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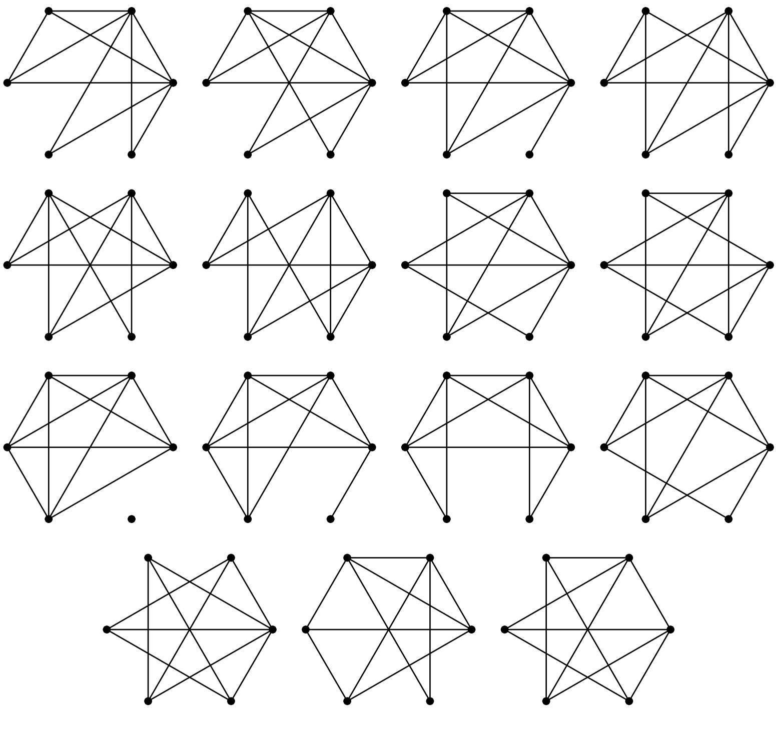

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Solution 277.6 – Rational sum

Suppose a, b and c are rational numbers and let

$$\begin{cases} d &=& 1+20a-10b+4c, \\ e &=& 3+38a-17b+5c, \\ f &=& 3+20a-8b+2c. \end{cases} \tag{1}$$

Show that

$$Z \;=\; \sum_{n=1}^{\infty} \frac{n^6+fn^5+en^4+dn^3+cn^2+bn+a}{n^4(n+1)^4}$$

is rational.

Bruce Roth

The denominator is a product; so some sort of partial fractions and method of differences look possible. There are four cases to consider.

Type 1 Because

$$\frac{1}{n}-\frac{1}{n+1} \;=\; \frac{1}{n(n+1)} \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) \;=\; 1$$

we consider fractions of the form

$$\frac{n^3(n+1)^3}{n^4(n+1)^4} \;=\; \frac{n^6+3n^5+3n^4+n^3}{n^4(n+1)^4}.$$

Type 2 Because

$$\frac{1}{n^2}-\frac{1}{(n+1)^2} \;=\; \frac{2n+1}{n^2(n+1)^2} \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right) \;=\; 1$$

we consider fractions of the form

$$\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4} \;=\; \frac{2n^5+5n^4+4n^3+n^2}{n^4(n+1)^4}.$$

Type 3 Because

$$\frac{1}{n^3}-\frac{1}{(n+1)^3} \;=\; \frac{3n^2+3n+1}{n^3(n+1)^3} \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right) \;=\; 1$$

we consider fractions of the form

$$\frac{n(n+1)(3n^2+3n+1)}{n^4(n+1)^4} = \frac{3n^4+6n^3+4n^2+n}{n^4(n+1)^4}.$$

Type 4 Because

$$\frac{1}{n^4} - \frac{1}{(n+1)^4} = \frac{4n^3+6n^2+4n+1}{n^4(n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\frac{1}{n^4} - \frac{1}{(n+1)^4}\right) = 1$$

we consider fractions of the form

$$\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}.$$

Now if we concentrate on the coefficients we need to aim for, we can construct the following table.

	n^6	n^5	n^4	n^3	n^2	n^1	n^0
type 1	1	3	3	1			
type 2		2	5	4	1		
type 3			3	6	4	1	
type 4				4	6	4	1

