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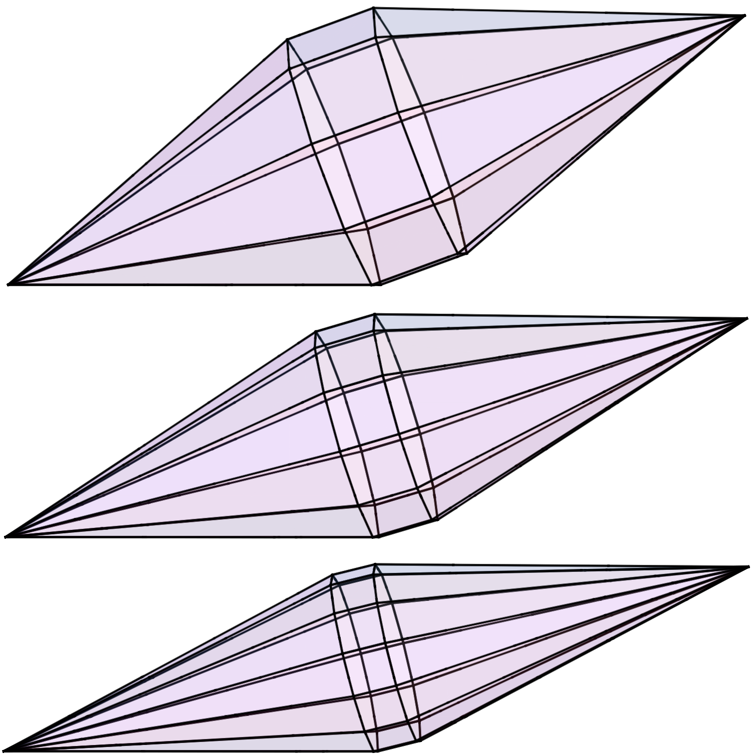
ISSN 1350-8539

M500 280



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Solution 274.3 – A nine-sided die

A 9-sided die is made using stuff of uniform density by sticking a regular tetrahedron of side 1 on to each end of a prism based on an equilateral triangle of side 1. How long must the prism be for the die to have zero probability of landing on a triangular face? Also solve the problem for 12-sided and 15-sided dice constructed in a similar manner from equilateral triangles and a 4- or 5-sided prism.

Chris Pile

(a) For the problem as stated I offer diagrams of 3-, 4- and 5-sided prisms terminated with equilateral triangular pyramids. The prisms must be longer than the minimum lengths of

$$\frac{1}{2\sqrt{6}} \approx 0.2041, \quad \frac{1}{\sqrt{2}} \approx 0.7071, \quad \text{and} \quad 1.8017 \approx \frac{1}{2}\sqrt{\frac{13}{2} + \frac{29}{2\sqrt{5}}}$$

respectively to ensure that they fall on to the face of the prism.

The resulting polyhedra make effectively a 3-, 4- or 5-sided die, although a regular tetrahedron would make a simplex 4-sided die and the triangular prism doesn't roll very well! The method of using *equilateral* triangles at the ends cannot be used for prisms with more than 5 sides. (With a little patience you can balance a hexagonal pencil on its end!)

(b) If the prism length is fixed at 1 (square faces) and the ends are terminated with *isosceles* triangular pyramids, the method can be extended to prisms of more than 5 sides. The problem is now to find the length of the sides of the triangular pyramids. For 3-, 4-, 5- and 7-sided *square-faced* prisms, the minimum edge length of the triangular pyramids is $\sqrt{13}/6 \approx 0.6009$, $\sqrt{3}/2 \approx 0.8660$, 1.2731 and 2.4446 respectively. For an n-sided square-faced prism, the minimum length is

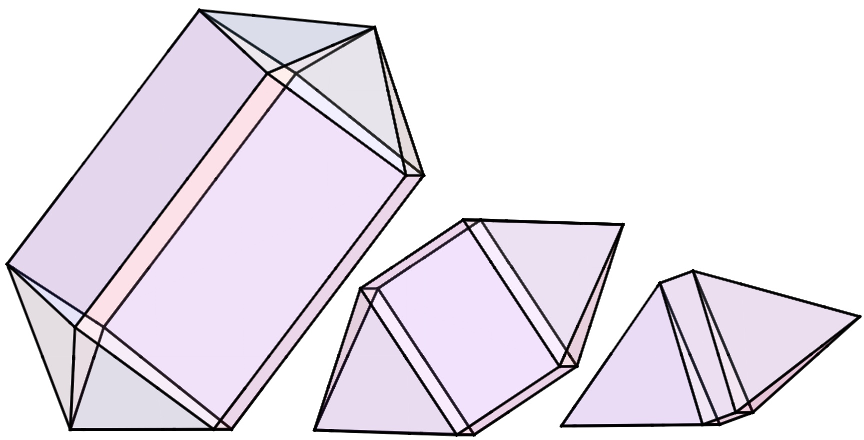
$$\frac{1}{2}\sqrt{1 + \cot^2(\pi/n)\operatorname{cosec}^2(\pi/n)}.$$

(c) The prism could be removed (length = 0) with the polyhedra made of two triangular pyramids of different lengths, arranged to fall on to the longer triangular face.

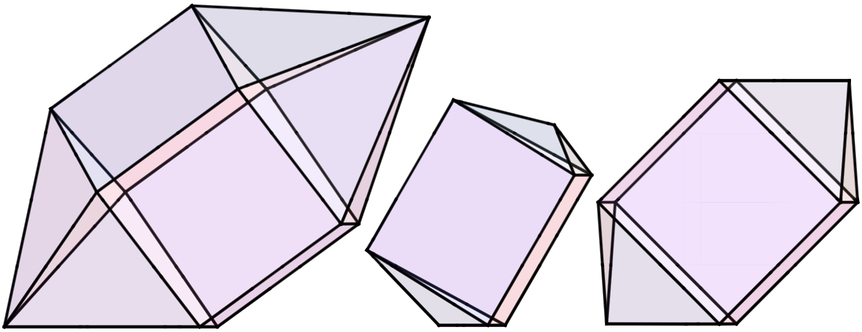
(d) Further comments. The prism/pyramid method could be extended to any odd (or even) number of sides. The regular polyhedra can be used as dice for 4, 6, 8, 12 or 20 sides. Maybe these could be used as the basis for more non-standard dice. The symmetry of each polyhedron ensures that the probability is the same for each of the faces on which it lands. The problem requires the possibilities to be calculable. Are there any dice where the probabilities are calculable but not equal? For example, consider a square pyramid (one square face and 4 equilateral triangles) with the probability of landing on the square or triangular face being S or T respectively. Then $S + 4T = 1$, but what are the values of S and T? In my search through back numbers of M500 I came across M500 **174**, 'Non-regular dice', by David Singmaster. In your introduction to the article you mentioned the difficulty of assigning probabilities to a cylinder or a brick. Has any progress been made since them? To your objects I would add an idealized drawing pin (thin disc with a central spike). What probabilities can be derived from various ratios of 'disc diameter' to 'spike length' to cause the pin to fall spike up or spike down?

