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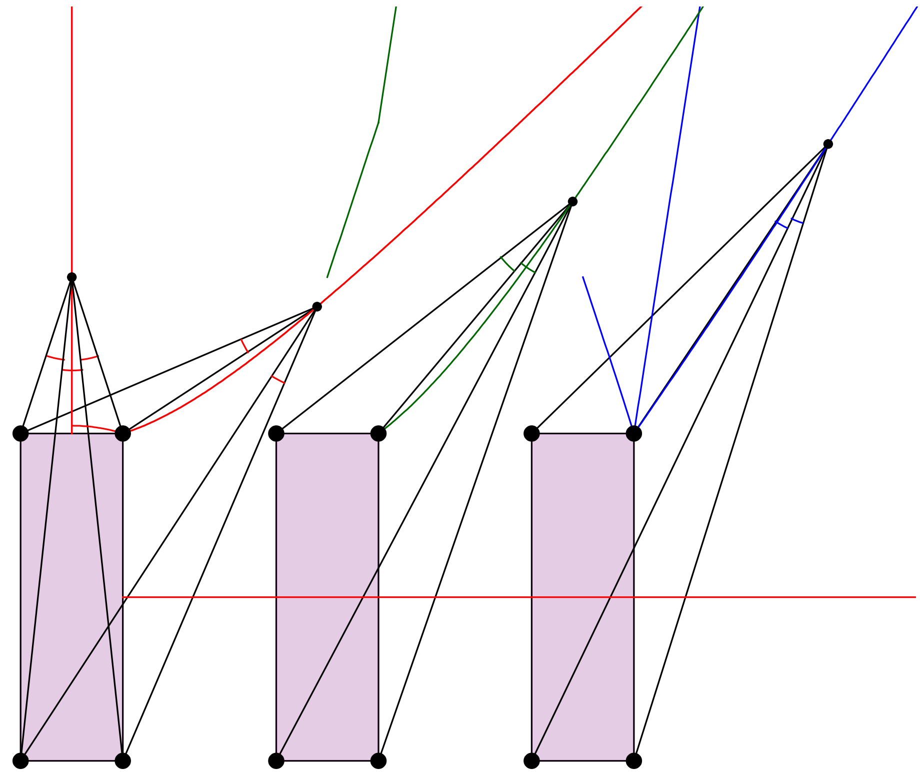
M500 260



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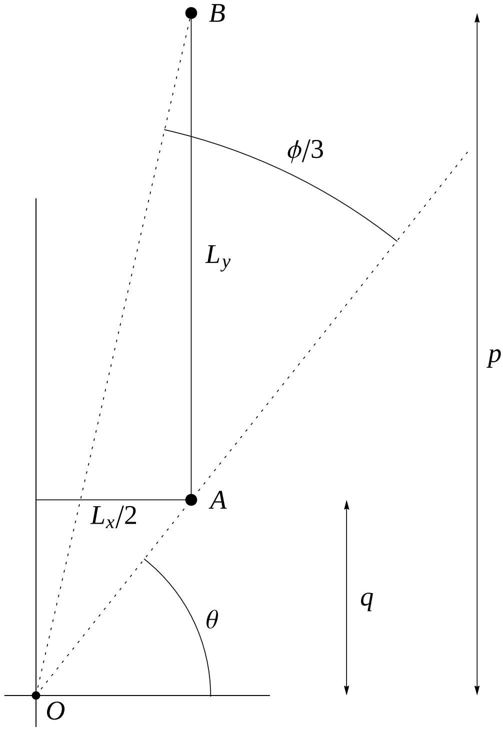
Solution 258.1 – Battersea Power Station

> Assuming that the chimneys of Battersea Power Station are at the corners of a 50 m × 160 m rectangle, from which points will they appear regularly spaced along the skyline?

Robin Marks

David Singmaster asks: '... It appears to me that there will be some point where the chimneys will appear regularly spaced along the skyline ... I want to be able to go to a correct viewpoint and take a photo.'



Let O be the origin; O is also the observation point for looking at the power station. The size of the rectangle formed by chimneys A, B, C, D is L_x by L_y. In this case we have chosen O to be on the line passing through the midpoints of the short sides, at a distance q away from the centre of one of the short sides, and a distance p away from the other short side. Let θ be the angle between OA and the abscissa. Let ϕ be the minimum angle of view from O containing all four chimneys.

We can see that $\phi = \pi - 2\theta$. When the chimneys are all lined up such that the three angles between adjacent chimneys are equal, the viewing angle between any two adjacent chimneys is $\phi/3$. This leads to two equations,

$$\tan\theta = \frac{2q}{L_x} \quad \text{and} \quad \tan\left(\theta + \frac{\phi}{3}\right) = \frac{2p}{L_x};$$

so

$$p - q \;=\; L_y \;=\; \frac{L_x}{2}\left(\tan\frac{\pi - 2\theta}{3} - \tan\theta\right);$$

hence

$$\frac{2L_y}{L_x} \;=\; \tan\frac{\pi - 2\theta}{3} - \tan\theta.$$

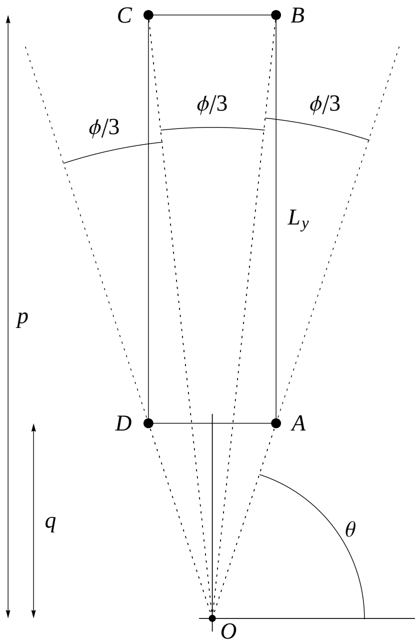
Substituting $L_y = 160\,\mathrm{m}$ and $L_x = 50\,\mathrm{m}$, and solving the equation using MATHEMATICA we get six solutions for θ, of which only two have an absolute value of less than π.

