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Ramblings round representations – Part 3

Roger Thompson

1 Polynomial selection

If an arbitrary polynomial has a large $\|U_1\|$, it can be difficult to identify sufficient prime representations to determine H reliably if this is large. Instead, we could explore how the coefficients of U_1, U_2, f constrain one another, in the hope of deriving polynomials with small $\|U_1\|$. Using

$$f \;=\; X^3 + PX^2 + QX + R$$

as usual,

$$U_1 \;=\; AX^2 + BX + C, \;\; U_2 \;=\; DX^2 + EX + F \;\;\text{and}\;\; U_1U_2 \equiv 1 \bmod f,$$

we get

$$\begin{aligned} CD + BE + AF - ADQ - P(BD + AE - ADP) \;&=\; 0,\\ CE + BF - ADR - Q(BD + AE - ADP) \;&=\; 0,\\ CF - R(BD + AE - ADP) \;&=\; 1. \end{aligned}$$

Since $P = 0$ or $P = 1$, we have three equations with two unknowns, so we cannot pick arbitrary $A, \ldots, F$. Since Q, R are integers, our choice is further constrained. We will investigate the two cases $D = 0$, $A = D = 1$.

For $D = 0$, we have

$$Q \;=\; \frac{CE + BF}{AE}, \quad R \;=\; \frac{CF - 1}{AE}, \quad \text{with } BE + AF \;=\; PAE.$$

For example, if $P = 0$ and $A = E^2$, $B = -EF$, $C = F^2$, we get $Q = 0$, $R = (F^3 - 1)/E^3$, which is an integer if $F^3 - 1 = kE^3$ for some integer k. The Appendix gives examples of such polynomials.

For $A = D = 1$, we have, recalling that $P = 0$ or 1,

$$\begin{aligned} Q \;&=\; C + BE + F + P(1 - B - E),\\ R \;&=\; P(B^2 + E^2 + 3BE + 1 + C + F - 2B - 2E)\\ &\qquad - (BC + B^2E + BE^2 + EF) \end{aligned}$$

and

$$R \;=\; \frac{CF - 1}{B + E - P}.$$

2 Finding representations

Given an arbitrary cubic polynomial

$$f = X^3 + PX^2 + QX + R,$$

the obvious way to start searching for representations up to some maximum M is to use the parameters μ, and ν (see Part 1, section 2, where we defined ν using one of two possible roots). For the chosen root, it can be shown that

$$\frac{\partial N}{\partial b} = \frac{\partial N}{\partial c} = 0 \text{ if } b = \mu a, \quad c = \nu a,$$

but not for the rejected root. If $b = \mu a + x$, $c = \nu a + y$ for small x/a, y/a, Taylor series expansion gives

$$N \approx \frac{1}{2}\left(\frac{\partial^2 N}{\partial b^2}x^2 + 2\frac{\partial^2 N}{\partial b \partial c}xy + \frac{\partial^2 N}{\partial c^2}y^2\right),$$

where the derivatives are evaluated at $b = \mu a$, $c = \nu a$, i.e.

$$N \approx \frac{a}{2}(k_1 x^2 + k_2 xy + k_3 y^2),$$

where

$$\begin{aligned} k_1 &= -6R\mu + 2PR + 2Q\nu, \\ k_2 &= 4Q\mu + 6R - 2PQ - 4P\nu \quad \text{and} \\ k_3 &= -2P\mu + 2P^2 - 4Q + 6\nu. \end{aligned}$$

For each $a > 0$, calculate x, y such that $k_1 x^2 + k_2 xy + k_3 y^2 \approx 2M/a$, evaluate integer pairs b, c with x, y within this ellipse. We evaluate $N(aX^2 + bX + c)$, using $-a$, $-b$, $-c$ in what follows if this is negative. This may reveal units. Alternatively, if $|N(aX^2 + bX + c)|$ is a prime, the corresponding root can be ascertained by checking if $ar^2 + br + c \equiv 0 \bmod p$ from a list of precalculated roots r of $f \equiv 0 \bmod p$. Alternatively, it is easy to show that

$$\begin{aligned} r &\equiv jk^{-1} \bmod p, \quad \text{where} \\ j &= c(aP - b) - a^2 R, \quad \text{and} \quad k = a(aQ - c) - b(aP - b). \end{aligned}$$

It may be possible to find all representations $\leq M$ using 64 bit arithmetic. If this is the case, the rate of discovery of new representations suddenly drops to zero. In other words, there are no significant ‘outliers’.

If any prime has three distinct roots, and representations for two of these have been found, then we know the third must exist, so the search is not complete. If this is so, even after having exhausted the μ, ν method using 64 bit arithmetic, we need another method, described in the remainder of this section. The search should have accumulated some representations with large magnitude. These are important for further discoveries, particularly if we haven't found any units yet.

