $(= 12)$ is not prime, therefore $p = 2$ does not satisfy the initial statement.

If $p = 3$, then $p^2 + 8$ $(= 17)$ is prime, therefore $p = 3$ does satisfy the initial statement.

For all $p > 3$, p is either one greater than a multiple of 3 or one less than a multiple of 3: $p = 3m \pm 1$. Then

$$p^2 + 8 \;=\; (3m \pm 1)^2 + 8 \;=\; 9m^2 \pm 6m + 1 + 8 \;=\; 3(3m^2 \pm 2m + 3).$$

So $p^2 + 8$ is a multiple of 3 and hence not a prime.

Therefore the initial statement, p and $p^2 + 8$ are both primes, is only true if $p = 3$. For $p = 3$, $p^3 + 4 = 31$, a prime. Therefore if p and $p^2 + 8$ are both primes, $p^3 + 4$ is also prime.

Ledger White

Proof 1 Clearly $p^2 + 8 = p^2 - 1 + 9$ and $(p^2 - 1) = (p + 1)(p - 1)$ for all $p \neq 3$. If p is not divisible by 3, then $p - 1$ or $p + 1$ must be divisible by 3. Then $p^2 - 1$ is divisible by 3 and so also $p^2 - 1 + 9$. Therefore $p^2 + 8$ is never prime; it is always divisible by 3.

Proof 2 Look at the squares of n:

n	0	1	2	3	4	5	6	...
n^2	0	1	4	9	16	25	36	...
$n^2 - (n-1)^2$	0	1	3	5	7	9	11	...

To find the square of n simply add the first n odd numbers. Reformat the series like this:

n	1	2	3	4	5	6	7	8	9	...
n^2	1	4	9	16	25	36	49	64	81	...
difference	1	3	5	7	9	11	19	21	3	...

A gentle study of the reformatted table shows the following. The table continues in the same way for all n. In column 1 and column 2 all the squares have a remainder of 1 when divided by 3. Squares in column 3 are all multiples of 3. Therefore all squares are (a) multiples of 3 or (b) have a remainder of 1 when divided by 3. Adding 8 to those gives a number which is divisible by 3. The n in columns 1 and 2 include all primes other than 3. Therefore for all primes other than 3, $p^2 + 8$ is never a prime; it is divisible by 3.

Tommy Moorhouse

The proof for M500 readers The strange thing about this problem is that it invites a search for a general solution. In fact this is a distraction. Consider the number p^2+8, which is prime by assumption. Reduce modulo 3, observing that for any $p \neq 3$ we have $p^2 \equiv 1 \pmod 3$:

$$p^2 + 8 \equiv 0 \pmod 3.$$

This tells us at once that $p = 3$, and we find that $p^3 + 8 = 31$ is indeed prime. By finding the only example we have proved the assertion.

The proof in everyday language Start by trying a simple experiment. Take any number, multiply this number by itself, and find the remainder when dividing by 3. For example $47 \times 47 = 2209 = 3 \times 736 + 1$. In fact if the number you start with is not divisible by 3 then the remainder is always 1. The proof is a matter of writing out the two possible cases: case one is that for which the number you start with leaves remainder 1 when divided by 3; in the second case it leaves remainder 2.

Now assume that the number p is prime, and form the number $p^2 + 8$. Dividing by 3 we find that we have a remainder from each term: 1 from p^2 and 2 from 8 ($8 = 3 \times 2 + 2$). These remainders can be added together to give another 3, so the sum $p^2 + 8$ is a multiple of 3. But hold on! if $p = 3$ (and only then) this argument fails, and $p^2 + 8 = 17$ is indeed prime. In this case $p^3 + 4 = 31$ is prime as well.

In the only case allowing $p^2 + 8$ to be prime we have shown that $p^3 + 4$ is also prime. Thus we are allowed to say we have proved the statement in general.