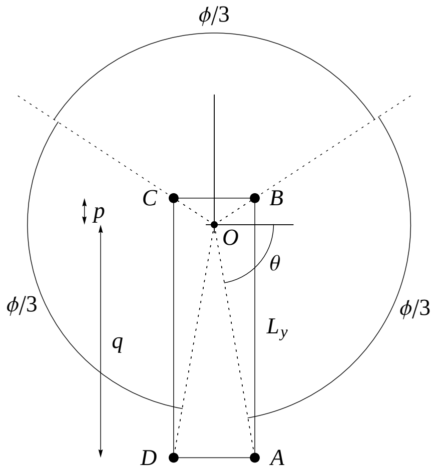
First solution (left diagram, below):

$$\theta = 71.9^\circ, \quad \phi/3 = 12.1^\circ, \quad p = 236.5\,\mathrm{m}, \quad q = 76.5\,\mathrm{m}.$$

Second solution (right):

$$\theta = -80.1^\circ, \quad \phi/3 = 113.4^\circ, \quad p = 16.4\,\mathrm{m}, \quad q = -143.6\,\mathrm{m}.$$





View from O in left-hand diagram, above.

It might seem difficult to take a single photograph of all four chimneys from the roof of the power station. However it should be possible using the setting ‘panoramic view’ on a camera or smartphone.

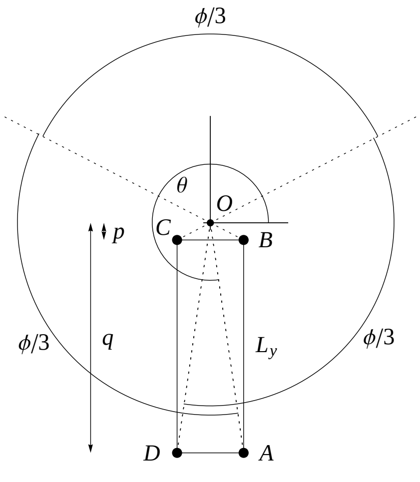
The other solutions to the equation do not lead to any further answers to the question posed by David but they are interesting to look at.

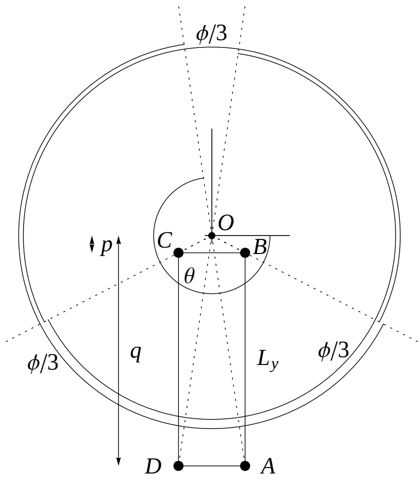
Left, below:

$$\theta = 278.2^\circ, \quad \phi/3 = -125.5^\circ, \quad p = -12.9\,\mathrm{m}, \quad q = -172.9\,\mathrm{m};$$

right, below:

$$\theta = -261.8^\circ, \quad \phi/3 = 234.5^\circ, \quad p = -12.9\,\mathrm{m}, \quad q = -172.9\,\mathrm{m}.$$

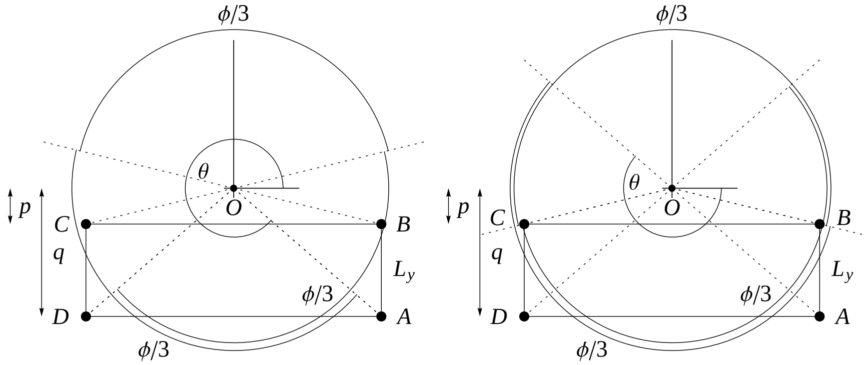




Other solutions to the equation (but not to the physical problem) with O on the line passing through the midpoints of the long sides:

$$\begin{aligned}
\theta &= 319.046^\circ, & q &= -69.4298\,\mathrm{m},\\
\theta &= -220.954^\circ, & q &= -69.4298\,\mathrm{m},\\
\theta &= (515.477 + 107.112\,i)^\circ, & q &= (-2.78508 + 77.4952\,i)\,\mathrm{m},\\
\theta &= (515.477 - 107.112\,i)^\circ, & q &= (-2.78508 - 77.4952\,i)\,\mathrm{m},\\
\theta &= (-24.5231 + 107.112\,i)^\circ, & q &= (-2.78508 + 77.4952\,i)\,\mathrm{m},\\
\theta &= (-24.5231 - 107.112\,i)^\circ, & q &= (-2.78508 - 77.4952\,i)\,\mathrm{m}.
\end{aligned}$$

The first two are illustrated on the next page. I would be interested if anyone could interpret the solutions θ which are not real numbers.



Tony Forbes

It is interesting to compute exact values for the solutions given in the previous article. They were obtained by MATHEMATICA but required quite a lot of cleaning up by hand before I felt they could be presented here. I shall concentrate on q, the distance of the observer from AD.