Let us consider just the numerator of our sum but use a, b, c, not d, e, f:

$$\begin{bmatrix}1\\ \\ \\ \\ \end{bmatrix} n^6 + \begin{bmatrix}3\\ +20a\\ -8b\\ +2c\end{bmatrix} n^5 + \begin{bmatrix}3\\ +38a\\ -17b\\ +5c\end{bmatrix} n^4 + \begin{bmatrix}1\\ +20a\\ -10b\\ +4c\end{bmatrix} n^3 + \begin{bmatrix}\\ \\ \\ c\end{bmatrix} n^2 + \begin{bmatrix}\\ \\ b\\ \\ \end{bmatrix} n + \begin{bmatrix}\\ a\\ \\ \\ \end{bmatrix}.$$

Clearly the constants fit what we are aiming for, as do the c coefficients:

$$\begin{aligned} & n^3\left(n^3+3n^2+3n+1\right) + cn^2\left(2n^3+5n^2+4n+1\right) \\ & \qquad + \begin{bmatrix}20a\\ -8b\end{bmatrix} n^5 + \begin{bmatrix}38a\\ -17b\end{bmatrix} n^4 + \begin{bmatrix}20a\\ -10b\end{bmatrix} n^3 + \begin{bmatrix}\\ b\end{bmatrix} n + \begin{bmatrix}a\\ \\ \end{bmatrix} \\ = {} & n^3(n+1)^3 + cn^2(n+1)^2(2n+1) \\ & \qquad + \begin{bmatrix}20a\\ -8b\end{bmatrix} n^5 + \begin{bmatrix}38a\\ -17b\end{bmatrix} n^4 + \begin{bmatrix}20a\\ -10b\end{bmatrix} n^3 + \begin{bmatrix}\\ b\end{bmatrix} n + \begin{bmatrix}a\\ \\ \end{bmatrix}. \end{aligned}$$

So we need to collect sums of multiples of the a and b terms in the form required. To find them it helps to start with the lowest powers of n and work to the left.

the a terms	n^5	n^4	n^3	n^2	n^1	n^0	
aiming for	20	38	20			1	
type 4			4	6	4	1	n^0 column correct
$-4\times$(type 3)		-12	-24	-16	-4		n^1 column correct
$+10\times$(type 2)	20	50	40	10			n^2 column correct

Thus we have our a terms as

$$a\left(\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}\right.$$
$$\left.-\,4\cdot\frac{n(n+1)(3n^2+3n+1)}{n^4(n+1)^4}+10\cdot\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4}\right).$$

the b terms	n^5	n^4	n^3	n^2	n^1	n^0	
aiming for	-8	-17	-10		1		
type 3		3	6	4	1		n^1 column correct
$-4\times$(type 2)	-8	-20	-16	-4			n^2 column correct

And the b terms are given by

$$b\left(\frac{n(n+1)(3n^2+3n+1)}{n^4(n+1)^4}-4\cdot\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4}\right).$$

So now we have that the sum is

$$\sum_{n=1}^{\infty}\left[\frac{n^3(n+1)^3}{n^4(n+1)^4}+a\left(\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}\right.\right.$$
$$\left.-\,4\cdot\frac{n(n+1)(3n^2+3n+1)}{n^4(n+1)^4}+10\cdot\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4}\right)$$
$$+\,b\left(\frac{n(n+1)(3n^2+3n+1)}{n^4(n+1)^4}-4\cdot\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4}\right)$$
$$\left.+\,c\cdot\frac{n^2(n+1)^2(2n+1)}{n^4(n+1)^4}\right],$$

or, more succinctly, since, as explained on pages 1 and 2, each of the fractions sums to 1:

$$1+a(1-4+10)+b(1-4)+c \;=\; 1+7a-3b+c.$$

As a, b and c are rational, we have shown that the infinite sum is rational.

Tony Forbes

I believe there is much more to this problem. First, let us get Mathematica to sum the series:

$$\begin{aligned} Z \;=\; & -35a + 20b - 10c + 4d - e \\ & + (1 + 20a - 10b + 4c - d)\,\zeta(2) \\ & + (-2 - 2b + 2c - d + f)\,\zeta(3) \\ & + (1 + 2a - b + c - d + e - f)\,\zeta(4). \end{aligned} \tag{2}$$

Considering the matrix of coefficients of a, b and c in (1), one might be tempted to compute its determinant:

$$\det \begin{bmatrix} 20 & -10 & 4 \\ 38 & -17 & 5 \\ 20 & -8 & 2 \end{bmatrix} \;=\; 24 \;=\; 4!.$$

Moreover, if you construct a matrix from the coefficients of a, b and c in the coefficients of $\zeta(2)$, $\zeta(3)$ and $\zeta(4)$ in (2), you get the same determinant:

$$\det \begin{bmatrix} 20 & -10 & 4 \\ 0 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix} \;=\; 4!.$$

Now if, like me, you get excited whenever you see a matrix whose determinant is factorial something, you might begin to think there is something other than coincidence at work. So I thought it would be interesting to see what happens when we generalize the problem.

