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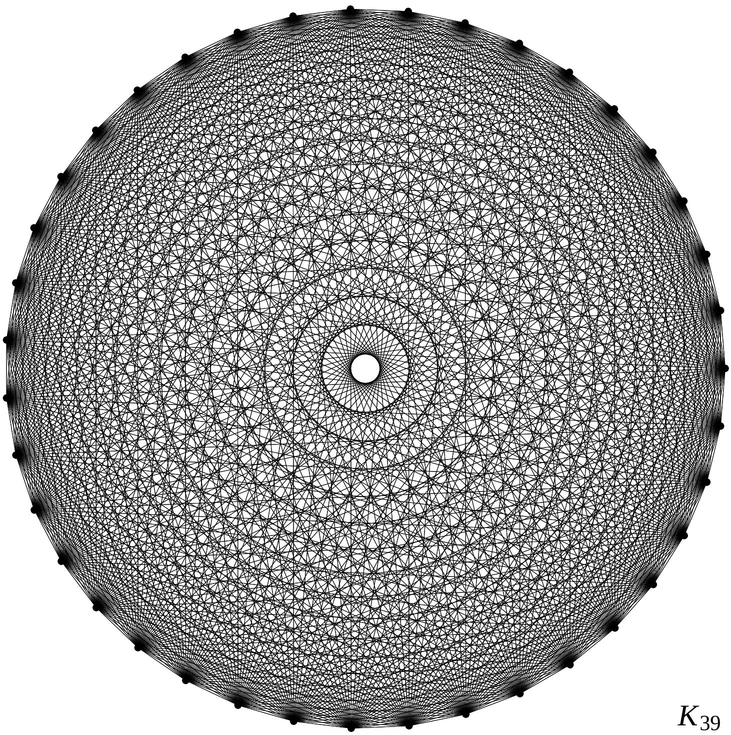

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Doing vector algebra properly

Dennis Morris

To be a vector, a mathematical object has to be part of a set of similar objects that satisfy the axioms of a linear space. Indeed, linear spaces are often called vector spaces. An algebraic field is such a linear space together with a multiplication operation, and so it is no surprise to find that algebraic fields can do everything that vectors can do and plus some.

Within physics, we often find the use of mathematical, arrow-like objects called vectors (usually written in boldface type or with a line underneath them) to describe various physical situations. These mathematical, arrow-like objects, together with two operations called the dot product (also known as inner product or scalar product) and the cross product (sometimes known as the outer product or (technically erroneously) exterior product) are referred to as vector algebra. This is a terminological inexactitude. These objects and these two operations are not an algebra. Neither of the two operations is a bona fide algebraic operation—neither of the operations is closed, for a start. Vector algebra lacks a vector product whereby two vectors multiplied together produce a vector. From a mathematical point of view, so called vector algebra is not an algebra—it is a mess. The only reason physicists like vector algebra is because it works perfectly[1].

This is not news to mathematicians; they have known about this mess since vectors were first invented (Grassman, 1845). Some 120 years ago, William K. Clifford attempted to rectify the situation by inventing a vector product and thereby producing the Clifford algebras. Unfortunately, the Clifford (vector) product is not an algebraic multiplication operation. The Clifford product multiplies two vectors together and produces both a real number and a bivector. Since then, mathematicians have failed to find a bona fide vector product and have had to content themselves with using the 'mess'. This article rectifies that situation.

Since we have within the complex numbers the two-dimensional euclidean space, and, since the complex numbers are an algebraic field, we can do every bona fide thing with this algebra that can be done with two-dimensional euclidean vectors. One of those bona fide things is multiply two complex numbers together to produce another complex number. In addition to doing the bona fide things, it turns out that we can also construct an operation resembling the dot product or cross product. This operation is not an algebraic operation; it is no more than a calculative procedure

[1] And, for this price, they sell themselves—bunch of tarts!

within the algebra.

There is a one-to-one correspondence between the two-dimensional vectors and the complex number matrices:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}.$$

The complex number matrix, as well as being a complex number, is a position vector matrix in two-dimensional euclidean space. The vector product is simply the product of two complex numbers.

> Nomenclature
>
> *Vector matrix*
>
> All the natural algebras are such that there is a bijective mapping between the algebraic matrix form and vectors of the same dimension. Thus, the term *vector matrix* means nothing more than algebraic matrix form. However, out of politeness to the reader, we use it when we are using the algebraic matrix forms as vectors.

We seek a means of calculating the angle subtended at the origin between two such position vectors. We take two position vector matrices:

$$\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} a+c & b+d \\ -(b+d) & a+c \end{bmatrix} \right\}.$$

These matrices have polar forms:

$$\left\{ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} \cos(\theta+\chi) & \sin(\theta+\chi) \\ -\sin(\theta+\chi) & \cos(\theta+\chi) \end{bmatrix} \right\}.$$

We normalize these position matrices by dividing by their respective lengths:

$$\left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos(\theta+\chi) & \sin(\theta+\chi) \\ -\sin(\theta+\chi) & \cos(\theta+\chi) \end{bmatrix} \right\}.$$

These two matrices are now positions upon the unit circle about the origin. The angle between these two position matrices is χ. We seek a way of combining these two matrices, $\odot$, in such a way that the result is:

$$\begin{bmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{bmatrix}.$$

We will then be able to write

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \odot \begin{bmatrix} \cos(\theta+\chi) & \sin(\theta+\chi) \\ -\sin(\theta+\chi) & \cos(\theta+\chi) \end{bmatrix} = \begin{bmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{bmatrix},$$

and for any two such position matrices, we can calculate the angle between them as $\{\arccos, \arcsin\}$ of the elements in the matrix.