There are three distinct values of q when the AD is the short side. Let

$$\sigma \;=\; \frac{1+\sqrt{3}i}{2},$$

the sixth root of 1 that has positive real and imaginary parts, and let

$$\delta \;=\; \left(3696 + \frac{5\sqrt{4960941}\,i}{9}\right)^{1/3}, \qquad \overline{\delta} \;=\; \left(3696 - \frac{5\sqrt{4960941}\,i}{9}\right)^{1/3} \;=\; \frac{743}{3\delta},$$

where in each case the cube root is the one nearest the real axis. Then

$$q \;=\; \begin{cases} 5(\delta+\overline{\delta}) - 80 & \approx\; 76.4627, \\ 5(-\sigma\delta+\sigma^2\overline{\delta}) - 80 & \approx\; -143.583, \\ 5(\sigma^2\delta-\sigma\overline{\delta}) - 80 & \approx\; -172.88. \end{cases}$$

When the observer is looking at the long side there is only one real value,

$$q \;=\; 5\left(\frac{\omega}{3} - \frac{181}{\omega}\right) - 25 \;\approx\; -69.4298,$$

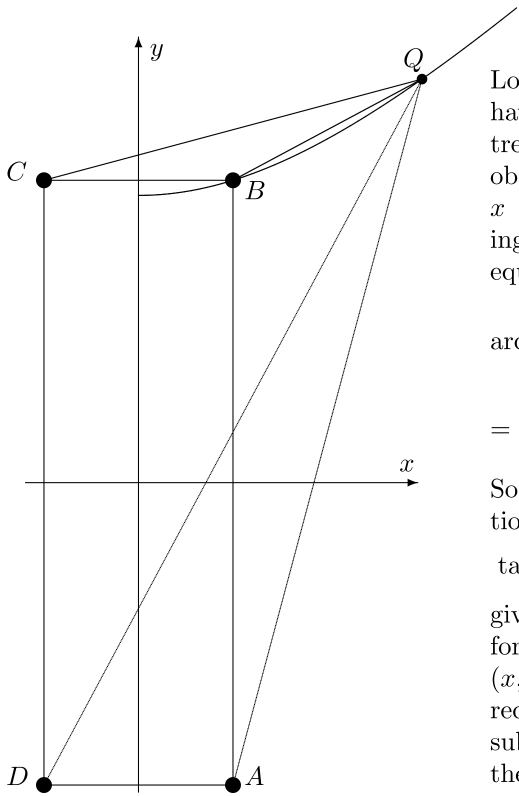
where $\omega = \left(3(-10395 + 16\sqrt{491583})\right)^{1/3} \approx 13.5162$.

Getting back to the original problem, there is one small point that is bothering me. In his analysis, Robin is clearly making the assumption that the observer must lie on the perpendicular bisector of one of the sides of the rectangle formed by the four chimneys. Then, ignoring positions inside the rectangle and positions where the chimneys are ordered in a bizarre manner, there is essentially a unique solution, Robin's first, where the observer is $5(\delta + \overline{\delta}) - 80 \approx 76.4627\,\mathrm{m}$ away from a short side.

What we seem to lack is a rigorous proof that there exist no others. So I am leaving the problem open.

Prove that there are no further solutions to David Singmaster's problem.

One might attempt to dismiss this possibility quickly by arguing that the two chimneys furthest away from the observer always subtend a smaller angle than the nearest pair. Unfortunately this doesn't always work.



Look at the diagram, where I have placed the origin at the centre of the rectangle. Assume the observer is at $Q = (x, y)$ with $x > 25$ and $y > 80$. Ignoring $\angle BQD$, we want to solve the equation $\angle CQB = \angle DQA$, or

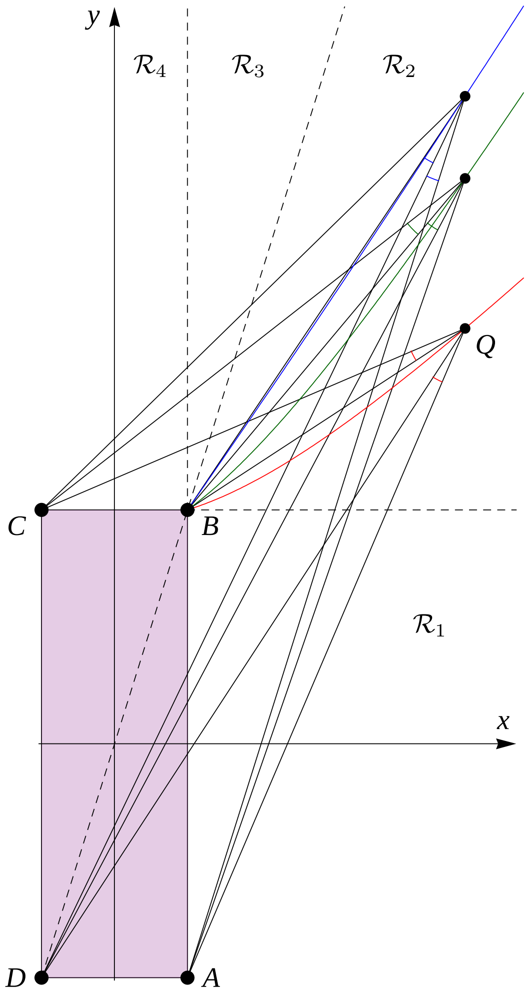
$$\arctan\frac{x+25}{y-80} - \arctan\frac{x-25}{y-80} = \arctan\frac{x+25}{y+80} - \arctan\frac{x-25}{y+80}.$$

Some straightforward manipulation using the sum formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - (\tan\alpha)(\tan\beta)}$$

gives $y^2 = x^2 + 5775$. Hence, for any point on the curve $(x, \sqrt{x^2 + 5775})$ and outside the rectangle $ABCD$, the angles subtended by AD and BC are the same.

We can perform a similar exercise for the other two pairs of angles. The results are illustrated by the diagram on the right. I have assumed the observer is in the wedge-shaped region $\mathcal{R}_2$ defined by $80 \le y \le 16x/5$, where he or she sees chimneys $ADBC$ in that order from left to right.



There are three curves. The leftmost one (blue if you are watching this in colour) is where chimneys A, D and B appear equally spaced, and on the middle curve (green) it is D, B and C that are equally spaced. The rightmost curve (red) is the one we have dealt with, where line segments AD and BC subtend equal angles at Q. From the drawing I imagine one could probably convince oneself that the curves do not meet in $\mathcal{R}_2$ except at B; nevertheless I would like to see a proof.

To complete the analysis we should also consider three more regions,

$$
\begin{aligned}
\mathcal{R}_1 &= \{\{x, y\} : x \ge 25,\ 0 \le y \le 80\} && (ADCB),\\
\mathcal{R}_3 &= \{\{x, y\} : x \ge 25,\ y \ge 16x/5\} && (ABDC),\\
\mathcal{R}_4 &= \{\{x, y\} : 0 \le x \le 25,\ y \ge 80\} && (BADC).
\end{aligned}
$$

However it is all becoming rather messy—so I shall gladly leave it for someone else to find a neater solution.