Problem 278.5 – Tan-gled trigonometry

William Bell

Find the general solution of the equation

$$\tan 9x - \tan 2x \;=\; \tan 9x \tan 6x \tan 3x + \tan 6x \tan 4x \tan 2x.$$

Solution 275.6 – Pistachio nuts

> There is a bowl containing n pistachio nuts. How many times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?
>
> (i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or it might be half of a pistachio nut shell.
>
> (ii) If it is a half-shell, you just return it to the bowl.
>
> (iii) Otherwise you split the shell into two halves, remove the kernel, which you eat, and return the two shell fragments to the bowl.

Tommy Moorhouse

An interesting point is how we quantify the idea of expectation here. One approach, the one taken below, is to say that if the probability of completing the task after N moves is greater than 1/2 we would expect, on average, to be successful after N moves. More sophisticated measures could be used.

Consider the simplest nontrivial case, that of two nuts in the bowl. The first nut is eaten (probability 1) and now there is one nut and two half shells. On the next try we might pick up a nut (probability 1/3) and finish, or pick a shell and be no better off (probability 2/3).

Continuing with this argument we conclude that the probability of finishing the nuts after $k+1$ 'dips' is zero if $k = 0$ and otherwise

$$p(k+1) \;=\; \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{k-1}\right)$$

and we find that $p(2) < 1/2$ but $p(k+1) \geq 5/9 > 1/2$ when $k > 2$, so that, most of the time, after three dips we will have eaten all the nuts.

The general case of n nuts in the bowl initially is more complicated. Let us denote the 'state' of the bowl with $n-k$ nuts and $2k$ shells as $(n-k, 2k)$. The probability of picking a nut is clearly $(n-k)/(n+k)$, after which the state will be $(n-(k+1), 2(k+1))$. The probability of picking a shell and staying in state $(n-k, 2k)$ is $2k/(n+k)$.

To keep track of the probabilities involved we will work as follows. We know that $p(k+1) = 0$ if $k+1 < n$, so we calculate below $p(n)$, the probability of eating all the nuts in n dips. A longer sequence of dips will generally be required, because we can get stuck in any of the earlier states, so we consider the total probability of eating all the nuts after $N \geq n$ dips, that is after exactly n dips or after exactly $n+1$ dips or ... or exactly N dips.

We can think of a sequence of dips as a path through the states (try drawing a diagram consisting of dots representing the states and lines joining the dots to represent the act of selecting an item from the bowl to see how it works: then a loop starting and ending at the same state represents picking a half-shell). The total probability of eating all the nuts after N dips, or equivalently reaching the state $(0, 2n)$ after N moves, is thus

$$p(N) \;=\; \frac{1}{\binom{2n-1}{n}} P(N)$$

where $P(N)$ is a function we will now investigate and the factor in front of $P(N)$ is the probability $p(n)$ of going directly through all the states (i.e. eating all the nuts). This factor occurs for every successful path and can be extracted as above.

The function $P(N)$ can be derived from the probabilities for getting stuck in a given state. Rather than try counting the number of possible paths a little trial convinces us that the function can be found from the coefficients of a rational function of a dummy variable x. Consider the functions

$$f_k(x) \;=\; \frac{1}{1 - 2kx/(n+k)}.$$

The expansion in x gives the probability of getting stuck in the kth state for m dips as the coefficient of x^m. Then the function $\prod_{k=0}^{n-1} f_k(x)$ covers the whole set of probabilities of getting stuck in any set of states along the path. To extract the coefficients of x we use the operator

$$F(x) \;\to\; \left[\frac{1}{k!}\frac{d^k F}{dx^k}\right]_{x=0}.$$

Finally we have

$$p(N) \;=\; \frac{1}{\binom{2n-1}{n}} \sum_{k=0}^{N-n} \frac{1}{k!} \left[\frac{d^k}{dx^k} \prod_{m=0}^{n-1} \frac{1}{1 - 2mx/(n+m)}\right]_{x=0}.$$

Here the sum for $N < n$ is zero. We now need to find the smallest value of N that gives $p(N) > 1/2$. This can be done on a case by case basis and is possibly a task for a computer or a keen reader. Starting with five nuts, sixteen dips will give a better than even chance of finishing the nuts, while seven nuts requires twenty six dips on average.