We have[2]

$$\left\{\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}, \begin{bmatrix}\cos\theta\cos\chi-\sin\theta\sin\chi & \cos\theta\sin\chi+\sin\theta\cos\chi\\ -(\cos\theta\sin\chi+\sin\theta\cos\chi) & \cos\theta\cos\chi-\sin\theta\sin\chi\end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}, \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{bmatrix}\right\}.$$

Clearly, the operation we need involves the conjugate of the $\theta+\chi$ matrix or the conjugate of the θ matrix. That operation is

$$\begin{aligned}&\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \odot \begin{bmatrix}\cos(\theta+\chi) & \sin(\theta+\chi)\\ -\sin(\theta+\chi) & \cos(\theta+\chi)\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos(\theta+\chi) & -\sin(\theta+\chi)\\ \sin(\theta+\chi) & \cos(\theta+\chi)\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{bmatrix} = \begin{bmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{bmatrix},\end{aligned}$$

or

$$\begin{aligned}&\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \odot \begin{bmatrix}\cos(\theta+\chi) & \sin(\theta+\chi)\\ -\sin(\theta+\chi) & \cos(\theta+\chi)\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos(\theta+\chi) & \sin(\theta+\chi)\\ -\sin(\theta+\chi) & \cos(\theta+\chi)\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{bmatrix} = \begin{bmatrix}\cos\chi & \sin\chi\\ -\sin\chi & \cos\chi\end{bmatrix}.\end{aligned}$$

The two results are the conjugates of each other. Thus, in general, we have

$$[\chi]\overline{[\theta]} = \overline{[\chi]\odot[\theta]}, \qquad \overline{[\chi]}[\theta] = [\chi]\odot[\theta],$$

where the bar over the matrix represents the complex conjugate matrix.

We have

$$\begin{aligned}[\chi]\odot[\theta] &= \overline{[\theta]\odot[\chi]},\\ (\alpha[\chi]+\beta[\lambda])\odot[\theta] &= \alpha[\chi]\odot[\theta]+\beta[\lambda]\odot[\theta],\\ [\chi]\odot[\chi] &= [\det[\chi]] > 0.\end{aligned}$$

[2]The multiple angle trigonometric relations all derive from matrix multiplication of two rotation matrices.

Thus the operation satisfies the axioms of an inner product[3]. However, there is more in this product than the conventional inner product, and therefore we shall not refer to it as the inner product. We shall refer to it as the angle product. This is a change of emphasis from conventional practice. Conventionally, mathematicians take the view that they introduce angles into a space by imposing an inner product upon that space. We already have the angles; we simply need to be able to calculate their values. We denote the angle product of two matrices $\{[A],[B]\}$ by the notation $[A]\odot[B]$.

Although we have gone through the above procedure with normalized matrices in polar form, we did this only for simplicity of presentation. The same procedure works with matrices that are not normalized or not in polar form or both.

Examples:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \odot \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \odot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & bc-ad \\ -(bc-ad) & ac+bd \end{bmatrix},$$

$$\begin{bmatrix} c & d \\ -d & c \end{bmatrix} \odot \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \odot \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ac+bd & -(bc-ad) \\ bc-ad & ac+bd \end{bmatrix}.$$

We found this inner product by seeking a way of calculating the angle between two position vectors. That method is:

$$\begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \odot \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Big/ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \odot \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \Big/ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Big/ \begin{bmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{bmatrix} \odot \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \Big/ \begin{bmatrix} \sqrt{c^2+d^2} & 0 \\ 0 & \sqrt{c^2+d^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2}} & \dfrac{b}{\sqrt{a^2+b^2}} \\ \dfrac{-b}{\sqrt{a^2+b^2}} & \dfrac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \odot \begin{bmatrix} \dfrac{c}{\sqrt{c^2+d^2}} & \dfrac{d}{\sqrt{c^2+d^2}} \\ \dfrac{-d}{\sqrt{c^2+d^2}} & \dfrac{c}{\sqrt{c^2+d^2}} \end{bmatrix},$$

leading to

[3] If anyone is interested, we thus have a complete Hilbert space.

$$\begin{bmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{bmatrix} = \frac{\begin{bmatrix}a & b\\ -b & a\end{bmatrix}\odot\begin{bmatrix}c & d\\ -d & c\end{bmatrix}}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$$
$$= \frac{\begin{bmatrix}a & b\\ -b & a\end{bmatrix}\begin{bmatrix}c & -d\\ d & c\end{bmatrix}}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}} = \frac{\begin{bmatrix}ac+bd & -(ad-bc)\\ ad-bc & ac+bd\end{bmatrix}}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}},$$

giving

$$\cos\chi = \frac{ac+bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}, \qquad \sin\chi = \frac{ad-bc}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

The reader should compare this to the expression in two-dimensional euclidean vector algebra for the angle between two vectors, $\{\vec{x} = [a,b],\ \vec{y} = [c,d]\}$:

$$\cos\chi = \frac{\langle\vec{x},\vec{y}\rangle}{|\vec{x}||\vec{y}|} = \frac{ac+bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

The expression in two-dimensional euclidean vector algebra for the magnitude of the cross product of two vectors is

$$|\vec{x}\times\vec{y}| = |\vec{x}||\vec{y}|\sin\chi, \qquad \sin\chi = \frac{|\vec{x}\times\vec{y}|}{|\vec{x}||\vec{y}|}.$$

In the case where the two vectors are coplanar, we have

$$\begin{bmatrix}a & b & 0\end{bmatrix}\times\begin{bmatrix}c & d & 0\end{bmatrix} = (ad-bc)\begin{bmatrix}0 & 0 & 1\end{bmatrix},$$

$$\sin\chi = \frac{ad-bc}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

It might seem that we have found both the vector dot product and the vector cross product within our angle product operation. This, however, is not so. The space we have found within the complex numbers is two-dimensional euclidean space; it has no third dimension. In keeping with the mathematics of Clifford algebras and the exterior algebra, we will refer to this 'cross product' as the wedge product and denote it by

$$\begin{bmatrix}a\\ b\end{bmatrix}\wedge\begin{bmatrix}c\\ d\end{bmatrix}.$$

We note that:

$$\begin{bmatrix}a\\ b\end{bmatrix}\wedge\begin{bmatrix}c\\ d\end{bmatrix} = -\begin{bmatrix}c\\ d\end{bmatrix}\wedge\begin{bmatrix}a\\ b\end{bmatrix}.$$