Let $d \ge 2$ be an integer and let

$$Z(d) \;=\; \sum_{n=1}^{\infty} \frac{A(n) + B(n) + n^{2d-2}}{n^d (n+1)^d},$$

where

$$A(n) \;=\; a_0 + a_1 n + a_2 n^2 + \cdots + a_{d-2} n^{d-2}$$

and

$$B(n) \;=\; b_0 n^{d-1} + b_1 n^d + \cdots + b_{d-2} n^{2d-3}$$

are polynomials in n with rational coefficients a_0, a_1, $\ldots$, a_{d-2}, b_0, b_1, $\ldots$, b_{d-2}. So the numerator of the thing being summed is a polynomial in n of degree $2d - 2$, two less than the degree of the denominator. Absolute convergence is therefore assured, and $Z(d)$ can be evaluated to obtain

$$Z(d) \;=\; R + \sum_{r=2}^{d} S_r\, \zeta(r),$$

where R and the S_r are rational linear combinations of 1 and the coefficients a_i and b_i.

We want to choose the coefficients b_i to eliminate the terms involving the zeta function. So we set up $d-1$ equations,

$$S_r \;=\; 0, \quad r = 2, 3, \ldots, d,$$

which we can write in matrix form as

$$\mathbf{S} \cdot (1, a_0, a_1, \ldots, a_{d-2}, b_0, b_1, \ldots, b_{d-2}) \;=\; \mathbf{0}. \tag{3}$$

Thus, writing $\mathbf{S}_r$ for the $(r-1)$th row of $\mathbf{S}$,

$$\mathbf{S}_r \cdot (1, a_0, a_1, \ldots, a_{d-2}, b_0, b_1, \ldots, b_{d-2}), \qquad r = 2, 3, \ldots, d,$$

is the coefficient of $\zeta(r)$ in $Z(d)$. Let $\mathbf{S}_A$ denote the matrix formed by columns 2, 3, ..., d of $\mathbf{S}$, i.e. the part of $\mathbf{S}$ that corresponds to the variables a_0, a_1, ..., a_{d-2}.

We solve (3) for b_0, b_1, ..., b_{d-2} in terms of a_0, a_1, ..., a_{d-2} to get a set of $d-1$ conditions that makes $Z(d)$ rational:

$$\left\{\begin{array}{rcl} b_0 &=& R_0 + M_{0,0}\, a_0 + M_{0,1}\, a_1 + \cdots + M_{0,d-2}\, a_{d-2}, \\ b_1 &=& R_1 + M_{1,0}\, a_0 + M_{1,1}\, a_1 + \cdots + M_{1,d-2}\, a_{d-2}, \\ && \ldots, \\ b_{d-2} &=& R_{d-2} + M_{d-2,0} a_0 + M_{d-2,1}\, a_1 + \cdots + M_{d-2,d-2}\, a_{d-2}. \end{array}\right. \tag{4}$$

Finally, let $\mathbf{M}$ be the matrix of the coefficients $M_{i,j}$. It is an astonishing fact (for which I have no explanation) that in every case where I have actually done the calculations I find that $\det \mathbf{M} = d!$ and $\det \mathbf{S}_A = (-1)^d\, d!$.

Turning our attention to the evaluation of $Z(d)$, we define the two-variable function

$$Z(d, k) \;=\; \sum_{n=1}^{\infty} \frac{n^k}{n^d (n+1)^d}.$$

Then

$$Z(d) \;=\; \sum_{k=0}^{d-2} \Big(a_k Z(d, k) + b_k Z(d, d-1+k) \Big) + Z(d, 2d-2)$$

and we can evaluate $Z(d,k)$ recursively in terms of the zeta function by

$$\begin{aligned}
Z(1,0) &= 1,\\
Z(0,k) &= \zeta(-k),\\
Z(d,d) &= \zeta(d)-1,\\
Z(d,k) &= \begin{cases} Z(d-1,k-1)-Z(d,k+1), & k<d,\\ Z(d-1,k-2)-Z(d,k-1), & k>d,\end{cases}
\end{aligned}$$

where to avoid trouble we should insist that $k \le 2d-2$. Note that k can be negative. Hence we have, after some rather messy computations the details of which are omitted,