Ralph Hancock

Many thanks for M500 **258**. I was intrigued by David Singmaster's Problem 258.1, as I often go past Battersea Power Station.

On the face of it, the question looks silly because, in a view such as the one illustrated, the rear two chimneys are farther away from the viewer than the front two and subtend a smaller angle of arc, so that their spacing is closer to the viewer's eye. The only way you could see an evenly spaced set would be from an infinite distance.

The perspective in the view supplied looks wrong. I thought at first it had been done from an architect's axonometric drawing with the vertical measurements fudged to create an illusion of natural perspective, but no, the rear two chimneys are indeed closer together. I think it's just inaccurately drawn – note the rooflines of the main building and the side extension, which are out of true with each other. Incidentally, it's a view from the south side, not from across the river.

Here is a photograph of the north side taken from roughly the same angle, and from the Embankment just east of Chelsea Bridge. It is about as close as you will get to even spacing. It is tempting to think that if you hooked up your helium-filled pig and advanced across the river towards the near corner of the building, your changing angle to the front of the building would bring the leftmost chimney in closer and you could find an equally spaced view. But I think that is an illusion: if you can only see the chimneys equally spaced from infinity, and the spacing gets more uneven as you get closer, getting closer still is not going to take you to a sweet spot where the spacing corrects itself.

There is one way to do it, which is to start at the midpoint of any side and advance towards the building until the rear pair of chimneys is framed by the front pair in such a way that the spacing is equal. If you did this while standing on the ground, the vast bulk of the building would probably hide the rear pair of chimneys, so you would need to be in midair.

Unfortunately Ralph's photograph was not suitable for publication; so I substituted one of my own, taken from the eastern edge of the railway crossing downstream from Chelsea Bridge.—TF

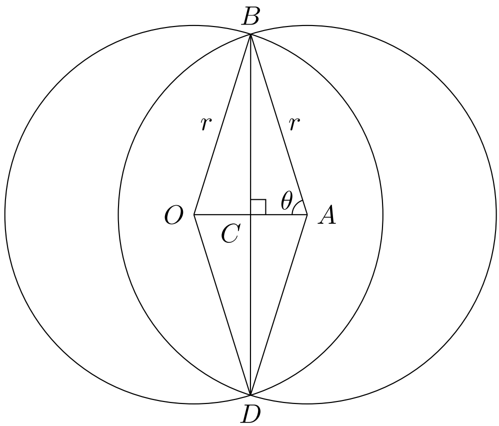
Solution 259.5 – Two darts

> Two darts are thrown and hit a dartboard of radius r at random. By this we mean that for each dart's landing point (x, y), x and y are chosen independently and uniformly at random on the interval $[-r, r]$ subject to $x^2+y^2 \le r^2$. Show that the probability of the darts' separation exceeding r is $3\sqrt{3}/(4\pi)$.

Reinhardt Messerschmidt

The dartboard can be represented as a circle with centre at the origin and radius r.

Step 1. We will first find the conditional probability that the darts are more than r apart, given that the distance of the first dart from the origin is s. Rotate the dartboard so that the first dart is at $A = (s, 0)$ and draw a circle with centre at A and radius r.



If $B = (x, y)$ then

$$x^2 + y^2 \;=\; (x-s)^2 + y^2 \;=\; r^2;$$

therefore $s^2 - 2sx = 0$; therefore $s = 0$ or $s = 2x$; therefore the length of AC is $s/2$. It follows that

$\theta = \arccos(s/(2r))$;

length of $AC = r\cos\theta$; length of $BC = r\sin\theta$;

area of slice BAD of circle $A = (2\theta/(2\pi))\pi r^2 = \theta r^2$;

area of triangle $ABD = (r\sin\theta)(r\cos\theta) = r^2\sin\theta\cos\theta$;

area of intersection of circle O and circle $A = 2(\theta r^2 - r^2\sin\theta\cos\theta)$.

The conditional probability is

$$1 - \frac{\text{Area of intersection of circle } O \text{ and circle } A}{\text{Area of circle } O} \;=\; 1 - \frac{2\theta - \sin(2\theta)}{\pi},$$

where $\theta = \arccos(s/(2r))$.

Step 2. Let S be the random variable for the distance of the first dart from the origin, and let $\Theta = \arccos(S/(2r))$. We will find the probability density function f_Θ of Θ. If $s \in [0, r]$ then

$$\mathbb{P}[S \le s] \;=\; \frac{\text{Area of a circle with radius } s}{\text{Area of a circle with radius } r} \;=\; \frac{s^2}{r^2}.$$

Note that

$$\mathbb{P}[\Theta \le \theta] \;=\; \mathbb{P}[S \ge 2r\cos\theta] \;=\; 1 - \mathbb{P}[S < 2r\cos\theta].$$

It follows that if $\theta \in [\pi/3, \pi/2]$ then

$$\mathbb{P}[\Theta \le \theta] \;=\; 1 - \frac{(2r\cos\theta)^2}{r^2} \;=\; 1 - 4\cos^2\theta$$

and

$$f_\Theta(\theta) \;=\; \frac{d}{d\theta}\mathbb{P}[\Theta \le \theta] \;=\; 8\cos\theta\sin\theta \;=\; 4\sin(2\theta).$$

Step 3. The probability that the darts are more than r apart is

$$\begin{aligned}
&\int_{\pi/3}^{\pi/2} \left(1 - \frac{2\theta - \sin(2\theta)}{\pi}\right) 4\sin(2\theta)\, d\theta \\
&= 1 - \frac{8}{\pi}\int_{\pi/3}^{\pi/2} \theta\sin(2\theta)\, d\theta + \frac{4}{\pi}\int_{\pi/3}^{\pi/2} \sin^2(2\theta)\, d\theta \\
&= 1 - \frac{2}{\pi}\Big[\sin(2\theta) - 2\theta\cos(2\theta)\Big]\Big|_{\pi/3}^{\pi/2} + \frac{1}{2\pi}\Big[4\theta - \sin(4\theta)\Big]\Big|_{\pi/3}^{\pi/2} \\
&= 1 - \frac{2}{\pi}\left(\pi - \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) + \frac{1}{2\pi}\left(2\pi - \frac{4\pi}{3} - \frac{\sqrt{3}}{2}\right) \\
&= \frac{3\sqrt{3}}{4\pi}.
\end{aligned}$$