The wedge product has the same magnitude as the cross product but is within the two-dimensional space. This is another way of saying that spatial curvature is intrinsic to the space (*theorema egregium*), or, equivalently, the equations of spatial curvature are 'mathematically isomorphic' to the equations of spatial stretchiness. The general angle product is thus

$$\left\langle \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \right\rangle = \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} & \begin{bmatrix} a \\ b \end{bmatrix} \wedge \begin{bmatrix} c \\ d \end{bmatrix} \\ \begin{bmatrix} c \\ d \end{bmatrix} \wedge \begin{bmatrix} a \\ b \end{bmatrix} & \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix}.$$

Since the trigonometric functions are projections on to the axes from the unit circle, the dot product is the projection from the normalized $[a \ \ b]$ vector on to the normalized $[c \ \ d]$ vector and vice-versa, and the wedge product is the perpendicular distance from the normalized $[a \ \ b]$ vector on to the normalized $[c \ \ d]$ vector and vice-versa. The expression $ad - bc$ is the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is thus the (oriented) area of the parallelogram formed by the vectors $\{[a \ \ b],\ [c \ \ d]\}$.

In the Clifford algebra, cl_2, we have the Clifford (vector) product:

$$\begin{aligned} (a_1\vec{e_1} + a_2\vec{e_2})\,(b_1\vec{e_1} + b_2\vec{e_2}) &= a_1b_1\vec{e_1}\vec{e_1} + a_1b_2\vec{e_1}\vec{e_2} + a_2b_1\vec{e_2}\vec{e_1} + a_2b_2\vec{e_2}\vec{e_2} \\ &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)\,\vec{e_1}\vec{e_2} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \end{aligned}$$

The angle product is thus the vector product of this Clifford algebra.

The important point is that both the dot product and the wedge product fall out of the algebra. They do not need to be imposed upon the space by a mathematician.

Thus, we have sorted out the vector algebra and constructed it properly as an algebraic field. It is now a bona fide algebra. The role of the vector product (the usual matrix multiplication) is to create the space. The dot product and the wedge product are simply calculative procedures.

The general form of the two-dimensional natural algebras is

$$\begin{bmatrix} a & b \\ jb & a \end{bmatrix} : j \neq 0 : |a| > |b| \text{ if } j > 0.$$

The general conjugate[4] is

$$\begin{bmatrix} a & -b \\ -jb & a \end{bmatrix}.$$

The general angle product is

$$\begin{bmatrix} a & b \\ jb & a \end{bmatrix} \odot \begin{bmatrix} c & -d \\ -jd & c \end{bmatrix} = \begin{bmatrix} ac - jbd & bc - ad \\ j(bc - ad) & ac - jbd \end{bmatrix}.$$

The general rotation matrix is

$$\begin{bmatrix} \cosh\left(\sqrt{j}\theta\right) & \dfrac{1}{\sqrt{j}} \sinh\left(\sqrt{j}\theta\right) \\ j\dfrac{1}{\sqrt{j}} \sinh\left(\sqrt{j}\theta\right) & \cosh\left(\sqrt{j}\theta\right) \end{bmatrix},$$

leading to the two-dimensional general dot product

$$\cosh\left(\sqrt{j}\theta\right) = \frac{ac - jbd}{\sqrt{a^2 - jb^2}\sqrt{c^2 - jd^2}}$$

and the general two-dimensional wedge product

$$\frac{1}{\sqrt{j}} \sinh\left(\sqrt{j}\theta\right) = \frac{bc - ad}{\sqrt{a^2 - jb^2}\sqrt{c^2 - jd^2}}.$$

When $j = 1$, we have the Minkowski space–time of special relativity. The Minkowski dot product is

$$\cosh(-\theta) = \cosh\theta = \frac{ac - bd}{\sqrt{a^2 - b^2}\sqrt{c^2 - d^2}}$$

and the wedge product is

$$\pm \sinh(\pm\theta) = \frac{bc - ad}{\sqrt{a^2 - b^2}\sqrt{c^2 - d^2}}.$$

The Minkowski dot product is established mathematics. I can find no reference to the Minkowski cross product or wedge product.

In the spaces of the three-dimensional natural algebras, similar procedures produce three angle identities corresponding to the three three-dimensional trigonometric functions of a particular space. The pattern continues upward into the higher-dimensional spaces.

[4]The conjugate is always the adjoint matrix within the natural algebras.

The box problem

Some thoughts in and outside the box

Ian Adamson

The classical integral cuboid has integral edges and integral face diagonals. The perfect integral cuboid also has an integral internal diagonal; it is the existence or not of this latter which we discuss here. We may seek sets $S = \{x, y, z\}$ of integers where (A) the sums of the squares of any two of S are square and (B) the sum of the squares of all three is square.

To satisfy (A), sets of equations in two or three parameters have been given by Saunderson (1740), Euler (1770 and 1772), Kraitchik (1943) and others. Colman has shown (1988) that there exists an infinite set of two-parameter formulae, but that no finite set generates all, which means that we cannot use them with constraint (B) to demonstrate non-existence of the perfect cuboid.