$$\begin{aligned}
Z(d,0) &= \begin{cases} 2\displaystyle\sum_{i=0}^{(d-2)/2}\binom{d-1+2i}{2i}\zeta(d-2i)-\binom{2d-1}{d}, & d \text{ even},\\[2ex] \displaystyle\binom{2d-1}{d}-2\sum_{i=0}^{(d-3)/2}\binom{d+2i}{2i+1}\zeta(d-1-2i), & d \text{ odd},\end{cases}\\
Z(d,k) &= (-1)^{d-k}\Bigg(\sum_{i=0}^{d-2}\Bigg(\binom{d-1-k+i}{i}\\
&\qquad + (-1)^{d-i}\binom{d-1-k+i}{i-k}\Bigg)\zeta(d-i)\\
&\qquad -\binom{2d-k-1}{d-k}\Bigg), \qquad 0<k<d,\\
Z(d,d) &= \zeta(d)-1,\\
Z(d,k) &= \sum_{i=0}^{k-d}(-1)^i\binom{k-d}{i}\zeta(2d-k+i), \qquad k>d.
\end{aligned}$$

Armed with these formulæ we should be in a good position to attack the general case. We can set up the linear equations (3), which we then solve to obtain the matrix $\mathbf{M}$ as in (4). Unfortunately the task seems to be beyond my capabilities. In fact I am beginning to wonder if a much better approach would be to generalize the method explained in Bruce Roth's contribution (page 1). I gladly leave it for someone else, and of course I and a few people

I know would be interested in the result. For now we finish by working a substantial example.

Put $d = 7$. Then

$$Z(7) \;=\; \sum_{n=1}^{\infty} \frac{a_0 + a_1 n + \cdots + a_5 n^5 + b_0 n^6 + b_1 n^7 + \cdots + b_5 n^{11} + n^{12}}{n^7 (n+1)^7}.$$

After a straightforward but tedious application of the recursion formula (or otherwise) we see that (3) becomes

$$\begin{bmatrix} 1 & -924 & 462 & -210 & 84 & -28 & 7 & -1 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 42 & -42 & 28 & -14 & 5 & -1 & 0 & 0 & 0 & 0 & 1 \\ 10 & -168 & 84 & -42 & 21 & -10 & 4 & -1 & 0 & 0 & 0 & 1 & -4 \\ -10 & 0 & 14 & -14 & 10 & -6 & 3 & -1 & 0 & 0 & 1 & -3 & 6 \\ 5 & -14 & 7 & -5 & 4 & -3 & 2 & -1 & 0 & 1 & -2 & 3 & -4 \\ -1 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_0 \\ \cdots \\ a_5 \\ b_0 \\ \cdots \\ b_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (5)$$

This is the matrix $\mathbf{S}$ of (3), and

$$\mathbf{S}_r \cdot (1, a_0, a_1, \ldots, a_5, b_0, b_2, \ldots, b_5)$$

is the coefficient of $\zeta(r)$ in $Z(7)$, $r = 2, 3, \ldots, 7$. From (5) this solution can be obtained almost immediately:

$$\begin{bmatrix} b_0 \\ b_1 \\ \cdots \\ b_5 \end{bmatrix} = \begin{bmatrix} 1 & -924 & 462 & -210 & 84 & -28 & 7 \\ 6 & -4438 & 2162 & -946 & 356 & -106 & 20 \\ 15 & -8722 & 4165 & -1773 & 641 & -179 & 30 \\ 20 & -8736 & 4102 & -1708 & 599 & -160 & 25 \\ 15 & -4452 & 2058 & -840 & 287 & -74 & 11 \\ 6 & -924 & 420 & -168 & 56 & -14 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ a_0 \\ a_1 \\ \cdots \\ a_5 \end{bmatrix}. \quad (6)$$

And if we extract the parts of each matrix in (5) and (6) that correspond to the variables a_0, a_1, …, a_5, we see that

$$-\begin{vmatrix} -924 & 462 & -210 & 84 & -28 & 7 \\ 0 & 42 & -42 & 28 & -14 & 5 \\ -168 & 84 & -42 & 21 & -10 & 4 \\ 0 & 14 & -14 & 10 & -6 & 3 \\ -14 & 7 & -5 & 4 & -3 & 2 \\ 0 & 1 & -1 & 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -924 & 462 & -210 & 84 & -28 & 7 \\ -4438 & 2162 & -946 & 356 & -106 & 20 \\ -8722 & 4165 & -1773 & 641 & -179 & 30 \\ -8736 & 4102 & -1708 & 599 & -160 & 25 \\ -4452 & 2058 & -840 & 287 & -74 & 11 \\ -924 & 420 & -168 & 56 & -14 & 2 \end{vmatrix} = 7!.$$

Partitions – an elementary approach to asymptotics

Tommy Moorhouse

Introduction In a previous article [M500 **248**] I stated that certain 'differential equations' were the starting point for developing, for example, asymptotic estimates for partition functions. In this article we will examine a simple example and indicate how it can be developed. This article does not attempt to produce a rigorous proof of the result, but aims simply to demonstrate that meaningful applications are possible, and to encourage the interested reader to investigate further.