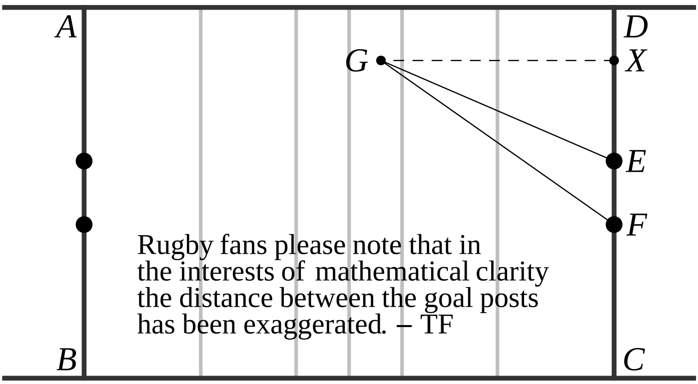
Problem 278.6 – Rugby conversions

John Bull

A rugby pitch, defined as a rectangle $ABCD$, has goal lines AB and CD. Line CD has goal posts at E and F.

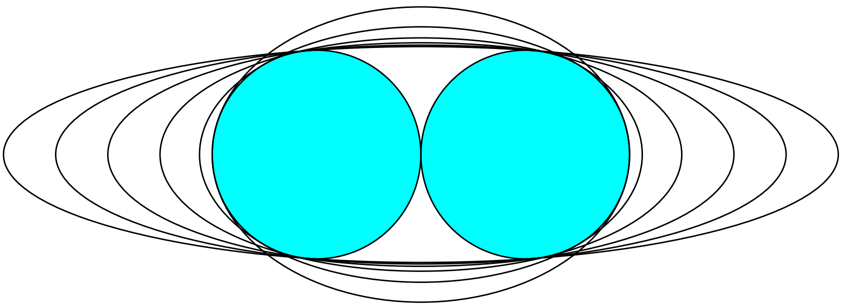
A player scores a try by touching the ball down at point X, outside the goal posts. To convert the try, he walks back from X towards AB along the line parallel to DA and stops at point G. He scores the conversion by kicking the ball between the goal posts. This will be easier when angle EGF is greatest.

How far should the player walk to see the largest angle EGF?



Problem 278.7 – Two circles and an ellipse

What is the smallest area of an ellipse containing two unit circles?



Solution 276.8 – Obdurate integral

Evaluate

$$I = \int \frac{\sqrt{a^2 + b^2 \cos^2 x}}{\cos x}\, dx.$$

Graham Lovegrove

Clearly

$$I = \int \sqrt{a^2 \sec^2 x + b^2}\, dx.$$

Put $c^2 = a^2 + b^2$. The two simple substitutions

$$a^2 \sec^2 x + b^2 = t^2, \quad \text{and} \quad t = \frac{c(u + u^{-1})}{2}$$

can be concatenated into one:

$$\tan x = \frac{c(u - u^{-1})}{2a}, \qquad \frac{dx}{du} = \frac{c(u + u^{-1})}{2au \sec^2 x}.$$