Clearly, $\{y, z\}$, $\{z, x\}$, $\{x, y\}$ are pairs of legs of Pythagorean triples, not necessarily primitive. Necessarily and sufficiently the legs are $t_i\{p_i^2 - q_i^2,\ 2p_iq_i\}$ where (C) $t_i, p_i, q_i \geqslant 1$, are integers and $p_i > q_i$ are coprime and not both odd. Existence of an S implies its existence when $\gcd(x, y, z) = 1$ so we shall assume this and, without loss of generality, that x is odd. In any triple no two legs are both odd; so y and z are even. We therefore have satisfying (A):

$$\begin{aligned} x &= t_2(p_2^2 - q_2^2) = t_3(p_3^2 - q_3^2), \\ y &= 2t_3p_3q_3 = t_1(p_1^2 - q_1^2) \ \ [\text{or } 2t_1p_1q_1], \\ z &= 2t_1p_1q_1 \ \ [\text{or } t_1(p_1^2 - q_1^2)] = 2t_2p_2q_2, \end{aligned}$$

where t_1 is even and t_2, t_3 are odd and we may ignore the alternatives.

To satisfy (B) also we have an additional constraint that for integral d,

$$d^2 = t_1^2(p_1^2 \pm q_1^2)^2 + t_2^2(p_2^2 \mp q_2^2)^2, \qquad (1)$$

whose average is

$$d^2 = t_1^2(p_1^4 + q_1^4) + t_2^2(p_2^4 + q_2^4),$$

which we may reformulate as

$$D^2 = \frac{d^2p_1^2q_1^2}{t_2^2} = p_2^2q_2^2(p_1^4 + q_1^4) + p_1^2q_1^2(p_2^4 + q_2^4),$$

which, when factored, gives

$$D^2 = (p_1^2p_2^2 + q_1^2q_2^2)(p_1^2q_2^2 + q_1^2p_2^2). \qquad (2)$$

Then, using Lagrange's relation, or (1),

$$D^2 \;=\; p_2^2q_2^2(p_1^2 \pm q_1^2)^2 + p_1^2q_1^2(p_2^2 \mp q_2^2)^2. \tag{3}$$

We could of course have first considered condition (B), as complete solutions for finding the sum of three squares equal to a square have been given by Desboves (1886), Mohapatra & Somayajulu (1967) and Bradley (1985), and then added the constraint (A).

All that needs to be done now is to find $T = \{p_i, q_i,\ i = 1, 2\}$ ignoring conditions (C) such that (2) or equivalently (3) is an integral square: assuming that the perfect integral cuboid exists. It would then be an easy exercise to determine S.

The only other conclusion is that it doesn't exist, which we can demonstrate (*if decidable)* by showing that equation (2) (or (3)), in four parameters, cannot be solved. Let us, for the sake of argument, assume decidability. Here we are helped by the constraints of conditions (C) since the set T is restricted. We could for example replace p_i by $2p_i$ and assume q_i to be odd, where again $\gcd(p_i, q_i) = 1$. But we would lose the assumption of the necessary positivity of some integral quantities with the 'risk' of varying the values of some residues as $s \equiv -s \pmod{2^n}$ necessarily only when $n = 1$.

To give a perfect cuboid (to date classical, not perfect, cuboids whose odd sides exceed 10 billion have been investigated) or show its non-existence, we may be aided by noting that $(p_1^2p_2^2 + q_1^2q_2^2)/t_0$ and $(p_1^2q_2^2 + q_1^2p_2^2)/t_0$ are both perfect squares, and consequently $(p_1^2p_2^2 + q_1^2q_2^2)/(p_1^2q_2^2 + q_1^2p_2^2)$ is a rational square, which one may think simpler than the necessary and sufficient equation given by Leech (1977), $p_2q_2(p_1^2 + q_1^2)/t$ and $p_1q_1(p_2^2 - q_2^2)/t$ are legs of a *primitive* Pythagorean triple. Here, $t_0 = \gcd(p_1^2p_2^2 + q_1^2q_2^2,\ p_1^2q_2^2 + q_1^2p_2^2)$ and $t = \gcd(p_2q_2(p_1^2 + q_1^2),\ p_1q_1(p_2^2 - q_2^2))$.

Does the perfect integral cuboid exist? In other words: Should we (i) try to find it or (ii) try to show that it doesn't exist? If (i) we may never succeed as it might be too big. Recall that Euler's conjecture took over 200 years for a relatively small counter-example: that $(20{,}615{,}673)^4$ is the sum of three fourth powers was discovered by Elkies (1988). If (ii), why didn't Euler accomplish this? He was no stranger to the use of elliptic functions, continued fractions, infinite descent and so on. But in an eighteenth century context. So we might consider such methods (known to Euler) with twenty-first century knowledge or some techniques as yet unknown.

A curious definition of a real number

Sebastian Hayes

'(2.1) **Definition**. A sequence (x_n) of rational numbers is *regular* if

$$|x_m - x_n| \;\leq\; \frac{1}{m} + \frac{1}{n}, \qquad m, n \in \mathbb{Z}^+.$$

A *real* number is a regular sequence of rational numbers.' (E. Bishop, *Foundations of Constructive Analysis*, p. 18)

If $m = n$, the result is trivial. But $1/m + 1/n = (m+n)/mn$ is greatest when mn is least and if $m > n$, this means $m \geq n + 1$. Thus, for given n, with $m, n \in \mathbb{Z}_+$ and $m > n$,

$$\frac{1}{m} + \frac{1}{n} \;\leq\; \frac{2n+1}{n(n+1)}.$$

The sequence $(x + n)$ is Cauchy. For given $k > 0$, we choose j from $\mathbb{Z}^+$ such that j is the first integer greater than $1/k$. Setting $n = N = 2j$, we have

$$x_m - x_n \;<\; \frac{1}{m} + \frac{1}{N} \;<\; \frac{2N+1}{N(N+1)} \;=\; \frac{4j+1}{4j^2+2j} \;<\; \frac{1}{j} \;<\; k$$

since $4j + 1 < 4j + 2 = (4j^2 + 2j)/j$. Thus we have found an N such that $x_m - x_n < k$ whenever $m, n > N$.