Some background As in earlier articles we start with a 'logarithm' on the integers. Consider an integer-valued function on the prime numbers, which we denote by p_i, where $p_1 = 2$, $p_2 = 3$ and so on. Call the function $\xi(p_i)$. Given the decomposition of an integer into prime factors $n = \prod_i p_i^{k_i}$ we define the logarithm

$$L_\xi(n) = \sum_i k_i \xi(p_i).$$

This is, as can easily be verified, an integer-valued logarithm on the integers.

The differential equation We start our asymptotic analysis of partition functions from the differential equation

$$\nabla p = c \circ p.$$

Here $\nabla p(n) = np(n)$ and $c(n) = \sum_{\xi(p)|n} \xi(p)$, the sum of those $\xi(p)$ dividing n. The reader can check that this is indeed the natural derivative for the product

$$a \circ b(n) = \sum_{k=0}^{n} a(k)b(n-k).$$

The solution of the differential equation can be expressed as

$$p = K \exp \circ\, \tilde{c},$$

where K is a constant, p is the partition function associated with ξ, and $\tilde{c}$ is the function

$$\tilde{c}(n) = \frac{c(n)}{n}.$$

Unrestricted partitions We will consider the unrestricted partitions of n, those partitions made up of other integers, without restricting them to be odd, prime, etc. In this case $\xi(p_k) = k$ and $\tilde{c}(n)$ is the sum of the reciprocals of the divisors of n, commonly denoted $\sigma_{-1}(n)$.

The expression $\exp \circ\, \tilde{c}$ is short for

$$\exp \circ\, \tilde{c} \;=\; 1 + \tilde{c} + \frac{1}{2!}\tilde{c} \circ \tilde{c} + \cdots + \frac{1}{k!}\tilde{c}^{(k)} + \cdots$$

(here $\tilde{c}^{(k)}$ is the k-fold $\circ$ product $\tilde{c} \circ \tilde{c} \circ \cdots \circ \tilde{c}$) and it soon becomes apparent that working out the terms on the right hand side is quite difficult in general. Here we will use an estimate that will allow us to come up with an asymptotic formula by elementary means.

A simplification In [Apostol] it is shown (in fact it is not difficult to deduce this) that the average value of σ_{-1} over the integers up to n is asymptotic to $\zeta(2) = \pi^2/6$. We think of this as the function $\zeta(2)u(n)$ where $u(n) = 1$ for all n. We can therefore (although this really needs more justification) estimate $\exp \circ\, \tilde{c}$ by replacing $\tilde{c}$ in our exponential by $\zeta(2)u$, so that

$$\tilde{c} \circ \tilde{c} \;=\; \zeta(2)^2 u \circ u(n)$$

and so on.

While the $\circ$-powers of σ_{-1} are hard to calculate, the powers of u, which we will denote u^k, are surprisingly easy. Later we will modify the definition of u to give us a better estimate. The first step is to note that

$$\frac{1}{1-x} \;=\; 1 + x + x^2 + \cdots \;=\; 1 + u(1)x + u(2)x^2 + \cdots + u(k)x^k + \cdots$$

so that, raising the series to the kth power,

$$\sum_k u^{(k)}(n)x^n \;=\; \frac{1}{(1-x)^k}.$$

We can extract the nth coefficient as

$$u^{(k)}(n) \;=\; \frac{1}{n!}D^n \frac{1}{(1-x)^k} \;=\; \binom{n+k-1}{n}.$$

Here D stands for d/dx. This gives an expression for $p(n)$ as a sum over binomial coefficients:

$$p(n) \;\sim\; \sum_{k=0}^{n} \frac{\zeta(2)^k}{k!}\binom{n+k-1}{n}.$$

We can check that this does indeed give an estimate of the right magnitude for large n, so we seem to be on the right track. With a little effort we can refine this estimate.