(e) A one-sided die. Facetiously, objects which are usually cited as landing on one side involve dropping a cat or a slice of buttered toast! However, there exist unistable polyhedra (solids of uniform density) found by Richard Guy [*The Penguin Dictionary of Curious and Interesting Geometry* by David Wells]. The first polyhedron found was a 17-sided prism with obliquely cut ends, which rolls over on to one face. I have since learnt that this has been 'improved' to a prism with 15 sides.

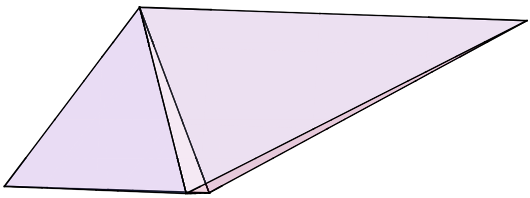
I enjoy making polyhedra (I have a large collection!), so I was happy to take your suggestion and build the 'dice' to gain a better understanding of their properties. It would be interesting (maybe!) to try to assign probabilities to the Archimedean polyhedra with two or three regular polygon faces.



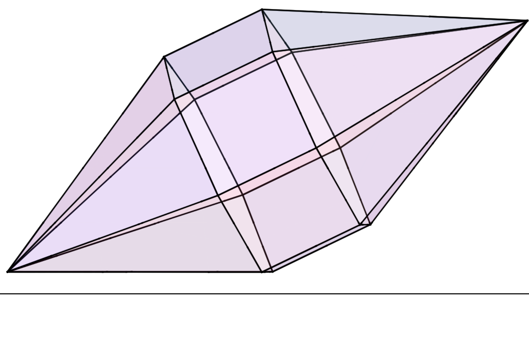
(f) Alternative dice with square faces on the prisms and isosceles triangles at each end. The 3-sided die (middle) has triangle edge length $\sqrt{13}/6$ and hence median $1/3$. The 4-sided die (right) has triangle edge length $\sqrt{3}/2$ and median $1/\sqrt{2}$. The 5-sided die (left) has triangle edge length 1.273 and median 1.1708.



A 3-sided die with 6 faces. This will land on the longer triangular face. On the left is a regular tetrahedron, median $\sqrt{3}/2$, and on the right the triangles have long side $\sqrt{3}$ and median $2\sqrt{2}/3$.



A 7-sided die with a square-faced prism. The triangles have edge length 2.4446 and median 2.3929.



The Chain Rule extended

Graham Lovegrove

Many will be familiar with the formula for differentiating a product of two functions n times:

$$\frac{d^n}{dx^n}(fg) \;=\; \frac{d^n f}{dx^n}g + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \ldots + \binom{n}{r}\frac{d^{n-r}f}{dx^{n-r}}\frac{d^r g}{dx^r} + \ldots + f\frac{d^n g}{dx^n}.$$

This is known as the Leibniz Rule, and is very similar to the Binomial Theorem if multiple differentiation is substituted for powers. It is easily proved by induction.

Recently, while wondering how to tackle a problem in M500 (Problem 279.1), I realized that I did not know of any counterpart of this for the Chain Rule. The Chain Rule states that, if f is a function of g, and g is a function of x, then

$$\frac{df(g(x))}{dx} \;=\; \frac{df}{dg}\frac{dg}{dx}.$$

The textbooks don't seem to provide any general formulae to extend this in a similar way to the Leibniz Rule, so I investigated a little further.

It will be helpful to switch to the notation where $u_v^{(n)}$ stands for $\dfrac{d^n u}{dv^n}$. Then in this notation the chain rule is written:

$$f_x^{(1)} \;=\; f_g^{(1)} g_x^{(1)}.$$

Applying it 2, 3, 4 and 5 times respectively, together with the product rule as required, we obtain:

$$\begin{aligned}
f_x^{(2)} &= f_g^{(2)}\left(g_x^{(1)}\right)^2 + f_g^{(1)}g_x^{(2)},\\
f_x^{(3)} &= f_g^{(3)}\left(g_x^{(1)}\right)^3 + 3f_g^{(2)}g_x^{(1)}g_x^{(2)} + f_g^{(1)}g_x^{(3)},\\
f_x^{(4)} &= f_g^{(4)}\left(g_x^{(1)}\right)^4 + 6f_g^{(3)}\left(g_x^{(1)}\right)^2 g_x^{(2)} + 3f_g^{(2)}\left(g_x^{(2)}\right)^2 + 4f_g^{(2)}g_x^{(1)}g_x^{(3)}\\
&\quad + f_g^{(1)}g_x^{(4)},\\
f_x^{(5)} &= f_g^{(5)}\left(g_x^{(1)}\right)^5 + 10f_g^{(4)}\left(g_x^{(1)}\right)^3 g_x^{(2)} + 15f_g^{(3)}g_x^{(1)}\left(g_x^{(2)}\right)^2\\
&\quad + 10f_g^{(3)}\left(g_x^{(1)}\right)^2 g_x^{(3)} + 10f_g^{(2)}g_x^{(2)}g_x^{(3)} + 5f_g^{(2)}g_x^{(1)}g_x^{(4)} + f_g^{(1)}g_x^{(5)}.
\end{aligned}$$

From this we can notice several things.

i The number of terms does not grow by the same number each time as it does in the Leibnitz Rule.

ii The constant factors are not the binomial coefficients, although there are similarities.