This is the definition of a real number used in the Intuitionist School of Mathematics founded by Brouwer. It still has a few followers in Holland but never caught on much in the English speaking countries. This school differs from all others by its insistence that a number, or other mathematical entity, be 'constructible'—though it is not always clear what 'constructible' means and why, for example, the above definition leads more readily to 'constructed' real numbers than the usual one.

Problem 212.1 – Fibonacci numbers

Tony Forbes

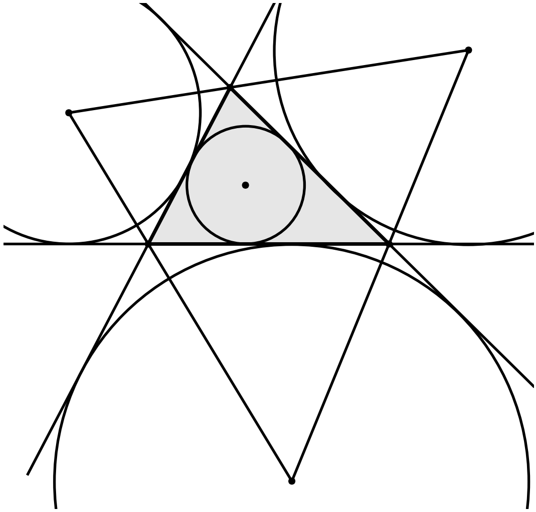
Let $F_1, F_2, F_3, \ldots$ denote the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, ... and let $f(x) = \cos(\arctan(\sin(\operatorname{arccot} x)))$. Show that

$$f(f(\ldots f(x)\ldots)) \;=\; \sqrt{\frac{F_{2n-1}x^2 + F_{2n}}{F_{2n}x^2 + F_{2n+1}}},$$

with n iterations of f.

Problem 212.2 – Area of a triangle

Draw a triangle with side lengths a, b and c. Extend the sides to infinity in both directions. (Not literally—just as far as necessary for the rest of the construction to work.) Draw the four circles each of which touches the three (extended) sides. One of these is inside the triangle (the in-circle); let this have radius r. The other three circles lie outside the triangles; join their centres to make a big triangle. Prove that the new triangle has area $abc/(2r)$.



Problem 212.3 – 100 seats

There are 100 seats on a plane, and 100 people have booked (different) seats. They form a queue. The first person to board the plane ignores the instructions on his ticket and chooses a seat at random. Thereafter each passenger goes to his/her allocated seat if it is unoccupied and otherwise chooses an unoccupied seat at random. What is the probability that the last person gets her booked seat?

Problem 212.4 – Integer density

ADF

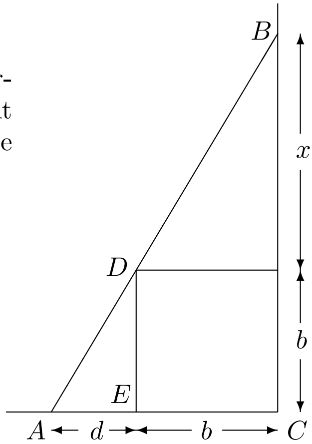
Let $\beta(x)$ denote the number of positive integers $n \le x$ such that

$$n = 3^{a_0} p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$$

for some $r \ge 1$ and primes $p_1, p_2, \ldots, p_r \equiv 3 \pmod 4$, where $a_0 \ge a_1 + a_2 + \cdots + a_r$.

Compute $\lim_{x\to\infty} \beta(x)/x$.

Solution 209.4 – Ladder



A ladder of length 1 stands against a vertical wall just touching a shed of height and width b. Find d, the distance of the ladder bottom from the shed.

Steve Moon

By similar triangles, ADE and ABC,

$$x = \frac{b^2}{d}.$$

Then Pythagoras on $\triangle ABC$ gives

$$(b+d)^2 + (b + b^2/d)^2 = 1.$$

Hence

$$(d+b)^4 - 2db(d+b)^2 - d^2 = 0, \tag{1}$$

a quadratic in $(d+b)^2$, which has solution

$$(d+b)^2 = d(b \pm \sqrt{b^2+1}). \tag{2}$$

But $\sqrt{b^2+1} > b$; so we take the positive square root in (2):

$$d + b = \sqrt{d}\sqrt{b + \sqrt{b^2+1}}.$$

Rearranging, we have

$$d - \sqrt{d}\sqrt{b+\sqrt{b^2+1}} + b = 0,$$

a quadratic in $\sqrt{d}$ with solution

$$\sqrt{d} = \frac{\sqrt{b+\sqrt{b^2+1}} \pm \sqrt{\sqrt{b^2+1} - 3b}}{2}. \tag{3}$$

On squaring and simplifying, we obtain

$$d = \frac{\sqrt{b^2+1} - b \pm \sqrt{1 - 2b^2 - 2b\sqrt{b^2+1}}}{2}. \tag{4}$$

For real d we require $\sqrt{b^2+1} \geq 3b$ to prevent the $\pm$ term in (3) from becoming negative. Hence $b \leq 1/(2\sqrt{2})$. When $b = 1/(2\sqrt{2})$ the ladder is at 45° and $d = b$.

Tony Forbes

Equation (1) is a quartic and therefore it must have exactly four solutions (even if some of them are equal). But, as we have seen, half of the solutions get spirited away when you take the positive square root in (2). Well, curiosity got the better of me; "What happens if you do consider all solutions?" I asked myself.

In fact, what happens is that you get two more solutions, just like the ones in (4) except that the first occurrence of $\sqrt{b^2+1}$ becomes $-\sqrt{b^2+1}$. For instance, when $b = 0.25$ the full solution set is (approximately)

$$d \;=\; -1.22996,\; -0.0508146,\; 0.09055,\; 0.690226.$$

The last two correspond to the ladder leaning against the shed at angles $70.0896°$ and $19.9104°$, two symmetrically opposite positions, and they are the ones obtained by plugging 0.25 into (4).