Improving the estimate To get a better estimate we define $u(n)$ to be 1 for $n > 0$ and take $u(0) = 0$. The generating function is then

$$\frac{1}{1-x} - 1 \;=\; x + x^2 + \cdots \;=\; u(1)x + u(2)x^2 + \cdots + u(k)x^k + \cdots .$$

Then

$$u^{(k)}(n) \;=\; \frac{1}{n!} D^n \left(\frac{1}{1-x} - 1 \right)^k \;=\; \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} \binom{n+r-1}{n}.$$

This last sum over r works out quite simply (you could try to prove this):

$$u^{(k)}(n) \;=\; \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} \binom{n+r-1}{n} \;=\; \binom{n-1}{k-1}.$$

Putting this together we have

$$p(n) \;\sim\; \sum_{k=1}^{n} \frac{\zeta(2)^k}{k!} \binom{n-1}{k-1}.$$

This form of the sum allows us to relate our results to the well-known asymptotic result of Hardy, Ramanujan and Rademacher, although our estimate will be coarser. There are a couple of alternative approaches. First we rewrite the sum as

$$\sum_{k=1}^{n} \zeta(2)^k \frac{(n-1)(n-2)\cdots(n-k+1)}{k!(k-1)!}.$$

We take the top line as a polynomial in n and retain only the leading power. Shifting k and extending the sum to infinity we obtain

$$p(n) \;<\; \zeta(2) \sum_{k=0}^{\infty} \frac{(\zeta(2)n)^k}{k!(k+1)!}.$$

The sum on the right is related to the Bessel function J_1 as

$$p(n) \;\sim\; -i\zeta(2) \frac{J_1\left(2i\sqrt{\zeta(2)n}\right)}{\sqrt{\zeta(2)n}}.$$

Now, asymptotically (see [Dettman, Exercise 10.3.6])

$$J(iy) \;\to\; \frac{ie^y}{\sqrt{2\pi y}}.$$

Putting this together we get an asymptotic estimate, where K is a constant,

$$p(n) \;<\; K\frac{e^{\pi\sqrt{2n/3}}}{n^{3/4}}.$$

The second approach involves noting that

$$\sum_{k=1}^{n} \frac{\zeta(2)^k}{k!}\binom{n-1}{k-1} \;=\; L_n^{(-1)}(-\zeta(2)),$$

the Laguerre polynomial of index -1 (see [Abramowitz and Stegun] or [Whittaker and Watson] – the notation varies). The asymptotic behavior in n of these polynomials reproduces the result above. The exponential term gives the essential character of the asymptotic behavior, and the full result proved using complex analysis is

$$p(n) \;\sim\; K\frac{e^{\pi\sqrt{2n/3}}}{n}.$$

Conclusion Using elementary methods we have derived a weaker version of the classic asymptotic result for the unrestricted partitions of a large integer. The steps in this analysis would benefit from a more rigorous approach, but hopefully it does demonstrate the utility of the 'differential equation' approach to partition functions by providing a non-trivial example application.

References

[Abramowitz and Stegun] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, 1965.

[Apostol] Tom Apostol, *An Introduction to Analytic Number Theory*, Springer, 1973.

[Dettman] John W. Dettman, *Applied Complex Variables*, Dover, 1984.

[Whittaker and Watson] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge, 1927 (reprinted 1992).

Singular matrices

Tommy Moorhouse

Introduction In [Venables & Smith] in the context of statistical analysis using a language called R, the authors consider two-by-two matrices with coefficients in the set $\{0, 1, \ldots, 9\}$. They observe on the basis of numerical results that 'about one in 20' such matrices are singular, which they consider surprising.

The problem Naturally, this calls for investigation. Can you find the exact number of solutions to

$$\det M \;=\; 0$$

where the coefficients of M are non-negative integers less than 10? What is the range R of possible values for $\det M$? If $x \in R$, is there a formula for the number of solutions to

$$\det M \;=\; x?$$

The results for $x = 1$ would be of particular interest in the context of matrix groups.

There seem to be many ways to generalize this investigation, including considering different ranges for the matrix elements, using symmetries to reduce the work, or working modulo a prime.

Reference

[Venables & Smith] *An Introduction to R*, W. N. Venables, D. M. Smith and the R Core Team, Vienna 2016, URL https://www.R-project.org/.

Problem 279.1 – Inverse functions

Let p and q be real numbers. Define

$$F(p,q,x) \;=\; x + \sum_{k=2}^{\infty} \left(\prod_{j=1}^{k-1} (p - jq) \right) \frac{x^k}{k!}, \qquad |x| < \frac{1}{|q|}.$$

Show that if $|x| < 1/|q|$, then

$$x \;=\; F(q,p,F(p,q,x)). \tag{1}$$

In other words, the functions $z \mapsto F(p,q,z)$ and $z \mapsto F(q,p,z)$ are inverses of each other. For example, with $p = 1$ and $q = 0$ we have $F(1,0,x) = e^x - 1$, $F(0,1) = \log(1 + x)$.