It will be convenient to abbreviate $(u + u^{-1})$ to v in the following manipulation. So

$$\frac{dx}{du} = \frac{2acv}{u(c^2v^2 - 4b^2)}.$$

With this substitution, the integral becomes

$$\begin{aligned}
I &= a \int \frac{c^2v^2}{u(c^2v^2 - 4b^2)} du = a \int \frac{1}{u} \left[1 + \frac{4b^2}{c^2v^2 - 4b^2}\right] du \\
&= a \ln(u) + a \int \frac{4b^2 du}{u(c^2v^2 - 4b^2)} \\
&= a \ln(u) + ab \int \left[\frac{1}{u(cv - 2b)} - \frac{1}{u(cv + 2b)}\right] du \\
&= a \ln(u) + ab \int \left[\frac{1}{cu^2 - 2bu + c} - \frac{1}{cu^2 + 2bu + c}\right] du \\
&= a \ln(u) + b \arctan\left[\frac{cu - b}{a}\right] - b \arctan\left[\frac{cu + b}{a}\right],
\end{aligned}$$

which can then be written in terms of x by substituting

$$u = \frac{a \tan x + \sqrt{a^2 \sec^2 x + b^2}}{c}.$$

Richard Gould

To integrate

$$I = \int \frac{\sqrt{a^2 + b^2 \cos^2 x}}{\cos x}\, dx,$$

proceed as follows, starting with the substitution $u = b \sin x$:

$$\begin{aligned} I &= b \int \frac{\sqrt{a^2 + b^2 - u^2}}{b^2 - u^2}\, du = b \int \frac{a^2 + b^2 - u^2}{(b^2 - u^2)\sqrt{a^2 + b^2 - u^2}}\, du \\ &= a^2 b \int \frac{du}{(b^2 - u^2)\sqrt{a^2 + b^2 - u^2}} + b \int \frac{du}{\sqrt{a^2 + b^2 - u^2}}. \end{aligned}$$

Now use

$$\frac{b}{b^2 - u^2} = \frac{1}{2}\left(\frac{1}{b+u} + \frac{1}{b-u}\right)$$

to give

$$\begin{aligned} I &= \frac{a^2}{2} \int \frac{du}{(b+u)\sqrt{a^2 + b^2 - u^2}} + \frac{a^2}{2} \int \frac{du}{(b-u)\sqrt{a^2 + b^2 - u^2}} \\ &\qquad + b \int \frac{du}{\sqrt{a^2 + b^2 - u^2}} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_1, let $v = 1/(b+u)$. Then $u = 1/v - b$ and $du = -dv/v^2$.

$$\begin{aligned} I_1 &= \frac{a^2}{2} \int \frac{-dv/v}{\sqrt{a^2 + b^2 - 1/v^2 + 2b/v - b^2}} \\ &= -\frac{a^2}{2} \int \frac{dv}{\sqrt{a^2 v^2 + 2bv - 1}} \\ &= -\frac{a}{2} \int \frac{dv}{\sqrt{v^2 + 2bv/a^2 - 1/a^2}} \qquad (1) \\ &= -\frac{a}{2} \int \frac{dv}{\sqrt{(v + b/a^2)^2 - (a^2 + b^2)/a^4}}. \end{aligned}$$

Now use the standard integral

$$\int \frac{dv}{\sqrt{(v+\alpha)^2 - \beta^2}} = \operatorname{arccosh}\left(\frac{v+\alpha}{\beta}\right) = \ln\left|\frac{v + \alpha + \sqrt{(v+\alpha)^2 - \beta^2}}{\beta}\right|$$

with

$$v+\alpha \;=\; v+\frac{b}{a^2} \;=\; \frac{1}{b+u}+\frac{b}{a^2} \;=\; \frac{a^2+b^2+bu}{a^2(b+u)}$$

and

$$\sqrt{(v+\alpha)^2-\beta^2} \;=\; \sqrt{\frac{1}{(b+u)^2}+\frac{2b}{a^2(b+u)}-\frac{1}{a^2}} \;=\; \frac{\sqrt{a^2+b^2-u^2}}{a(b+u)}$$

(from (1)) to give

$$I_1 \;=\; -\frac{a}{2}\ln\left|\frac{a^2+b^2+bu+a\sqrt{a^2+b^2-u^2}}{a^2(b+u)}\times\frac{1}{\beta}\right|.$$