The first solution has the ladder leaning against the wall on the *opposite* side to the shed. The point D, where the roof and the flank wall of the shed meet, is collinear with the ladder but some way off it.

In the second solution the ladder is on the same side as the shed but entirely underneath it. One end of the ladder is between E and C, and some way below C the other end touches the wall's foundations, which we assume go quite deep. Again, D is collinear but not incident with the ladder.

Ralph Hancock

To answer Tony's question under Problem 209.4 – Ladder:

Welded rail is actually quite flexible—indeed, if it weren't, it would soon break in service. Also, consider that points are shifted by flexing a section of rail. Hundred-metre lengths are simply carried on a series of long wagons, on which they are restrained by a strong transverse grid at each end, and they bend (with a certain amount of jostling) when the train goes round a curve. I have often been in a passenger train overtaking a rail-carrying train, and you can see the lengths of rail flexing with the swaying of the wagons.

The rails are loaded by dragging them endwise off the ground and on to the train. You can see a rail being unloaded at www.lalrr.com/photos.html; note how much it curves.

Hugh McIntyre

Norman Graham's ladder problem fair takes me back. In the drawing office, before I went to sea as an engineer, we often knocked our heads against that and other problems in lieu of useful work. I don't recall ever solving it. At an OU Maths Summer School in Stirling a bright spark posted the problem on the notice board, asking a tutor in rather provocative terms if he could solve it. 'I can—can you?' was the posted up response.

In the 80s I was engaged on the design of automatic access hatches to an equipment decontamination area at Torness and Heysham Nuclear Power Stations. The firm declined to pay a representative to stay on site looking intelligent, hence information didn't always come through when needed. So one day, on a site visit, I noted that the large 'entrance door' we and others had been using for equipment access was no longer there, having been replaced by a small man-sized door. The 'entrance door' had been an entire wall left unbuilt for convenience—a fine time to find that out. The small door gave entry to a narrowish corridor with a right angle corner part way along, and a daunting notice saying radioactivity might be present due to testing. In the vicinity of the door were various long bits and pieces belonging to various sub-contractors caught out by the disappearance of the usual way in. What they did I never learned—what I did was introduce a bolted flange at the appropriate point on a couple of the long bits, to enable them to be taken round the corner in two sections. Not possible with railway lines. Happy days.

There's one problem from my drawing office days that might interest some. It's more a case of setting the conditions than solving a problem. Consider a ship floating in an enclosed dock (e.g. a filled dry dock with the gate closed). A weight is removed from the ship and dropped into the water. Under what conditions will the dock water level rise, fall, or stay the same?

Dingbats (Eddie Kent)

(i) SENSATION
WORLD

(ii) ME HISTO HELES

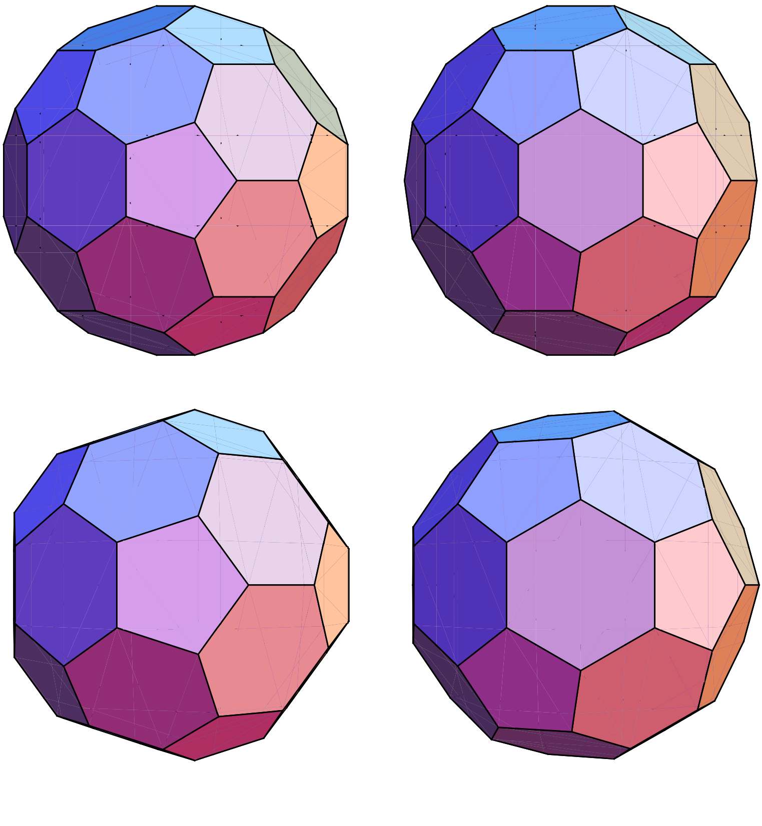
(iii) dlrowehtfopotnognileeF
dekciwehtrofecaepoN

Problem 212.5 – Truncated icosahedron

ADF

Behold, four views of a truncated icosahedron, the football-like Archimedean solid involving 12 pentagons and 20 hexagons. The top row shows the thing from above a pentagon and above a hexagon as it would appear infinitely far away (through a very powerful telescope). Bottom left shows the solid as viewed from the special distance D_1 where five hexagons clearly visible in the upper picture have degenerated into lines. Similarly, in bottom right the view is from distance D_2 where three pentagons have become lines.

Assuming side length 1, what are D_1 and D_2? I am curious. After a little experimentation I see that $D_1 \approx D_2$.