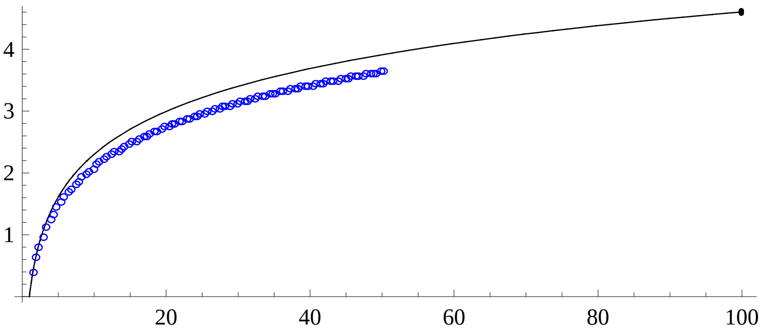
Prime density and centre of mass

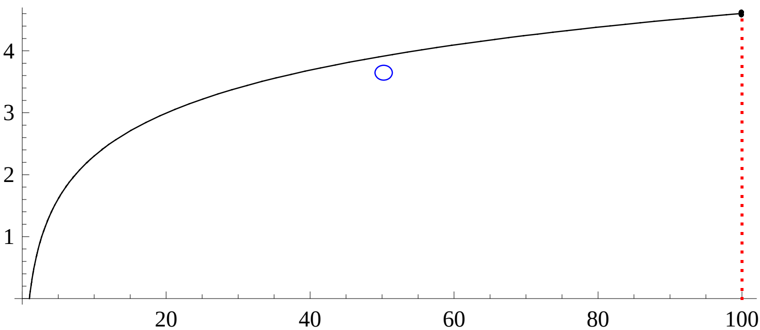
Robin Whitty

Suppose that $(\overline{x}, \overline{y})$ is the centre of mass of the arc of the log curve in the interval $[1, x]$. We denote by $\pi(x)$ the number of primes not exceeding x. We observe that $\frac{1}{2}\pi(x)$ is asymptotic to $\overline{x}/\overline{y}$.

This follows from the prime number theorem $\pi(x) \sim x/\log x$ and the fact that $(\overline{x}, \overline{y}) = (x/2, -1 + \log x)$, in the limit, as $x \to \infty$. This is presumably a well-known fact but perhaps not an obvious one without recourse to mathematical software.

The centres of mass of arcs on $\log(x)$ shadow the log curve. The centres of mass for the integers $2, \ldots, 100$ are plotted against $\log x$ in the first picture below, with the centre for $x = 100$ plotted in the second picture.





We apply the formulae

$$\overline{x} = \frac{1}{L}\int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \qquad \overline{y} = \frac{1}{L}\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

to the curve $y = \log x$, where L is arc length in the interval $[a, b]$. In our ratio $\overline{x}/\overline{y}$ the $1/L$ factors cancel, but it seems worthwhile to investigate the two coordinates separately. Calculating from 1 to x:

$$\overline{x} = \frac{\int_1^x t\sqrt{1+1/t^2}\,dt}{\int_1^x \sqrt{1+1/t^2}\,dt} \quad \text{and} \quad \overline{y} = \frac{\int_1^x \log t\sqrt{1+1/t^2}\,dt}{\int_1^x \sqrt{1+1/t^2}\,dt}.$$

The common integrand in the denominators has an elementary antiderivative which, with the assumption that $t > 0$, and ignoring the constant of integration, may be expressed as:

$$\begin{aligned}
\int \sqrt{1+1/t^2}\,dt &= \sqrt{1+t^2} - \tanh^{-1}\left(1/\sqrt{1+t^2}\right) \\
&= \sqrt{1+t^2} - \frac{1}{2}\log\left(\frac{1+\sqrt{1+t^2}}{-1+\sqrt{1+t^2}}\right) \\
&= \sqrt{1+t^2} - \log\left(1+\sqrt{1+t^2}\right) + \log t.
\end{aligned}$$

We observe that

$$\lim_{t\to\infty}\left(-t+\sqrt{1+t^2} - \log\left(1+\sqrt{1+t^2}\right) + \log t\right) \;=\; 0,$$

so that $\int_1^x \sqrt{1+1/t^2}\,dt$ may be replaced with $x + C$ in the limit, for a constant C.

The numerator of $\overline{x}$, again because $t > 0$, simplifies to

$$\int_1^x \sqrt{1+t^2}\,dt \;=\; \frac{1}{2}\left[t\sqrt{1+t^2} + \sinh^{-1} t\right]_1^x.$$

Since $\displaystyle\lim_{x\to\infty}\frac{\sinh^{-1} x}{x} = 0$ we find that, in the limit, $\overline{x}$ is $x/2$.

The integrand in the numerator of $\overline{y}$ does not have a closed form antiderivative; however MATHEMATICA offers one involving a hypergeometric series:

$$\begin{aligned}
\int \log t\sqrt{1+1/t^2}\,dt &= \log(t)\left(t\sqrt{1+\frac{1}{t^2}} - \sinh^{-1}\left(\frac{1}{t}\right)\right) \\
&\quad -t\cdot {}_3F_2\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2};-\frac{1}{t^2}\right) \\
&= \log(t)\left(\sqrt{1+t^2} - \log\left(1+\sqrt{1+t^2}\right) + \log t\right) \\
&\quad -t\cdot {}_3F_2\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2};-\frac{1}{t^2}\right) \quad (t > 0).
\end{aligned}$$

Now

$$\lim_{t\to\infty} {}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{1}{t^2}\right) = 1,$$

so $\int_1^x \log t\sqrt{1+1/t^2}\,dt$ may be replaced, in the limit, with $x\log x - x + D$ for D a constant.