Solution 276.4 – Symmetric binary matrix

Let $\mathbf{B}$ be a symmetric $\{0,1\}$ matrix with diagonal elements $\mathbf{d}$. Show that $\mathbf{Bx} \equiv \mathbf{d}$ (mod 2) is solvable for $\mathbf{x}$, or find a counter-example.

Graham Lovegrove

This is a generalization of Problem 276.5 – Put that light out!

Recall that $\mathrm{Ker}(B)$ is the subspace of column vectors v such that $Bv = 0$. We first show that any vector that is orthogonal to every element of $\mathrm{Ker}(B)$ (i.e. each dot product is zero) is in the image of B. Taking any basis of $\mathrm{Ker}(B)$, we can view this set of vectors as the coefficients of a set of homogeneous equations, the complete solution set of which is the vector space spanned by the rows of B. This is the same thing as saying that every vector that is orthogonal to every element of $\mathrm{Ker}(B)$ is a combination of rows of B. However, B is symmetric, so every vector orthogonal to all elements of $\mathrm{Ker}(B)$ is a combination of columns of B. But the space spanned by the columns is the image space of B, so we have proved that any vector that is orthogonal to every element of $\mathrm{Ker}(B)$ is in the image of B.

Now we show that d is in the image of B by showing that it is orthogonal to every element of $\mathrm{Ker}(B)$. Suppose the vector v is in the kernel of B, so $Bv = 0$ in GF(2). Then for any row r of B, $r \cdot v \equiv 0 \pmod 2$, where the dot signifies the dot product. This means that an even number of 1s in v match with 1s in r. Suppose the non-zero elements of v are indexed by $L = \{i_1, i_2, \ldots, i_q\}$, and consider the square submatrix of B consisting of rows and columns from L. This is again a symmetric matrix. Each row r must sum to zero, because the sum is the same as $r \cdot v$. Therefore the sum of the row-sums is zero. However, since the matrix is symmetric, each off-diagonal value occurs twice, and so contributes zero, leaving the diagonal, which must therefore also sum to zero. But inspection shows that this is the same as $d \cdot v$, so d is orthogonal to every element of $\mathrm{Ker}(B)$, and so is in the image of B.

Problem 279.2 – Limit

Does the following limit exist? If so, compute it.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^{2n}}\, dx.$$

Solution 275.4 – Hidden die

> A special X-ray scanner can detect the results of a hidden die rolled inside a box. The results are 90 per cent accurate. The scanner shows that a six has been rolled. What is the probability that the die in the box actually shows a six?

Ted Gore

Let s be the case that a six has been thrown. Let r be the case that a six has been registered. We want the probability that a six has been thrown given that a six has been registered; i.e. $\Pr(s|r)$.

Using Bayes's Theorem,

$$\Pr(s\,|\,r)\Pr(r) \;=\; \Pr(s) \cap \Pr(r) \;=\; \frac{1}{6} \times \frac{9}{10} \;=\; \frac{9}{60}.$$

But

$$\Pr(r) \;=\; \Pr(r\,|\,s) + \Pr(r\,|\,\text{not}\,s) \;=\; \frac{9}{10} \times \frac{1}{6} + \frac{1}{10} \times \frac{5}{6} \;=\; \frac{14}{60}.$$

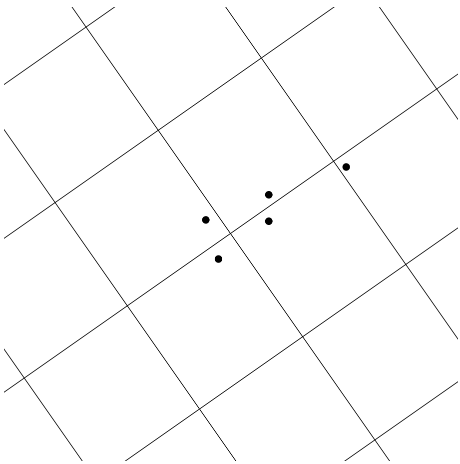
Therefore

$$\Pr(s\,|\,r) \;=\; \frac{9}{14}.$$

Problem 279.3 – Five points

Tony Forbes

There are five points in the plane. Is there a simple criterion to determine whether or not you can draw a suitably orientated integer grid such that each point is alone in one of the grid squares?



Obviously this is not a problem if the points are spread out. Otherwise, however, it can be a little difficult, especially if some of them are collinear.