Solution 204.4 – Ones

Show that

$$\frac{11}{10}\cdot\frac{1111}{1110}\cdot\frac{111111}{111110}\cdot\frac{11111111}{11111110}\cdot\ldots \;=\; 1.101001000100001000001\ldots.$$

Show that this is true in any number base, not just 10. For example, when the base is 2 we have (using decimal notation)

$$\frac{3}{2}\cdot\frac{15}{14}\cdot\frac{63}{62}\cdot\frac{255}{254}\cdot\ldots \;=\; \sum_{n=0}^{\infty} 2^{-n(n+1)/2}.$$

Tony Forbes

Although it appears in M500 **204** sandwiched between two straightforward high-school geometry problems, this is a difficult one—especially if you haven't seen it before.

The problem is a special case of

$$\prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}} \;=\; \sum_{k=0}^{\infty} q^{k(k+1)/2}, \tag{1}$$

obtained by setting the number base to $1/q$. In fact (1) holds for any complex q, $|q|<1$. It occurs as Entry 22(ii) in Chapter 16 of Ramanujan's second notebook and is in turn a special case of Jacobi's triple product identity,

$$\prod_{n=1}^{\infty}\left(1-x^{2n}\right)\left(1+zx^{2n-1}\right)\left(1+\frac{x^{2n-1}}{z}\right) \;=\; \sum_{k=-\infty}^{\infty} x^{k^2}z^k, \tag{2}$$

valid for $|x|<1$.

Actually, getting from (2) to (1) is by no means a trivial exercise. To make a start, observe that for the substitutions $x=q^{\alpha}$ and $z=q^{\beta}$ to match the exponent of q on the right of (1) we need to put $x=z=\sqrt{q}$. Then (2) becomes

$$\prod_{n=1}^{\infty}\left(1-q^{n}\right)\left(1+q^{n}\right)\left(1+q^{n-1}\right) \;=\; \sum_{k=-\infty}^{\infty} q^{k(k+1)/2},$$

or after some gathering together of factors on the left and terms on the right,

$$\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+q^{n-1}\right) \;=\; 2\sum_{k=0}^{\infty} q^{k(k+1)/2}, \tag{3}$$

which is beginning to look vaguely like (1). In fact (3) will look exactly like (1) if we can prove that

$$\prod_{n=1}^{\infty}(1+q^{n-1})(1-q^{2n-1}) \;=\; 2. \tag{4}$$

This is one of those product formulae where the proof consists of writing down the factors and looking at them until it becomes obvious how it works. If you multiply out the left-hand side of (4) to yield

$$2(1-q)(1+q)(1-q^3)(1+q^2)(1-q^5)(1+q^3)(1-q^7)(1+q^4)(1-q^9)\ldots,$$

I think you should be able to see what is happening. The factors $1-q$, $1+q$, $1+q^2$, $1+q^4$, $\ldots$, $1+q^{2^m}$ combine to make $1-q^{2^{m+1}}$, which tends to 1 as m tends to infinity. So $(1-q)(1+q)(1+q^2)(1+q^4)\cdots = 1$. Similarly, $(1-q^3)(1+q^3)(1+q^6)(1+q^{12})\cdots = 1$, $(1-q^5)(1+q^5)(1+q^{10})(1+q^{20})\cdots = 1$, and so on.

Thus (4) is proved. So we can plug it into (3) to get (1). Hence (2) implies (1). As for the hard part, Jacobi's formula, (2), there is a proof in G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*. Alternatively, you could sign up for the Open University M.Sc. course in analytic number theory, the one based on the text by Tom Apostol, *Introduction to Analytic Number Theory*. Here, (2) is Theorem 14.6 with a proof that goes something like this.

Put $w = \sqrt{z}$,

$$F_x(w) \;=\; \prod_{n=1}^{\infty}(1+x^{2n-1}w^2)(1+x^{2n-1}w^{-2}) \tag{5}$$

and

$$G_x(w) \;=\; F_x(w)\prod_{n=1}^{\infty}(1-x^{2n}). \tag{6}$$

Then one can quite easily prove the identity $xw^2F_x(xw) = F_x(w)$ from which it follows that

$$xw^2G_x(xw) \;=\; G_x(w). \tag{7}$$

Now $G_x(w)$ is the left-hand side of (2) and moreover it has a Laurent expansion of the form

$$G_x(w) \;=\; \sum_{k=-\infty}^{\infty} a_k(x)w^{2k}, \tag{8}$$

where the coefficients $a_k(x)$ are functions of x which satisfy $a_{-k}(x) = a_k(x)$ (since $G_x(w) = G_x(w^{-1})$). But from (7) and (8) we see that the $a_k(x)$ satisfy the recursion

$$a_k(x) \;=\; x^{2k-1}a_{k-1}(x).$$

Therefore $a_k(x) = a_0(x)x^{k^2}$. Hence

$$G_x(w) \;=\; a_0(x)\sum_{k=-\infty}^{\infty} x^{k^2}w^{2k}, \tag{9}$$

and therefore

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) \;=\; G_x(\sqrt{z}) \;=\; a_0(x)\sum_{k=-\infty}^{\infty} x^{k^2}z^k,$$

which is (2) provided we can prove that $a_0(x) = 1$ for $|x| < 1$.

Put $w = \sqrt{i}$ in (9) to get

$$\frac{G_x(\sqrt{i})}{a_0(x)} \;=\; \sum_{k=-\infty}^{\infty} x^{k^2}i^k \;=\; \sum_{m=-\infty}^{\infty}(-1)^m x^{4m^2} \;=\; \frac{G_{x^4}(i)}{a_0(x^4)}. \tag{10}$$

Then from (5) and (6) we have

$$\begin{aligned}
G_x(\sqrt{i}) \;&=\; \prod_{n=1}^{\infty}(1-x^{2n})(1+x^{4n-2}) \\
&=\; \prod_{n=1}^{\infty}(1-x^{4n})(1-x^{4n-2})(1+x^{4n-2}) \\
&=\; \prod_{n=1}^{\infty}(1-x^{4n})(1-x^{8n-4}) \\
&=\; \prod_{n=1}^{\infty}(1-x^{8n})(1-x^{8n-4})(1-x^{8n-4}) \;=\; G_{x^4}(i).
\end{aligned}$$

Hence (10) implies $a_0(x) = a_0(x^4)$, and therefore $a_0(x) = a_0(x^{4^m})$ for $m = 1, 2, \ldots$. Furthermore, $|x| < 1$ implies $x^{4^m} \to 0$ as $m \to \infty$. Hence $a_0(x) = a_0(0)$.