iii Whereas in the Leibnitz Rule each term consists of the product of a constant, the $(n-r)^{\text{th}}$ derivative of f, and the r^{th} derivative of g for some r between 0 and n, each term here is rather more complicated. It could be summarized as

$$Cf_g^{(\kappa)}\left(g_x^{(\lambda_1)}\right)^{\mu_1}\left(g_x^{(\lambda_2)}\right)^{\mu_2}\cdots\left(g_x^{(\lambda_p)}\right)^{\mu_p},$$

where C is a constant.

iv In each case, $\kappa=\sum_{i=1}^{p}\mu_i$.

v In each case, $n=\sum_{i=1}^{p}\lambda_i\mu_i$, that is, the λ_i, each repeated μ_i times, represent a *partition* of n. So for instance the terms for $n=5$ yield the partitions: $1+1+1+1+1$, $1+1+1+2$, $1+2+2$, $1+1+3$, $2+3$, $1+4$, and 5. These represent all the partitions of 5. A partition is usually written for convenience in a power notation, viz. 1^5, $1^3 2^1$, etc. The number of terms is then seen to be the number of partitions of n.

So that explains everything about the n^{th} derivative of $f(g)(x)$, except for the coefficients. The coefficients are the same whatever the functions f and g, and it's easy to see that the coefficients for the n^{th} derivative can be calculated by examining what happens when the $(n-1)^{\text{th}}$ expression is differentiated term by term.

Considering the typical term (where C stands for the corresponding coefficient)

$$Cf_g^{(\kappa)}\left(g_x^{(\lambda_1)}\right)^{\mu_1}\left(g_x^{(\lambda_2)}\right)^{\mu_2}\cdots\left(g_x^{(\lambda_p)}\right)^{\mu_p},$$

differentiating it using the Chain and Product rules produces a number of new terms, one for each factor in the original term.

First, using the chain rule on $f_g^{(\kappa)}$ produces a term

$$Cf_g^{(\kappa+1)}g_x^{(1)}\left(g_x^{(\lambda_1)}\right)^{\mu_1}\left(g_x^{(\lambda_2)}\right)^{\mu_2}\cdots\left(g_x^{(\lambda_p)}\right)^{\mu_p},$$

and for each factor of the form $\left(g_x^{(\lambda_i)}\right)^{\mu_i}$, a term

$$\mu_i C f_g^{(\kappa)} \left(g_x^{(\lambda_1)}\right)^{\mu_1} \ldots \left(g_x^{(\lambda_i)}\right)^{\mu_i - 1} g_x^{(\lambda_i+1)} \ldots \left(g_x^{(\lambda_p)}\right)^{\mu_p}.$$

At this point, we need a notation for each constant. Instead of specifying the λ and μ for each factor appearing in the term, we specify the value of μ for every λ between 1 and n, putting $\mu = 0$ where there is no factor for a value of λ. Thus we denote the coefficients for $n = 5$ as

$$\begin{array}{lll} C_5(5,0,0,0,0) = 1; & C_5(3,1,0,0,0) = 10; & C_5(1,2,0,0,0) = 15; \\ C_5(0,2,1,0,0) = 10; & C_5(0,1,1,0,0) = 10; & C_5(1,0,0,1,0) = 5; \\ C_5(0,0,0,0,1) = 1. & & \end{array}$$

The value of κ for each term constant is the sum of the μ, as determined above. We can now use the identities we have just derived to make a recurrence relation between the constants for the $(n-1)^{\text{th}}$ derivative and the n^{th}:

$$\begin{aligned} C_n(\mu_1, \mu_2, \ldots, \mu_n) \; &= \; C_{n-1}(\mu_1 - 1, \mu_2, \ldots, \mu_{n-1}) \\ &+ \sum_{\mu_i > 0, \; i < n-1} (\mu_{i-1} + 1) C_{n-1}(\mu_1, \ldots, \mu_{i-1} + 1, \mu_i - 1, \ldots, \mu_{n-1}) \\ &+ (\mu_{n-1} + 1) C_{n-1}(\mu_1, \mu_2, \ldots, \mu_{n-1} + 1), \end{aligned}$$

where it is understood that the first term on the right-hand side is omitted if $\mu_1 = 0$, and the last term is omitted if $\mu_n = 0$. Also there will be no corresponding term in the sum in the right-hand side if $\mu_i = 0$.

This recurrence relation can be used to calculate the coefficients for the n^{th} derivative of $f(g)(x)$ from the coefficients for $n - 1$. But it is very cumbersome, and it would be much better to have a formula to calculate the coefficients directly. However, it is difficult to see how the recurrence relation could be solved to construct the formula. Some other approach is therefore necessary.

Since we only want the constants, we could perhaps try to find a particular pair of functions f and g for which we can express $f_x^{(n)}$ in terms of $f_g^{(i)}$, $g_x^{(j)}$, etc.

This is in fact what I did, but before proceeding we will need a brief diversion to mention the *Multinomial Theorem*. This says that

$$(a_1 + a_2 + \ldots + a_p)^N \; = \; \sum_{\sum t_i = N} Q_{t_1, t_2, \ldots, t_p} a_1^{t_1} a_2^{t_2} \ldots a_p^{t_p},$$

where

$$Q_{t_1,t_2,\ldots,t_p} \;=\; \frac{N!}{t_1!t_2!\ldots t_p!}, \quad \text{with } \sum t_i = N.$$

These coefficients are the numbers of ways of dividing N identical objects into piles of sizes $t_1, t_2, \ldots, t_p$, and the theorem is proved in a similar way to the Binomial Theorem.

To find the coefficients for the extended Chain Rule, we select g to be

$$g \;=\; 1 + \nu_1 x + \frac{\nu_2 x^2}{2!} + \frac{\nu_3 x^3}{3!} + \ldots + \frac{\nu_p x^p}{p!},$$

where $\nu_1, \nu_2, \ldots, \nu_p$ are arbitrary parameters, and we choose $f = g^q$, where p and q are positive integers that are both larger than n.

We can use the Multinomial Theorem to express $f(g)(x)$ as a polynomial in x. We are going to calculate the coefficient of x^n in this polynomial. The reason for this is that the coefficient of x^n is equal to $\dfrac{1}{n!}\dfrac{d^n}{dx^n}(fg)$ evaluated at $x = 0$.