Similarly

$$I_2 \;=\; \frac{a}{2}\ln\left|\frac{a^2+b^2-bu+a\sqrt{a^2+b^2-u^2}}{a^2(b-u)}\times\frac{1}{\beta}\right|.$$

Setting $z \;=\; \sqrt{a^2+b^2-u^2}$ gives

$$I_1+I_2 \;=\; \frac{a}{2}\ln\left|\frac{(b+u)\bigl(a^2+b^2-bu+az\bigr)}{(b-u)\bigl(a^2+b^2+bu+az\bigr)}\right|.$$

Which reduces to

$$\begin{aligned} I_1+I_2 \;&=\; \frac{a}{2}\ln\left|\frac{bz^2+a(b+u)z+a^2u}{bz^2+a(b-u)z-a^2u}\right| \;=\; \frac{a}{2}\ln\left|\frac{(bz+au)(z+a)}{(bz-au)(z+a)}\right| \\ &=\; \frac{a}{2}\ln\left|\frac{b\sqrt{a^2+b^2-u^2}+au}{b\sqrt{a^2+b^2-u^2}-au}\right| \qquad (z\neq -a) \\ &=\; \frac{a}{2}\ln\left|\frac{\sqrt{a^2+b^2\cos^2 x}+a\sin x}{\sqrt{a^2+b^2\cos^2 x}-a\sin x}\right|. \end{aligned}$$

The third, I_3, is a standard integral giving

$$I_3 \;=\; b\arcsin\left(\frac{u}{\sqrt{a^2+b^2}}\right) \;=\; b\arcsin\left(\frac{b\sin x}{\sqrt{a^2+b^2}}\right),$$

and

$$I \;=\; b\arcsin\left(\frac{b\sin x}{\sqrt{a^2+b^2}}\right)+\frac{a}{2}\ln\left|\frac{\sqrt{a^2+b^2\cos^2 x}+a\sin x}{\sqrt{a^2+b^2\cos^2 x}-a\sin x}\right|.$$

Solution 276.4 – Symmetric binary matrix

> Let $\mathbf{B}$ be a symmetric $\{0,1\}$ matrix with diagonal elements $\mathbf{d}$. Show that $\mathbf{Bx} \equiv \mathbf{d} \pmod 2$ is solvable for $\mathbf{x}$, or find a counter-example.

Tony Forbes

If $\mathbf{B}$ is non-singular (mod 2), then $\mathbf{x} \equiv \mathbf{B}^{-1}\mathbf{d} \pmod 2$ and there is nothing more to say. So we need only consider singular (mod 2) $\mathbf{B}$; however, I cannot immediately see how that helps. The problem is actually well known (to everyone except me) and the following is based on a neat solution attributed to **Noga Alon**. Thanks to **Robin Whitty** for my enlightenment.

We attempt to solve

$$\mathbf{Bx} \equiv \mathbf{d} \pmod 2 \tag{1}$$

by reducing the augmented matrix $[\mathbf{B}\ \mathbf{d}]$ to row echelon form modulo 2 by repeated use of the operation of adding one row to another row (mod 2).

If you have done any linear algebra, you might recall that a $\{0,1\}$ matrix is in *row echelon form* when (i) treating them as binary numbers, the rows are sorted into non-increasing order, and (ii) if a row has a 1 in it, then all entries below the leading 1 are zero. To illustrate this important concept we show a typical augmented binary symmetric matrix on the left, below, and a row echelon form on the right. Here you would begin by adding row 1 to rows 2, 5 and 6, then swap rows 2 and 3, add row 2 to rows 5, 6 and 7, add row 3 to rows 4, 5, 6, 7, 8 and 14, add row 4 to rows 6, 7, 9 and 13, add row 5 to rows 7, 8, 10 and 14, swap rows 6 and 13, add row 6 to row 14, swap rows 7 and 14, swap rows 8 and 11, . . . , you get the idea.