To finish, therefore, all we have to do is show that $a_0(0) = 1$. But this follows by putting $x = 0$ in (9) since, by (5) and (6), $G_0(w) = 1$.

Problem 211.6 – Sudoku verification

Tony Forbes

Here's a sudoku puzzle. Fill in the blanks such that each of the 27 regions (rows, columns, 3×3 boxes) contains the symbols $\{1, 2, \ldots, 9\}$.

		1					5	7
			7	8	6	1		
	9	3			1		2	
4			3		7			1
	7		4			8	3	
		6	1	9	3			
8	1					2		

Now for the problem. Wise sudoku fans are well aware of the importance of checking a region as soon as it is completed just to make sure that it really does contain precisely one each of the symbols 1 to 9.

But suppose you are given a completed puzzle. *What is the minimum number of regions you must check to verify that a completed sudoku grid is a valid solution?*

[Hint: the answer is ≤ 26.]

Problem 212.7 – Dice

Start by throwing 6 (or n, if you prefer) dice and remove any that have landed six-side up. Repeat until no more dice are left. What is the expected number of throws?

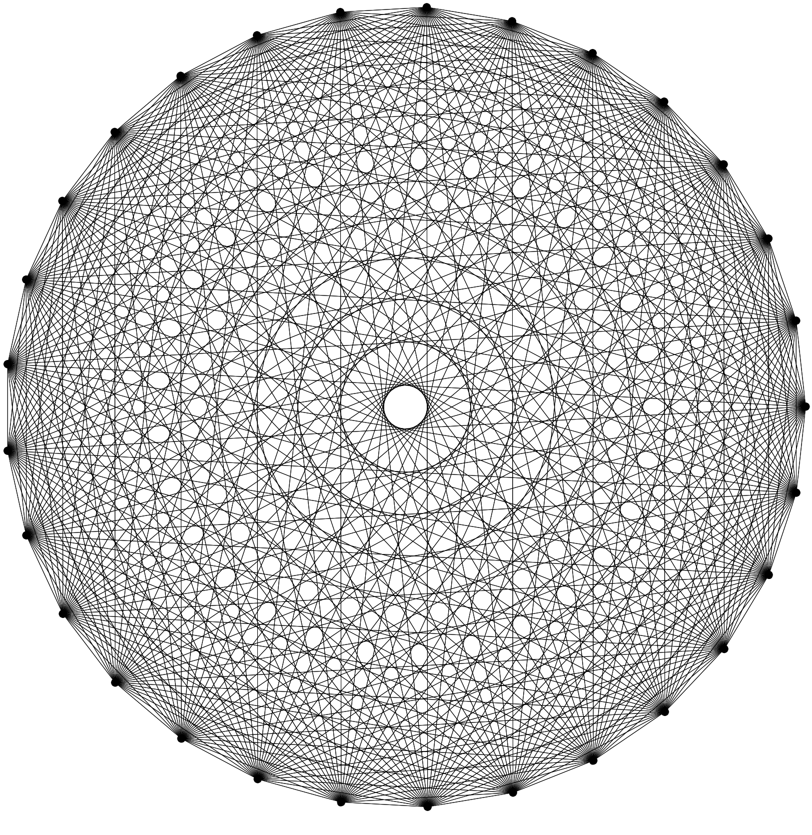
Algebra is X minus Y equals Z plus Y—and things like that. And all that time you are saying they are equal, you feel in your heart, 'Why should they be?'

[J. M. Barrie]

Problem 208.3 – Concentric circles, revisited

Recall that I (ADF) asked for an explanation of the circles in the cyclic representation of the complete graph K_n. One or two people stated the fact, which I now realize is more or less obvious, that the sets of chords of constant length define a sequence of concentric 'best fit' circles which tend to become less well-defined as the radius increases.

But look at K_{29}, below. It is plain that the fourth circle from the centre is significantly more prominent than the others, and the same is true for the third circle of K_{23} [M500 **208**]. Also the 5th circle of K_{39} and the 6th of K_{49}, to give two more examples. So now I ask a harder question: Where do *these* circles come from?



M500 Winter Weekend 2007

The twenty-sixth M500 Society Winter Weekend will be held on **Friday 5th to Sunday 7th January 2007** at **NOTTINGHAM UNIVERSITY**.

This is an annual residential weekend to dispel the withdrawal symptoms due to courses finishing in October and not starting again until February. It's an excellent opportunity to get together with acquaintances, new and old, and do some interesting mathematics in a leisurely and friendly atmosphere. We are trying something new this year. There will be no overall theme—instead we will be providing a mix of mathematical attractions from amongst ourselves.

Cost: £185.00. This includes standard accommodation and all meals from dinner on Friday to lunch on Sunday. M500 members get a £5 discount. For full details and a booking form, send a stamped, addressed envelope to

Diana Maxwell.

E-mail enquiries to diana@m500.org.uk. Full details and a booking form are available at www.m500.org.uk.

	2	7	9					
								6
9				7				
				4				2
3								
					4	6	8	

A *quasi-magic* sudoku-like puzzle. Fill in the blanks according to the usual rules ($\{1, 2, \ldots, 9\}$ in every row, column and 3×3 box) with the further constraint that within each 3×3 box the rows, columns and diagonals sum to any of the numbers 13, 14, 15, 16 or 17 (i.e. 15 ± 2). So you can think of the little boxes as quasi-magic squares of order 3.

Tony Forbes

Hint: 5s must go in either box centres or box corners.

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