We conclude that, in the limit,

$$\overline{y} \to \frac{x\log x - x + D}{x + C} \to -1 + \log x.$$

The ratio $x/(-1+\log x)$ is preferred as an estimate for $\pi(x)$ over the classic $x/\log x$ of the Prime Number Theorem, but obviously does not compete with $\mathrm{Li}(x)$ for large x.

Solution 258.5 – Integral

Suppose $a, b > 0$. Show that

$$I(a,b) = \int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}.$$

John Davidson

Define

$$J(a,b,c) = \int_0^\infty \frac{e^{-cx}(\cos ax - \cos bx)}{x}\,dx, \tag{1}$$

where $c \ge 0$. Then

$$\lim_{c\to\infty} J(a,b,c) = 0 \tag{2}$$

and

$$J(a,b,0) = I(a,b). \tag{3}$$

From (1),

$$\frac{\partial J}{\partial c} = -\int_0^\infty e^{-cx}(\cos ax - \cos bx)\,dx. \tag{4}$$

It is readily shown that

$$\int_0^\infty e^{-cx}\cos px\,dx = \frac{c}{p^2+c^2} \tag{5}$$

and

$$\int \frac{c}{p^2+c^2}\,dc = \frac{1}{2}\log(p^2+c^2) + A(p). \tag{6}$$

Substituting equation (5) in equation (4):

$$\frac{\partial J}{\partial c} = \frac{c}{b^2+c^2} - \frac{c}{a^2+c^2}. \tag{7}$$

Integrating equation (7) and making use of equation (6):

$$\begin{aligned} J(a,b,c) &= \frac{1}{2}\log(b^2+c^2) - \frac{1}{2}\log(a^2+c^2) + A(a,b) \\ &= \log\sqrt{\frac{b^2+c^2}{a^2+c^2}} + A(a,b). \end{aligned} \tag{8}$$

Using the condition (2) in equation (8) and noting that

$$\lim_{c\to\infty} \log\sqrt{\frac{b^2+c^2}{a^2+c^2}} = 0,$$

we have $A(a,b) = 0$ and so from (8)

$$J(a,b,c) = \log\sqrt{\frac{b^2+c^2}{a^2+c^2}}. \tag{9}$$

Setting $c = 0$ in equation (9) and using (3), it follows that

$$I(a,b) = \log\sqrt{\frac{b^2}{a^2}} = \log\frac{b}{a}.$$

Problem 260.1 – Iterated trigonometric integral

Tony Forbes

For positive integer n, define $F_n(x)$ by

$$F_1(x) = \sin(\arctan x), \qquad F_{n+1}(x) = \sin(\arctan F_n(x)).$$

Show that for $a \ge 0$,

$$\int_0^a F_n(x)\,dx = \frac{\sqrt{na^2+1}-1}{n}.$$

MATHEMATICIAN: “Look at all these awards we can get nowadays – Nevanlinna, Gauss, Abel, Wolf, etc. etc.”

ANOTHER MATHEMATICIAN: “Yes – I can remember when it was just Fields around here.”

[Sent by Jeremy Humphries]

Rational fundamental constants

Tony Forbes

Recall (from *Wikipedia*, or elsewhere) that (i) one metre is the distance travelled by light in a vacuum in 1/299792458 seconds, (ii) one second is the total time elapsed by 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of caesium-133, and (iii) one kilogram is the mass of a certain collection of atoms, the *international prototype kilogram*, consisting of 90% platinum, 10% iridium and approximately 0% impurities (mainly oxides of platinum and iridium as well as a small amount of Os-186 from the decay of Pt-190).

Therefore c, the speed of light, is rational when measured in metres per second. Is it possible there might be other fundamental constants that are rational, or rational expressions involving π, say?

Well, it seems that we can look forward to seeing a few more if a suggestion arising from the 2011 meeting of the Conférence Générale des Poids et Mesures goes ahead. Amongst their recommendations is a proposal to abandon (iii) above as well as the present SI definitions of the ampere, kelvin and mole, and instead assert the following *equalities*:

(1) Planck's constant h is exactly $6.62606\text{X} \times 10^{-34}$ joule–seconds,

(2) the electron charge e is exactly $1.60217\text{X} \times 10^{-19}$ coulombs,

(3) Boltzmann's constant k is exactly $1.38065\text{X} \times 10^{-23}$ joules per kelvin,

(4) Avogadro's number N_{A} is exactly $6.02214\text{X} \times 10^{23}$ mole^{-1},

in each case the X denoting 0 or more possible additional digits that the CGPM have yet to decide. And presumably once the Xs have been determined, the fundamental constants h, e, k and N_{A} (as well as c) will be fixed for the rest of eternity—provided new physics does not discover anything that would excessively compromise their static nature. Four new rational numbers! Thus Planck's constant (1) together with (i) and (ii) will provide a new SI definition of the kilogram. And with this in place, new SI definitions of the ampere, kelvin and mole arise from e, k and N_{A} respectively.

In common with just about everyone else with a scientific mind, I have often puzzled about the nature of the dimensionless number α, the fine structure constant. Is it rational, or perhaps some rational expression involving mathematical objects like π, e (not the electron charge) and possibly

Euler's constant,

$$\gamma \;=\; \lim_{n\to\infty}(1+1/2+\cdots+1/n-\log n) \;\approx\; 0.57721566\,?$$

Nobody knows, although Eddington [1] once built a substantial cosmological theory from first principles on the basis that $\alpha = 1/137$.

The good news is that with the new definitions of h and e (assuming for now there are no new digits X in each case) as well as the current value of the permeability of free space, a rational multiple of π,

$$\mu_0 \;=\; 4\pi \times 10^{-7} \text{ henries per metre,}$$

we can show that fine structure constant is also a rational multiple of π:

$$\alpha = \frac{e^2 c\mu_0}{2h} = \frac{549679902143612483\,\pi}{236645000000000000000} = 0.0072973033125581753 98\ldots.$$

Great news indeed! The nature of the mysterious α is determined.

And now the bad news. This is nonsense. The number α is handed to us by God and mere mortals have no business meddling around with it. So obviously something has gone wrong. Readers might like to pause here and ponder for a few trillion Cs-133 hyperfine transition periods before continuing.