Problem 279.4 – Unique Fibonacci sum

David Wild

Let F_1, F_2, F_3, ... denote the Fibonacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots.$$

Show that each positive integer can be represented uniquely in the form

$$a_2F_2 + a_3F_3 + \cdots + a_nF_n + \ldots,$$

where each of the a_i is either 0 or 1, and no adjacent a_i are both equal to 1.

This problem was set in the Enumerative Combinatorics course on the *Coursera* platform.

Problem 279.5 – Half

Bruce Roth

Using the digits from 1 to 9 once only, how many equivalent fractions of 1/2 can you make. Can you be sure you have got them all? For example,

$$\frac{7329}{14658}.$$

Problem 279.6 – Sharing a cake

Tony Forbes

A cake with weight $w = w_1 + w_2$ and volume $v = v_1 + v_2$ is to be divided into two equal halves, C_1 and C_2, where C_i has weight w_i and volume v_i, $i = 1, 2$. Can this always be done?

By *equal* we mean that the two halves must be identical according to some metric. By *some metric* we mean a weighted combination of weight and volume. So there is a parameter α, $0 \le \alpha \le 1$, and we weight weight with weight α and volume with weight $1 - \alpha$. Then the 'value' of C_i will be $\alpha w_i + (1 - \alpha)v_i$, $i = 1, 2$.

This might be helpful in real life. When one of the recipients (of half of a real cake) complains, "Ya! Boo! Your piece is better than mine!", you can resolve the argument by saying that they are identical—for some parameter α in the metric as defined above. For example, if $v_1 = v_2$, put $\alpha = 0$.

Letter

Holey cube

Dear Eddie,

Many thanks for the copy of M500 **276**.

The holey cube (Problem 274.2) reminds me Harry Gordon Selfridge's megalomaniac plan of 1915 to build a 1000 ft tower on top of his store in Oxford Street, as a British riposte to the Eiffel Tower of 1889, which is only 984 ft tall. His grandiose scheme had to be abandoned when the engineers pointed out to him that the substructure for this thing would have involved thickening the walls and floors of the shop to such an extent that the rooms would have been reduced to little cells in a mass of masonry.

Selfridge became madder and madder, and in 1941 was ousted from the board of the shop he founded. His pension was cut several times, and he was exiled to a small house in Putney. He used to come in on the number 22 bus every day and wander around on the pavement in front of his shop, to which eventually he was no longer admitted, accosting passers by and telling them that he had founded the place and was its owner. Once he was arrested for vagrancy. Fixed in his belief and increasingly fuddled, he died in 1947, aged 91.

Best wishes,

Ralph Hancock

Problem 279.7 – Two or three dice

Tony Forbes

This is like Problem 274.1 – Two dice, except that there might be three dice. I offer you the following game, which is repeated many times. The rules are simple. How should you play?

I throw a die. If it is a 6, I throw a second die.
  If the second die is a 6, I pay you £180; otherwise you pay me £30.

If the first die is not a 6, I invite you to give me £1.
  If you accept, the game ends; Otherwise I throw two more dice.
  If double 6 appears, I pay you £180.
  If precisely one 6 appears, you pay me £30.
  If no 6 appears, nobody pays anybody anything.

M500 Mathematics Revision Weekend 2018

The forty-fourth M500 Revision Weekend will be held at

Yarnfield Park Training and Conference Centre,
Yarnfield, Staffordshire ST15 0NL
from Friday 18th to Sunday 20th May 2018.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £265 for single occupancy, or £230 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £150.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2018 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 18th April 2018. The Late Booking Fee for applications received after 18th April 2018 is £20, with no membership discount applicable.

A shuttle bus service will be provided between Stone station and Yarnfield Park on Friday and Sunday. This will be free of charge, but seats will be allocated for each service and must be requested before 1st May.

There is free on-site parking for those travelling by private transport. For full details and an application form after 1st November, see the Society's web site:

www.m500.org.uk.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

Please note that the venue is not the same as last year.

Problem 279.8 – Arithmetic

(i) Show that

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 5 \cdot 11 \; = \; 9^2 + 0. \qquad (*)$$

(ii) Translate $(*)$ into limerick form.

Contents

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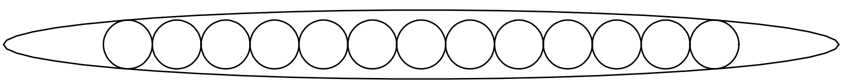
Problem 279.9 – Circles and an ellipse

There are n unit circles arranged in a straight line. An ellipse encloses them. Show that the area of the ellipse is at least $2\pi(n-1)$.



Front cover Graphs with 6 vertices and 10 edges