Expanding g^q by the Multinomial Theorem, a typical term is of the form

$$Q_{t_0,t_1,t_2,\ldots,t_p}\frac{\nu_1^{t_1}\nu_2^{t_2}\ldots\nu_p^{t_p}\; x^{\sum it_i}}{(1!)^{t_1}(2!)^{t_2}\ldots(p!)^{t_p}} \;=\; \frac{q!\nu_1^{t_1}\nu_2^{t_2}\ldots\nu_p^{t_p}\; x^{\sum it_i}}{t_0!t_1!t_2!\ldots t_p!\; 1^{t_0}(1!)^{t_1}(2!)^{t_2}\ldots(p!)^{t_p}},$$

where t_0, t_1, t_2, etc., count the numbers of factors of type x^0, $\nu_1 x^1$, $\nu_2 x^2$, etc. present, and $\sum_{i=0}^{p} t_i = q$. If $x^{\sum it_i} = n$, then $\sum it_i$ is a partition of n, so the whole term in x^n is the sum of a number of these terms, one for each partition of n. Hence

$$\frac{1}{n!}\left(\frac{d^n}{dx^n}(fg)\right)_{x=0} \;=\; \sum_{\sum it_i = n}\frac{q!\nu_1^{t_1}\nu_2^{t_2}\ldots\nu_p^{t_p}}{t_0!t_1!t_2!\ldots t_p!\; 1^{t_0}(1!)^{t_1}(2!)^{t_2}\ldots(p!)^{t_p}}.$$

Now we note that $\nu_i \;=\; g^{(i)}_{x=0}$. Also, $t_0 = q - \sum_{i=1}^{p} t_i$, so

$$\frac{q!}{t_0!} \;=\; \frac{q!}{(q - \sum_{i=1}^{p} t_i)!},$$

which is equal to $\left(f_g^{(\sum t_i)}\right)_{g=1}$, and $g = 1$ when $x = 0$. So

$$\left(\frac{d^n}{dx^n}(fg)\right)_0 \;=\; \sum_{\sum it_i = n}\frac{n!\;((g^{(1)})_0)^{t_1}((g^{(2)})_0)^{t_2}\ldots((g^{(p)})_0)^{t_p}}{t_1!t_2!\ldots t_p!\;(1!)^{t_1}(2!)^{t_2}\ldots(p!)^{t_p}}\left(f_g^{\sum t_i}\right)_0,$$

where the suffix 0 denotes evaluation at $x = 0$. As we have assumed $p > n$ and $\sum it_i = n$, we can infer that $t_{n+1} = t_{n+2} = \ldots = t_p = 0$. From which it is clear that

$$C_n(\mu_1, \mu_2, \ldots, \mu_n) \;=\; \frac{n!}{\mu_1!\mu_2!\ldots\mu_n!\,(1!)^{\mu_1}(2!)^{\mu_2}\ldots(n!)^{\mu_n}},$$

where $1^{\mu_1}2^{\mu_2}\ldots n^{\mu_n}$ is any partition of n. It is straightforward to show that these coefficients satisfy the recurrence relation that was derived above, so this is left as an exercise for the reader!

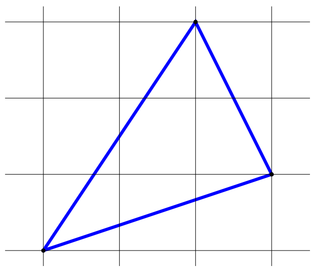
Problem 280.1 – Triangles

Tony Forbes

We know from Pick's Theorem (see, for example, *Theorem of the Day* number 77, or M500 **253**) that a triangle drawn on an integer grid has area which is half an integer. Moreover, its sides must be square roots of integers. Is the converse true? Well, no. For instance, a $(\sqrt{7}, \sqrt{7}, \sqrt{14})$ triangle has area 7/2 but there is no way you can place its vertices on an integer grid. This is because $7 = a^2 + b^2$ has no solution in integers. Bearing that in mind we ask:

> *Can a triangle with half-integer area and sides which are of the form $\sqrt{a^2 + b^2}$, a, b integers, be placed in the (x, y)-plane such that the coordinates of the three vertices are integers?*

Obviously this is true for any Pythagorean triangle, $(a, b, \sqrt{a^2 + b^2})$, a, b integers. Less obviously, we see that a $(\sqrt{5}, \sqrt{10}, \sqrt{13})$ triangle has area 7/2 and with a bit of experimentation—perhaps using a cardboard cut-out as a guide—you find that you can, for example, place the vertices at $(0, 0)$, $(3, 1)$ and $(2, 3)$. So in this particular case the assertion is also true.



Problem 280.2 – Integral

Show that

$$\int_0^\infty \frac{\sin x}{e^{2\pi x} - 1} \;=\; \frac{1}{4}\coth\frac{1}{2} - \frac{1}{2} \;\approx\; 0.0409884.$$

Four blocks

Chris Pile

You may have seen *The Crystal Maze*, the Channel 4 television game show in which Richard Ayoade sends the players through the maze's Aztec, Medieval, Industrial and Future zones to tackle tasks of various difficulties. One member of the team is selected to attempt a task while the others watch and give advice. One challenge is to stack four differently coloured blocks to form a 'totem pole' in front of a fire-breathing dragon. The blocks have to be in the 'right' order to win a crystal. At each attempt the dragon is asked how many blocks are in the correct positions, and it responds with one, two or three bursts of flame.

There are 24 permutations and the number of attempts is limited by the speed of construction and the overall time limit of 3 minutes. The team is not told which blocks are correct, and must remember the orders of previous attempts as the structure is assembled to construct the next attempt. The correct order can be found by good fortune or good reasoning, but the team often fails after about seven attempts. Is there an optimum strategy? What is the minimum number of attempts needed to ensure success?