In fact it didn't take me more than a day or two of concentrated effort to realize that $\mu_0 = 4\pi\times 10^{-7}$ H/m is simply incompatible with the new h and e. It will have a slightly different value in terms of the new kilogram and ampere (together with the old metre and second). If we work backwards from the currently accepted value of the fine structure constant, which I quote from *Wikipedia*,

$$\alpha \;\approx\; 0.0072973525698,$$

we find that (1) and (2) imply $10^7\mu_0 \approx 12.566455438$, differing significantly from the true value of $4\pi \approx 12.566370614$. So the obvious rationality of μ_0/π when expressed in current SI units must be sacrificed along with that of its companion, $\pi\epsilon_0$, where $\epsilon_0 = 1/(\mu_0 c^2)$, the permittivity of free space.

Finally, I offer this meaningless curiosity for you to think about:

$$chkN_{\mathrm{A}}^2 \;=\; \frac{1243286942372990528443236647 19}{1250000000000000000000000000000} \;\approx\; 0.99463\,\frac{\mathrm{kg}^2\mathrm{m}^5}{\mathrm{mol}^2\mathrm{s}^4\mathrm{K}}.$$

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[1] Arthur S. Eddington, *Fundamental Theory*, Cambridge, 1953.

Solving a nonlinear ordinary differential equation

Tommy Moorhouse

We consider a transformation that allows a class of first order inhomogeneous nonlinear ordinary differential equations (ODEs) to be solved. The first order nonlinear equation becomes a second order homogeneous linear equation, which we know how to solve using elementary or special functions.

The class of equations to be considered here is defined by

$$\dot{u} \;=\; au^2 + bu + c.$$

Here $u = u(t)$ is a function of t and the dot denotes differentiation with respect to t. Also a cannot be zero, otherwise the equation is linear. We take a to be independent of t. The substitution

$$u \;=\; \alpha \frac{\dot{\psi}}{\psi},$$

where α is a constant to be determined later, leads to

$$\alpha \frac{\ddot{\psi}}{\psi} - \alpha \frac{\dot{\psi}^2}{\psi^2} \;=\; a\alpha^2 \frac{\dot{\psi}^2}{\psi^2} + b\alpha \frac{\dot{\psi}}{\psi} + c.$$

Choosing $\alpha = -1/a$ we find that the squared terms cancel and we can rearrange to get

$$\ddot{\psi} - b\dot{\psi} + ac\psi \;=\; 0.$$

Up to this point we have not stipulated the properties of b and c. The above substitution works if b and c are functions of t, and a slight generalization allows a to depend on t. If a, b and c are in fact constants we can solve the linear equation for ψ to find

$$\psi = A_+ \exp(\rho_+ t) + A_- \exp(\rho_- t),$$

where $\rho_\pm$ are the solutions to $\lambda^2 - b\lambda + ac = 0$ and $A_\pm$ are constants. This leads to

$$u \;=\; -\frac{1}{a}\,\frac{A_+\rho_+ \exp(\rho_+ t) + A_-\rho_- \exp(\rho_- t)}{A_+ \exp(\rho_+ t) + A_- \exp(\rho_- t)}.$$

As usual if there is a repeated root ($\rho_+ = \rho_- \equiv \rho$) we get the solution $\psi = (A + Bt)\exp(\rho t)$ and

$$u \;=\; -\frac{1}{a}\,\frac{B + (A + Bt)\rho}{(A + Bt)}.$$

It isn't too hard to check that this is indeed a solution (try $a = 1, b = 2, c = 1$ for a simple case).

If we allow b and c to be functions of t we can still use our substitution and see whether the equation for ψ is a familiar one. As an example take

$$\dot{u} \;=\; n(n+1)u^2 + \frac{2t}{1-t^2}u + \frac{1}{1-t^2}$$

for $-1 < t < 1$. We find

$$(1-t^2)\ddot{\psi} - 2t\dot{\psi} + n(n+1)\psi \;=\; 0,$$

which is the differential equation for the Legendre polynomials. For example, if $n = 2$ we have $\psi(t) = P_2(t) = (3t^2 - 1)/2$ and $u = -t/(3t^2 - 1)$; u satisfies the nonlinear equation

$$\dot{u} \;=\; 6u^2 + \frac{2t}{1-t^2}u + \frac{1}{1-t^2}.$$

Many second degree nonlinear first order ODEs can be solved using this transformation. *Applied Complex Variables* by John W. Dettman (Dover, 1970) gives some background on special second order ODEs.

Problem 260.2 – Right-angled triangle

Show that for $|a| < 1$,

$$\arctan a \;\approx\; \frac{3a}{1 + 2\sqrt{1+a^2}}.$$

If you prefer not to work in radians, show that there is an even better approximation:

$$\arctan a \;\approx\; \frac{172a}{1 + 2\sqrt{1+a^2}} \text{ degrees.}$$

Observe that this gives a convenient way of calculating the angle opposite the smallest side of a right-angled triangle with sides a, b, c, $a \le b \le c$:

$$\arctan \frac{a}{b} \;\approx\; \frac{172a}{b + 2c} \text{ degrees.}$$

For example, with a $(3, 4, 5)$ triangle the angle we want will be $\arctan 3/4 \approx 258^\circ/7$, about 36.8571°, the true value being nearer 36.8699°.

Fork handles revisited

Tony Forbes

Attach two candles to each other with sticky tape and bring their wicks together so that when lit they produce a single flame. Recall that in M500 **241** Judith Furner's young grand-niece Lily asked: Is the light from this arrangement significantly brighter than the combined light from two separated candles?

Then a few years later I discovered amongst my possessions four candles of no great value to me. So I was willing to sacrifice them in the interests of scientific enquiry. Behold the illustrations below. Even from just the first picture (where the candles are about 20 cm long) you can see that the massive single flame on the left must surely outshine the other two. So Lily's question seems to have a positive answer. However, without an accurate illumination-measuring device to verify this observation, I thought it would be a good idea to continue the experiment for a few tens of minutes and thereby confirm that the big flame does indeed consume the wax in its candles more rapidly than the other two. It is pleasing to conduct a scientific experiment and get the result you want first time and without cheating.

You might notice that the fourth candle is not necessary if you only want to measure the rates of burning. I included it because I was quite keen to get a rough visual indication of the relative brightness—and of course the *Two Ronnies* joke won't work without it.