$$
\begin{bmatrix}
1\,1\,0\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
1\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\\
1\,0\,0\,1\,1\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
1\,0\,0\,0\,0\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,1\,0\,0\,0\,0\,1\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,1\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,0\,1\\
0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,1\\
0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,1\,1\,0\,0\,0\,0\,1\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,1\,0\,1\,1\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,1\,0\,1\,1\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,1\,0\,1\,0\,1\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,1\,0\,1\,1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1\,1\,0\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,1\,0\,1\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,0\,1\,1\,1\,1\,0\,0\,1\,1\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,1\,1\,0\,0\,0\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,1\,1\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,0\,1\,0\,1\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,1\,1\,1\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0
\end{bmatrix}
$$

Observe that (i) a row echelon form obtained in this manner is not necessarily unique (in our example we could have begun by swapping rows 1 and 2), (ii) each row of a row echelon form is a linear combination of rows of the original matrix, and (iii) we haven't really introduced a new operation—to swap rows a and b, you add row a to row b, then row b to row a, then row a to row b.

We now return to the original problem. The only way (1) can fail to have a solution is if a row echelon form of $[\mathbf{B}\ \mathbf{d}]$ contains the row $[0\ 0\ \ldots\ 0\ 1]$. We therefore want to show that this cannot happen. So let us suppose that it does. Then there is a vector $\mathbf{z}$ such that

$$\mathbf{z}^{\mathrm{T}}[\mathbf{B}\ \mathbf{d}] \equiv [0\ 0\ \ldots\ 0\ 1] \pmod 2.$$

Therefore

$$\mathbf{z}^{\mathrm{T}}\mathbf{B} \equiv [0\ 0\ \ldots\ 0] \quad \text{and} \quad \mathbf{z}^{\mathrm{T}}\mathbf{d} \equiv 1 \pmod 2.$$

On the other hand,

$$\begin{aligned} \mathbf{z}^{\mathrm{T}}\mathbf{B}\mathbf{z} &= \sum_{i,j} z_i B_{i,j} z_j = \sum_i z_i B_{i,i} z_i + \sum_{i<j} z_i B_{i,j} z_j + \sum_{i>j} z_i B_{i,j} z_j \\ &= \sum_i z_i B_{i,i} z_i + \sum_{i<j} z_i B_{i,j} z_j + \sum_{i<j} z_i B_{i,j} z_j \qquad (2) \\ &\equiv \sum_i B_{i,i}\, z_i^2 \equiv \sum_i B_{i,i} z_i \equiv \mathbf{z}^{\mathrm{T}}\mathbf{d} \equiv 1 \pmod 2, \end{aligned}$$

where to get (2) we have used the fact that $\mathbf{B}$ is symmetric. Hence

$$\mathbf{z}^{\mathrm{T}}\mathbf{B} \not\equiv [0\ 0\ \ldots\ 0] \pmod 2,$$

a contradiction.

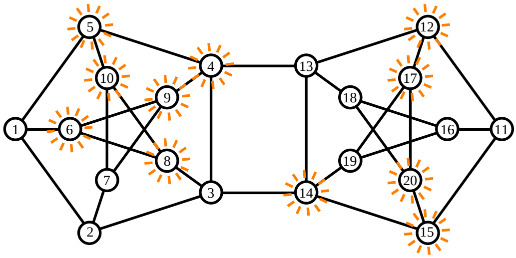
Incidentally, we appear to have solved Problem 276.5 – Put that light out! Suppose there are n lamps, let $\mathbf{I}$ be the $n \times n$ identity matrix and let $\mathbf{A}$ be the adjacency matrix of the graph G representing the arrangement of the switches. Thus $\mathbf{A}_{i,j} = 1$ if $\{i, j\}$ is an edge of G, $\mathbf{A}_{i,j} = 0$ otherwise. Then Problem 276.5 is equivalent to showing that

$$(\mathbf{A} + \mathbf{I})\,\mathbf{x} = [1\ 1\ \ldots\ 1]^{\mathrm{T}} \qquad (3)$$

always has a solution. And when you have solved (3) you operate the switches indicated by the vector $\mathbf{x}$.