Ignoring the time limit, the problem is, essentially, to find the 'correct' order for four objects in the least number of attempts. Of the 24 permutations, 9 are all wrong, 8 have one correct and 6 have two correct. For example, using playing cards from different suits, a typical situation might look like this.

none correct									one correct								two correct						target
♣	♣	♣	♢	♢	♢	♠	♠	♠	♡	♡	♣	♣	♢	♢	♠	♠	♡	♡	♡	♣	♢	♠	♡
♡	♢	♠	♡	♠	♠	♡	♢	♢	♢	♠	♢	♠	♡	♣	♡	♣	♣	♢	♠	♡	♣	♣	♣
♠	♠	♡	♠	♡	♣	♣	♡	♣	♠	♣	♡	♢	♣	♠	♢	♡	♠	♣	♢	♢	♡	♢	♢
♢	♡	♢	♣	♣	♡	♢	♣	♡	♣	♢	♠	♡	♠	♡	♣	♢	♢	♠	♣	♠	♠	♡	♠

I think five attempts are sufficient, and the majority of cases can be successful after four or fewer. However, my 'strategy' is neither simple nor elegant.

If the initial attempt is 'all wrong', it seems logical to change the positions of all four cards. If one is correct, then changing three or two may be appropriate, and if two are correct, changing two cards would seem to be the best option. (At least two cards must be changed!) The third attempt will depend on the previous two attempts, etc. Is it worth considering extending the problem to five objects (120 permutations) or more?

Solution 274.5 – 27 cubes

> There are 27 cubes each face of which is coloured either red, blue or green. Moreover, the 27 cubes can be assembled in three ways to form either a red, blue or green $3 \times 3 \times 3$ cube. Interestingly, this can be achieved in essentially only one way. How?

Chris Pile

I had a feeling of 'déjà vu' when I saw this problem! I consulted my back numbers of M500 and I found it in M500 **109** (September 1988!), and I was among those who supplied a solution in M500 **110**. A general solution for n^3 cubes was given by John Reade in M500 **111**.

The $3 \times 3 \times 3$ cube has 6 faces, each comprised of 9 faces of the smaller cubes. Therefore $6 \times 9 = 54$ faces are either red, blue or green, and 54×3 colours $= 162$ faces.

The 27 cubes each have 6 faces $= 27 \times 6 = 162$ faces. Therefore each face is displayed once and only once.

Each small cube has a vertex (V), edge (E) or face (F) on the $3 \times 3 \times 3$ cube, with one small cube hidden internally. By symmetry, the hidden cube must have only two colours (no face exposed on the $3 \times 3 \times 3$ cube) and must therefore be a vertex cube for the other two arrangements. Thus three cubes of 27 are 2-colour vertex cubes.

In each $3 \times 3 \times 3$ cube, six small cubes have only one face displayed. The other five faces must therefore form one vertex cube and one edge cube in different colours. This accounts for $6 \times 3 = 18$ small cubes. The remaining six small cubes must therefore be edge cubes in each of the $3 \times 3 \times 3$ arrangements. The cubes are coloured thus:

VV: RRRBBB, RRRGGG, GGGBBB;

VEF: RRRGGB $\times$ 6, BBBRRG $\times$ 6, GGGBBR $\times$ 6;

EEE: RRBBGG $\times$ 6.

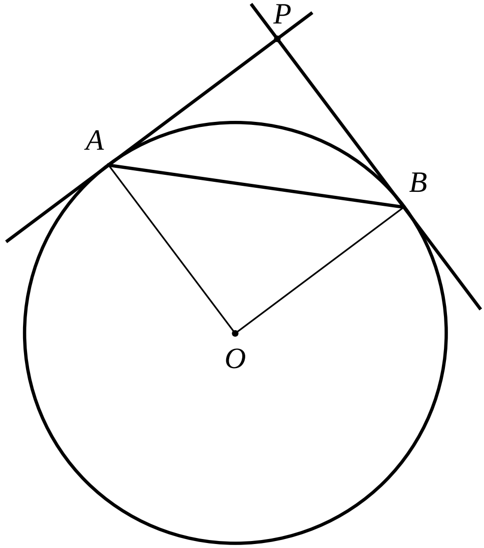
The colours of the 18 VEF cubes can be permuted as long as each cube has a vertex, edge and face in a different colour, and there are six of each.

Problem 280.3 – Digits

Tony Forbes

List all the d-digit numbers in base $d + 1$ (actually any base larger than d will do) that satisfy all of the following. (i) The digits are in non-increasing order reading left to right, (ii) the difference between consecutive digits is either 0 or 1; (iii) the units digit is 0 or 1. How long is the list?

Solution 277.2 – Circle



The points A and B lie on the circle with equation

$$x^2 + y^2 = 25.$$

The tangents to the circle at A and B meet at the point $P = (1, 7)$. Show that the chord AB has equation

$$x + 7y = 25.$$

In what ratio does this chord divide the area of the circle?

Edward Stansfield

I find that sometimes a problem looks easy to solve until I start to work on a solution—and then I give up as it becomes too difficult. This one nagged me for days as I was sure there was an easy solution—and my persistence prevailed because there is an easy solution. Here goes.

Referring to the diagram, the length of the line OP is $\sqrt{1^2 + 7^2} = 5\sqrt{2}$. The tangents to the circle of radius 5 at points A and B are necessarily at right angles to the lines OA and OB. Applying Pythagoras's Theorem to the right angled triangles OAP and OBP, we deduce that the lines AP and BP are both of length 5, and hence the rectangle $OAPB$ is a square of side 5.

The line OP has slope 7, and since chord AB is orthogonal to OP, it must have slope $-1/7$. The equation of a line along AB is thus given by

$$y = -\frac{x}{7} + \text{constant}, \quad \text{or} \quad x + 7y = c.$$

The point Q on AB lies is at the mid-point of the line OP, and hence has the coordinates $Q = (1/2, 7/2)$. Substituting this point into the equation determines the constant c to be given by

$$c = \frac{1}{2} + 7 \cdot \frac{7}{2} = 25.$$

Hence the chord AB has the equation $x + 7y = 25$, as was to be shown.

Since the area of $OAPB$ is $5^2 = 25$, the area of the triangle OAB is $25/2$. The area of the circle is 25π and the square $OAPB$ covers exactly one quarter of this. Hence the area of the circle segment above the chord AB is

$$\frac{25\pi}{4} - \frac{25}{2} = \frac{25(\pi - 2)}{4}.$$

The ratio of the area of this segment to that of the whole circle is therefore $(\pi - 2)/(4\pi)$.

I hope M500 readers agree with me that, unlike many other maths problems, this one really was easy! Q.E.D. (Quite Easily Done!)

Solved in a similar manner by **Bruce Roth** and **Ted Gore**.