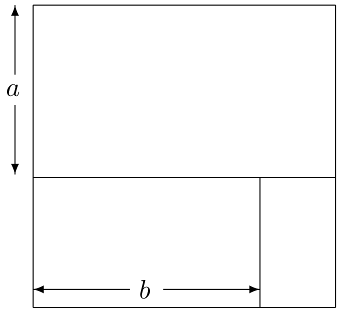



Solution 252.1 – Three pieces

Divide a square into three same-shape pieces but different sizes.

Steve Moon

I offer one possible division into three similar rectangles. Consider the unit square on the right. To establish the longer and shorter sides of each rectangle we need $b > a > 1/2$. Then



$$\frac{1}{a} = \frac{b}{1-a} = \frac{1-a}{1-b}$$

for similarity. Hence $b = (1-a)/a$ and $(1-a)^2 = b(1-b)$. Therefore

$$a^3 - a^2 + 2a - 1 = 0,$$

which has solution

$$a = \left(\sqrt{\frac{23}{108}} + \frac{11}{54}\right)^{1/3} - \left(\sqrt{\frac{23}{108}} - \frac{11}{54}\right)^{1/3} + \frac{1}{3},$$

giving $a \approx 0.57$ and $b \approx 0.75$.

Problem 260.3 – Three dice

Tony Forbes

One lasting memory of my visit to a summer fête in Guernsey was of boisterous, overweight and not exceedingly intelligent punters gambling quite large amounts of money at the Crown and Anchor stalls of which there were considerably more than a just a few. In this game players bet on numbers 1–6 (usually represented by ace, king, queen, jack, crown and anchor) and three dice are thrown. If a player has bet x on n, and n appears i times amongst the three dice, he loses x if $i = 0$ and wins ix if $i > 0$.

As is well known, the game is hideously biased against the player, which does not readily explain its popularity in places where it is legal. So to redress the balance we alter the rules. Now players bet on numbers 1–5 and, as before, three dice are thrown. But if a 6 shows, everyone loses. Otherwise, a successful player wins double the amount, $2ix$ instead of ix. Analyse the game and hopefully deduce that it is much less unfair.

Simon Singh, his book

Eddie Kent

Readers might be interested to know that Simon Singh, an old friend of M500, has produced another volume to go with *Fermat's Last Theorem* (or *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem* in the US), *The Code Book*, *Big Bang* and *Trick or Treatment? Alternative Medicine on Trial*. This new book is called *The Simpsons and their Mathematical Secrets*. I have not read it yet (and I really mean yet) but have seen a review of it in the London Mathematical Society's *Newsletter* (no. 437, 2014).

Among the facts and numbers included, we are told, are that many of the Simpsons' writers have advanced degrees in mathematics and related fields: Harvard, Princeton and Berkeley are involved. Also that, as a counterexample to Andrew Wiles's efforts, Homer Simpson showed that $3987^{12} + 4365^{12} = 4472^{12}$. Of course this isn't exactly true but it ain't half close. More numbers that are used include 87539319 which (cf Hardy and Ramanujan) is the smallest number that is the sum of two cubes in three different ways, and 8191, 8128 and 8208—given as the multiple-choice possible numbers of people attending a particular ball game. You might notice that $8191 = 2^{13} - 1$ and is thus Mersenne, that 8128 is the fourth perfect number (being $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 127$) and is also the sum of the first 127 integers, and that $8208 = 8^4 + 2^4 + 0^4 + 8^4$, which makes it narcissistic.

What is more, although Homer's daughter Lisa is the most gifted mathematically of the Simpsons, she has to disguise herself as a boy in order to attend maths classes. She then wins the prize, so it turns out that 'The best math student in the whole school is a girl!', thus mirroring the case of Sophie Germain who had to pretend to be a man in order to get her results in number theory accepted. Simon uses the maths in *The Simpsons* to explain some topics in an elementary way, including prime numbers, perfect numbers, topology, higher-dimensional geometry, P v NP, countability. I expect, therefore, that it won't be greatly useful as a textbook to a maths undergraduate; but it all sounds great fun. It is published by Bloomsbury at £12.91 or £6.29 paper.

My memory was jogged. Then I did a search through back numbers and found another almost-counter-example to Fermat's Last Theorem at the bottom of page 15 in M500 **221**: $1782^{12} + 1841^{12} = 1922^{12}$, nearly, the difference being only about 7×10^{29}. And just in case you were wondering, $87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$. — TF

M500 Winter Weekend 2015

The **thirty-fourth M500 Society Winter Weekend** will be held at

Florence Boot Hall, **Nottingham University**

from Friday 9$^{\text{th}}$ to Sunday 11$^{\text{th}}$ January 2015.

Cost: £205 to M500 members, £210 to non-members. This includes accommodation and all meals from dinner on Friday to lunch on Sunday. You can obtain a booking form either from the M500 web site,

http://www.m500.org.uk/winter/booking.pdf,

or by emailing the Winter Weekend Organizer at winter@m500.org.uk.

The Winter Weekend provides you with an opportunity to do some non module based, recreational maths with a friendly group of like-minded people. The relaxed and social approach delivers maths for fun. And as well as a complete programme of mathematical entertainments, on Saturday we will be running a pub quiz with Valuable Prizes.

M500 Mathematics Revision Weekend 2015

The M500 Revision Weekend 2015 will be held at

Yarnfield Park Training and Conference Centre,

Yarnfield

Staffordshire ST15 0NL

from Friday 15th to Sunday 17th May 2015.

We expect to offer tutorials for most undergraduate and postgraduate mathematics Open University modules, subject to the availability of tutors and sufficient applications. Application forms will be sent via email to all members who included an email address with their membership application or renewal form, and are included with this magazine mailing for those who did not.

Contact the Revision Weekend Organizer, **Judith Furner**, if you have any queries about this event. (Please note that the organizer is different from last year.)

ALPHA PARTICLE: "Doctor, doctor, I've just discovered I am missing a couple of electrons."

DOCTOR: "Are you sure?"

ALPHA PARTICLE: "I'm positive."

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Front cover: Battersea Power Station observed from points along various curves. See pages 1–7.