Solution 276.5 – Put that light out!

An illumination facility is based on a simple graph G with n vertices. With each vertex v of G we associate a lamp and a push-button switch. When the switch at v is operated the lamps at v and all the neighbours of v change their states, on to off, off to on. The picture shows a typical graph with 20 vertices after switches 1, 4, 9 and 16 have been operated. Prove that, whatever the graph, if all the lights are initially on, there is a set of vertices such that pushing their buttons will turn all the lights off.



Graham Lovegrove

To shorten the explanation, we shall refer to a combination of key presses that toggles all the lights in a chosen subgraph as a *keying* of that subgraph. The proof is by induction. The proposition is easily checked to be true for all graphs of order $n = 1, 2, 3$.

First of all let n be even, and assume that the proposition is true for all graphs of order $n - 1$. Then for each subgraph of $n - 1$ vertices, there is a keying that turns all the lights off. If for one of these subgraphs, the keying also turns off the remaining vertex, we are done. So assume none of the keyings turns off the light of any remaining vertex. Perform all the keyings in turn. Since each vertex is toggled an odd number of times, every light in the graph will be turned off.

Now consider the case of n odd, and again assume that there is a keying to toggle every subgraph of $n - 1$ lights. Once again we can assume that none of the keyings that toggle $n-1$ lights toggles the remaining light. Since the number of vertices is odd, there exists a vertex v of even degree, say $2m$, since as is well known the sum of the degrees of any graph is equal to twice the number of edges. Consider the $2m + 1$ keyings for the subgraphs that exclude in turn v and each of the vertices in its neighbourhood. Apply all of these keyings. Since $2m + 1$ is odd, all of the remaining lights are toggled off. However, the lights for vertex v and its $2m$ neighbours remain on, since each is toggled only $2m$ times. Now push the button on v to turn off its own light and the lights of all its neighbours, and we are done.

Things you can't buy in shops

You are innocently wasting time in a shop, idly browsing and minding your own business, when suddenly you are awakened by that dreaded question, "Can I help you?" Here are some suggestions of useful, everyday items you might want to try asking for.

A protractor marked in radians
A hand-held geiger counter
A thermometer marked in kelvins
An anticlockwise pencil sharpener
Green flowers
An ultra-violet torch
An Ordnance Survey map of the moon
A set of left-handed drill bits
A wristwatch that runs on petrol
A scientific calculator that has at least 12 digits precision

I (TF) am most interested in the last one. I've looked everywhere, including online, but without success—except in charity shops, where old 12-digit models do turn up from time to time. And by '12 digits' I really do mean 12 digits. If I enter 999999 $\times$ 999999, I expect to see the exact answer, 999998000001, including the 1 at the end. I say this because retailers who describe their products as '12-digit calculators' are often making claims that are consistent with my understanding of falsehood. I believe the economy will collapse when consumers eventually discover that suppliers are not supplying the things they actually want to buy.

M500 Winter Weekend 2018

The **thirty-seventh M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

Friday 5$^{\text{th}}$ – Sunday 7$^{\text{th}}$ January 2018.

Cost: £215 to M500 members, £220 to non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. You can obtain a booking form either from the M500 web site,

http://www.m500.org.uk,

or by emailing the Winter Weekend Organizer.

The Winter Weekend provides you with an opportunity to do some non-module-based, recreational maths with a friendly group of like-minded people. The relaxed and social approach delivers maths for fun. And as well as a complete programme of mathematical entertainments, on Saturday we will be running a pub quiz with Valuable Prizes.

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04.00

Front cover The graph $K_4 \vee K_3$.