John C. Davidson

The circle has the equation

$$x^2 + y^2 = 25 \tag{1}$$

and hence

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{2}$$

Since the tangents to the circle pass through the point $P = (1, 7)$, it follows from (2) that

$$-\frac{x_A}{y_A} = \frac{7 - y_A}{1 - x_A} \quad \Rightarrow \quad x_A + 7y_A = x_A^2 + y_A^2. \tag{3}$$

Since A is a point on the circle, it follows from (1) and (3) that

$$x_A + 7y_A = 25 \quad \Rightarrow \quad x_A = 25 - 7y_A. \tag{4}$$

Substituting (4) in (3):

$$\frac{7y_A - 25}{y_A} = \frac{7 - y_A}{1 - (25 - 7y_A)} \quad \Rightarrow \quad y_A^2 - 7y_A + 12 = 0. \tag{5}$$

The two roots of equation (5) are 3 and 4, so (from the sketch on page 11) $y_A = 4$ and $y_B = 3$. It follows from (4) that $x_A = -3$ and $x_B = 4$. The gradient m_{AB} is then given by

$$m_{AB} = \frac{3 - 4}{4 + 3} = -\frac{1}{7},$$

so the equation of AB is

$$y - 3 \;=\; -\frac{1}{7}(x - 4) \quad\Rightarrow\quad \underline{\underline{x + 7y \;=\; 25}}, \qquad \text{Q.E.D.}$$

It is clear from equation (1) that the centre O of the circle is the origin $(0, 0)$. Since A is the point $(-3, 4)$ and B is the point $(4, 3)$, $\mathbf{OA} = -3\mathbf{i} + 4\mathbf{j}$, $\mathbf{OB} = 4\mathbf{i} + 3\mathbf{j}$ and $|\mathbf{OA}| = |\mathbf{OB}| = 5 = r$. Expanding the scalar product:

$$\begin{aligned} \mathbf{OA} \cdot \mathbf{OB} \;&=\; |\mathbf{OA}|\,|\mathbf{OB}| \cos A\hat{O}B \\ \Rightarrow\quad 0 \;&=\; 25 \cos A\hat{O}B \quad\Rightarrow\quad A\hat{O}B \;=\; \frac{\pi}{2}, \end{aligned}$$

and so the region $AO\overset{\frown}{BA}$ is a quadrant of the circle with area $\pi r^2/4$. Also, the area of right-angled $\triangle AOB \;=\; r^2/2$ and the area of the minor segment is

$$\frac{1}{4}\pi r^2 - \frac{1}{2} r^2 \;=\; \frac{(\pi - 2) r^2}{4}. \tag{6}$$

The area of the major segment is

$$\pi r^2 - \frac{(\pi - 2) r^2}{4} \;=\; \frac{(3\pi + 2) r^2}{4}. \tag{7}$$

From (6) and (7) it follows that the chord AB divides the area of the circle in the ratio $(\pi - 2)/(3\pi + 2) \approx 0.1$.

William R. Bell

The first result can be shown via implicit differentiation of the circle equation, or (more elegantly) by a geometric argument. In the second part, one finds the desired ratio as $(\pi - 2) : (3\pi + 2)$.

This is an original problem of mine, which draws on knowledge from the core A-Level mathematics syllabus, although it requires a little more thought than a standard problem. Those starting a degree in mathematics should certainly find this interesting.

Tony Forbes

Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and suppose the chord AB has equation $x + my \;=\; c$. The points where the line $x + my \;=\; c$ meets the circle $x^2 + y^2 = 25$ are obtained by solving $(c - my)^2 + y^2 = 25$ for y:

$$y_A \;=\; \frac{cm + \sqrt{25 - c^2 + 25m^2}}{1 + m^2}, \quad y_B \;=\; \frac{cm - \sqrt{25 - c^2 + 25m^2}}{1 + m^2}. \tag{1}$$

Implicit differentiation of the circle equation gives $dy/dx = -x/y$. Therefore

$$-\frac{c - my_A}{y_A} = \frac{7 - y_A}{1 - (c - my_A)}, \quad \text{and} \quad -\frac{c - my_B}{y_B} = \frac{7 - y_B}{1 - (c - my_B)}.$$

Using y_A, y_B from (1) and solving for m and c gives three solutions:

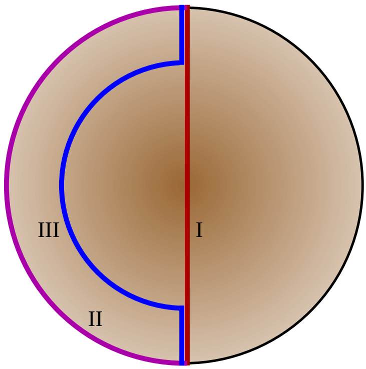
$$\begin{array}{llllll}
m = 7, & c = 25, & x_A = -3, & y_A = 4, & x_B = 4, & y_B = 3, \\
m = -3/4, & c = -25/3, & x_A = -3, & y_A = 4, & x_B = -3, & y_B = 4, \\
m = 3/4, & c = 25/4, & x_A = 4, & y_A = 3, & x_B = 4, & y_B = 3.
\end{array}$$

The first solution, the only case where A and B are actually distinct, is the one we want to solve the problem.

The reader might find it instructive to draw the diagrams corresponding to the second and third solutions.

Problem 280.4 – Mud

Tony Forbes



There is a circular field of radius 1. The field is quite muddy on its circumference and the muddiness increases as you go towards the centre. More precisely, if x is the distance from the centre of the field, the muddiness (in some suitable units) is given by

$$m(x) = \begin{cases} \dfrac{12(\pi - 2)(1 - x)}{5(1 + x)^4} + 1, & x < 1, \\ 1, & x \geq 1. \end{cases}$$

In particular, at the centre of the field $m(0) = (12\pi - 19)/5 \approx 3.73982$. You want to get from a point on the circumference to the diametrically opposite point. What path will minimize your exposure to mud?

The picture shows some typical routes. Your exposure to mud whilst walking in a straight line across the diameter (I) is $2\int_0^1 m(x)dx = \pi$, the same as by going around the circumference (II); hence the bizarre nature of the formula for $m(x)$. But if you combine two straight paths and a semicircle, you can do better (III). However, the optimum route is likely to be of a completely different nature.

Solution 277.1 – Cooling towers

> Why do power station cooling towers have the shape that they have?

Ralph Hancock

Problem 277.1 – Cooling towers reminded me of a QBasic program I wrote with Eddie Kent's help many years ago called SHAPE.BAS, which drew a curve based on one of several simple mathematical operations and rotated it to create a perspective drawing on the screen. I managed to make quite a convincing cooling tower by inputting the variable `R` to this:

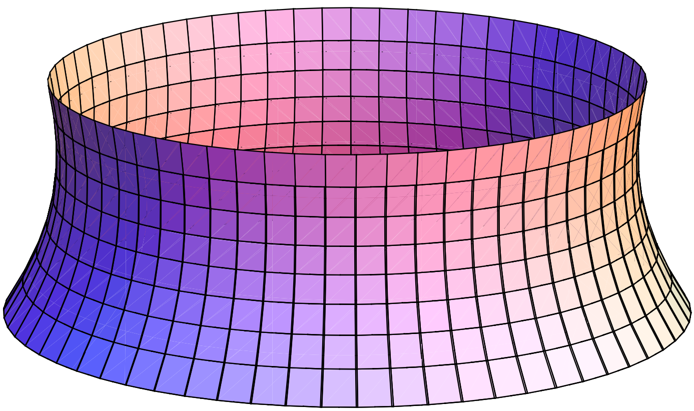
```
200 * R / (SIN(N + 1)).
```

The value I chose for `R` was 0.004, so the actual calculation was $0.8/\sin(N+1)$. The shape is too wide because the simple program has no control over the radius of the circle.

I attach the drawing that the program made. I'm keeping my old QBasic programs alive, as far as possible, with the QB64 program that compiles them in Windows.

The reason cooling towers are this shape is that it keeps the reinforced concrete of which they are made in compression all the way down the structure, thus allowing the tower to be made with a minimum of material.

The trouble about old programs is that when you look at them again you have completely forgotten what's going on.



Solution 275.6 – Pistachio nuts

There is a bowl containing n pistachio nuts. How many times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?

(i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or it might be half of a pistachio nut shell.

(ii) If it is a half-shell, you just return it to the bowl.

(iii) Otherwise you split the shell into two halves, remove the kernel, which you eat, and return the two shell fragments to the bowl.

Graham Lovegrove

If a number m nuts have already been consumed, then the probability that the next selection will be a nut is $p_m = \dfrac{n-m}{n+m}$, and the probability that it will be a half-shell is $q_m = \dfrac{2m}{n+m}$. So the probability of a particular 'scenario', in which there are k_1 unsuccessful selections after the first nut, k_2 after the second, etc., until all the nuts are consumed is

$$\Pr_{k_1,k_2,\ldots,k_{n-1}} = p_0 p_1 \ldots p_{n-1}\, q_1^{k_1} q_2^{k_2} \ldots q_{n-1}^{k_{n-1}}.$$

This scenario sees the nuts consumed in $n+\sum_{i=1}^{n-1} k_i$ moves. The probability that the nuts will be consumed in a certain number of moves is the sum of these probabilities over all scenarios with the same number of moves. We want to find a way of generating all these sums and calculating the expectation for the number of moves required. Consider the function

$$F(x) = p_1 p_2 \ldots p_{n-1}\, x^n\, Q_1(x) Q_2(x) \ldots Q_{n-1}(x),$$

where

$$Q_i(x) = 1 + xq_i + (xq_i)^2 + \ldots (xq_i)^k + \ldots = \frac{1}{(1-xq_i)} = \frac{n+i}{n+i-2ix}$$

since $Q_i(x)$ is the sum of an infinite geometric progression, which converges when $x = 1$, as $q_i < 1$ for all i. It is quite easy to see from what we have said above that the coefficient of x^k in $F(x)$ is the probability that all the nuts will be consumed in k moves. Note that $k \geq n$ because of the initial factor of x^n. So

$$F(x) = \sum_{k=n}^{\infty} x^k P_k,$$

where P_k represents the probability that all the nuts will be consumed in k moves.

Now we want to know the expected number of moves,

$$E(n) \;=\; \sum_{k=n}^{\infty} k \cdot P_k,$$

which is $F'(1)$. We calculate this by the product rule:

$$F'(1) \;=\; p_1 p_2 \ldots p_{n-1}\, Q_1(1) Q_2(1) \ldots Q_{n-1}(1) \left(n + \sum_{i=1}^{n-1} \frac{Q_i'(1)}{Q_i(1)} \right).$$

But

$$Q_i(1) \;=\; \frac{n+i}{n-i} \;=\; \frac{1}{p_i}, \quad \text{and} \quad Q_i'(x) \;=\; \frac{2i(n+i)}{(n+i-2ix)^2};$$

so

$$\frac{Q_i'(1)}{Q_i(1)} \;=\; \frac{2i}{n-i}.$$

Thus

$$E(n) \;=\; F'(1) \;=\; n + 2 \sum_{i=1}^{n-1} \frac{i}{(n-i)}.$$

This yields $E(2) = 4$, $E(3) = 8$, $E(4) = 12^2/_3$, etc.

Problem 280.5 – Pistachio nuts

For completeness, here is our third (the other two being Problems 275.6 and 278.1 for which we have had solutions to 275.6 (Tommy Moorhouse, **278** and Graham Lovegrove, this issue (above)) but not yet 278.1) and final version of the pistachio nuts problem. We again modify the eating strategy.

There is a bowl containing n pistachio nuts. How many times would you expect to perform the following procedure in order to consume all of the edible material in the bowl?

(i) You select uniformly at random one object from the bowl. It might be a whole pistachio nut in its shell, or just half of a pistachio nut shell.

(ii) If it is a half-shell, you return it to the bowl.

(iii) Otherwise you split the shell into two halves, remove the kernel, which you eat, and discard the two shell fragments.

Solution 278.5 – Tan-gled trigonometry

Find the general solution of the equation

$$\tan 9x - \tan 2x = \tan 9x \tan 6x \tan 3x + \tan 6x \tan 4x \tan 2x.$$

Bruce Roth

Rearranging the original equality gives

$$\tan 9x(1 - \tan 6x \tan 3x) = (\tan 2x)(\tan 6x \tan 4x + 1).$$

Using the compound angle formula

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

gives

$$\tan 6x + \tan 3x = \tan 6x - \tan 4x,$$

or

$$\tan 3x = -\tan 4x.$$

Therefore

$$\sin 3x \cos 4x + \sin 4x \cos 3x = 0.$$

Using the compound angle formula

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

gives $\sin 7x = 0$, and therefore $7x = n\pi$, $n \in \mathbb{Z}$. So the general solution is

$$x = \frac{n\pi}{7}, \qquad n \in \mathbb{Z}.$$

Solved in an almost identical manner by **William Bell**, the problem's originator.

Problem 280.6 – Four numbers

Find all solutions in positive integers a, b, c, d of

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = d.$$

Problem 280.7 – Convex to concave

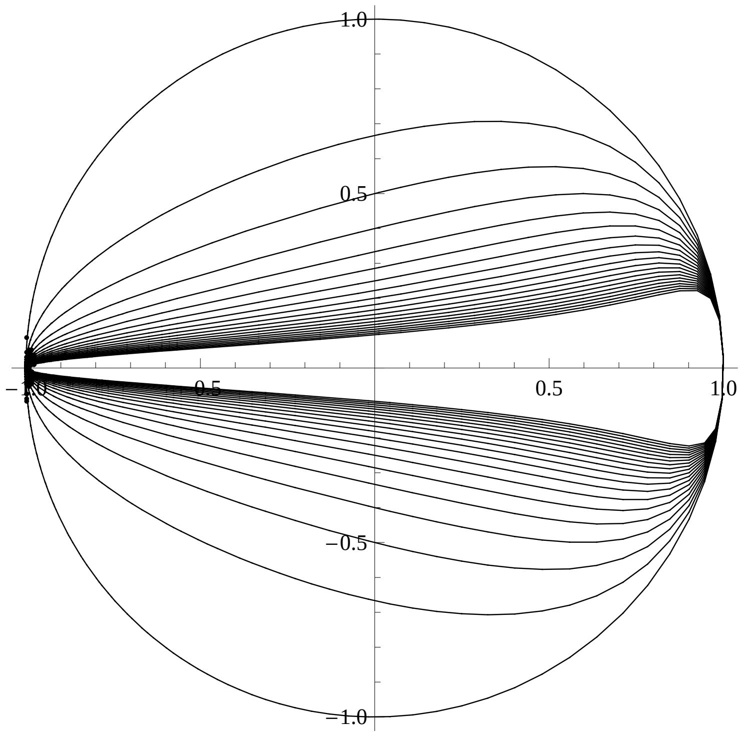
Tony Forbes

Let w be a positive number and consider the mappings $f_w(t)$ defined by

$$f_w(t) \;=\; \left(\frac{1-t^2}{1+t^2},\; \frac{2t}{1+wt^2}\right).$$

To show you what they look like, we have plotted $f_w(t)$, $-\infty < t < \infty$, for $w = 1, 2, \ldots, 20$. Recall that $f_1(t)$ gives the familiar parametrization of a circle of radius 1 centred at $(0,0)$, and this is indeed the outermost closed curve in the picture. Thereafter the curves shrink as the parameter w increases.

Observe that the first few curves are convex and the last few are not. So there must be a special number W, where the curve is convex if $w < W$ and not if $w > W$. What is the value of W?



Problem 280.8 – Regular graphs of girth 5

Tony Forbes

Suppose $k \geq 2$ and let A be the adjacency matrix of a simple graph G. Show that G is k-regular (each vertex has k neighbours) and has girth at least 5 (no triangles, no squares) if and only if each row of $A^2 - A$ has precisely k occurrences of -1, one of k, $k(k-1)$ of 1, and whatever is left over, if any, of 0. Or find a counter-example.

Proverbs and well-known sayings

Continuing a trend we started some time ago (M500 **192**, **194**, **211**) ...

1. Whilst performing financial calculations it is always sufficient to work modulo 1.

2. It is wise to preserve a significant proportion of ones liquid assets in order to deal with the consequences of the onset of precipitation.

3. Utilization of the appropriate Macintosh computer application will serve to create the required daily exclusion order against the Medical Officer.

4. A doubling in ornithological-annihilation efficiency can be achieved with suitable common-place geological material.

5. With regard to financial calculations, one has to question the wisdom of working modulo 1 and the foolishness of applying the floor function.

One day I (TF) noticed that my W-10 computer was plagued by frequent short-duration stoppages, where the keyboard and mouse freeze for a few seconds—not a minor inconvenience you may well imagine. Task Manager shows WMI Provider Host hogging an entire core. Whilst researching the problem I found a 6-minute You-Tube video explaining what to do. And in the Comments section someone suggested you can save yourself the trouble of watching the video by reading these simple instructions.

run services.msc

right click Windows Management Instrumentation

select Restart

That's 9 words (counting services.msc as one). Assuming 25 frames per second, 6 minutes of film works out at 9000 pictures. So here we have an interesting example of that well-known saying:

A word is worth a thousand pictures.

{An Apple a day keeps the doctor away; Look after the pennies, the pounds will take care of themselves; Penny wise, pound foolish; Kill two birds with one stone; Save for a rainy day}

M500 Mathematics Revision Weekend 2018

The forty-fourth M500 Revision Weekend will be held at

Yarnfield Park Training and Conference Centre,
Yarnfield, Staffordshire ST15 0NL
from Friday 18th to Sunday 20th May 2018.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £265 for single occupancy, or £230 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £150.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2018 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 18th April 2018. The Late Booking Fee for applications received after 18th April 2018 is £20, with no membership discount applicable.

A shuttle bus service will be provided between Stone station and Yarnfield Park on Friday and Sunday. This will be free of charge, but seats will be allocated for each service and must be requested before 1st May.

There is free on-site parking for those travelling by private transport. For full details and an application form after 1st November, see the Society's web site:

www.m500.org.uk.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

Please note that the venue is not the same as last year. We go back to the 2016 location.

Die Mathematiker sind eine Art Franzosen; redet man zu ihnen so übersetzen sie es in ihrer Sprache und dann ist es alsobald etwas anders.'

— Goethe

[The mathematician is a type of Frenchman; If you talk to them, they translate it into their language and then it is something different.]

04.00

Front cover Dice which can't make up their minds whether to land on a square or a triangle. See page 1.