Suppose

$$N(iX^2 + jX + k) \;=\; n \;=\; q\prod_i p_i^{a_i},$$

where q and the p_i are primes. If we know representations for each root of $f \equiv 0 \bmod p_i$, then we can find at least a fractional representation for q. This is done by calculating the reciprocal for each representation. This is done as follows. If

$$\rho(p_r) \;=\; aX^2 + bX + c,$$

we want to find rationals t, u, v such that

$$(tX^2 + uX + v)(aX^2 + bX + c) \;\equiv\; 1 \bmod f.$$

Multiplying out the left hand side modulo f and equating the coefficients of X^2, X^1, X^0 with the right hand side gives

$$\begin{pmatrix} (aP-b)P - aQ + c & b - aP & a \\ (aP-b)Q - aR & c - aQ & b \\ (aP-b)R & -aR & c \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(I am grateful to Tony Forbes for pointing this out originally.) The determinant of the above 3×3 matrix is $N(aX^2 + bX + c) = p$ (see the definition of norm in Part 1, Introduction); so

$$tX^2 + uX + v \;=\; \frac{t'X^2 + u'X + v'}{p},$$

where t', u', v' are integers. In other words, division by a polynomial amounts to multiplication by some other polynomial divided by an integer.

As an example of how all this helps to find representations, we consider $f = X^3 + 141$, for which

$$\mu \;=\; -5.2048278633942, \quad \nu \;=\; 27.0902330875646$$

and

$$U_1 \;=\; 3356998050X^2 - 17472596988X + 90941859649.$$

Suppose we want to find $\rho(p_r)$ for $p = 83869$, $r = 20417$. This is done by factorizing the norm of $aX + b$ for small a, b values such that $ar + b \equiv 0 \bmod p$, i.e. $b = kp - ar$. For $a = -92$, $k = -1$, giving $b = -50623$, we find that

$$N(aX + b) = -23^3 \times 127133 \times 83869.$$

We will suppose we have already found that

$$\rho(127133_{31233}) = 1781X^2 - 9270X + 48248$$

using the μ, ν method. From the above, we get

$$\frac{1}{\rho(127133_{31233})} = \frac{3212X^2 + 12459X - 22166}{127133}.$$

Now

$$\frac{-92X - 50623}{\rho(127133_{31233})} \equiv -23(56X^2 + 215X - 398) \bmod f;$$

so

$$\rho(83869_{20417}) = 56X^2 + 215X - 398.$$

Since $215/56$, $-398/56$ are nowhere near μ, ν, the μ, ν method would only find this as

$$(56X^2 + 215X - 398)U_1 = 50564024X^2 - 263177041X + 1369791196.$$

If we find an integer representation, we can use it in further searches. It is also worth doing this even if q is composite, since it may be the product of prime / root combinations without representations. Searches for representations within a particular list of prime / root combinations may require multiple iterations, particularly where the units have large magnitude, since it requires the product of many reciprocal representations of small magnitude to find a representation of large magnitude.

For polynomials with units of small magnitude, I have typically searched for representations for primes less than 10^{12}, and less than a few million for those with units of very large magnitude. Some with very large H (notably $X^3 + 6876X + 573$) have required searches up to 4.9×10^{13}.

3 Notes

The interested reader may like to explore this topic in the context of quadratics. From the limited work I have done, there are interesting properties to find.

4 Appendix

Parameters for some polynomials

The n in the fourth column is the power of 3 described in Part 2, section 3 for cases where H is divisible by 3, and is zero unless otherwise specified. $H/h = 1$ unless otherwise specified. Polynomials are grouped by root pattern.

Polynomial$-X^3$	μ	ν	$H, n, H/h$
126	-5.01329793496458	25.1331561847202	27, 2, 3
$-126X + 714$	-13.3910166632046	53.3193272742243	18, 1, 2
$126X + 462$	-3.36442079327547	137.319327274224	18, 1, 2
$252X - 420$	1.64887714168911	254.718795828385	18, 1, 2
$-252X + 1596$	-18.4043145981692	86.7187958283848	18, 1, 2
$756X + 1302$	-1.71554365158636	758.943090020498	27, 1, 3
$-252X + 4830$	-21.7687353914447	221.877840542737	9, 1, 1

Polynomial$-X^3$	U_1	U_2
126	$X^2 - 5X + 25$	$-X - 5$
$-126X + 714$	$438977085X^2$ $-5878349460X$ $+23405962861$	$1065X^2 - 13590X - 372959$
$126X + 462$	$480768762662280X^2$ $-1617508421858295X$ $+6601884306324552$1	$-742455X^2$ $-21414105X$ -63641969
$252X - 420$	See below	See below
$-252X + 1596$	$236653278X^2$ $-4355441379X$ $+20522287297$	$507X^2 - 10701X - 368675$
$756X + 1302$	See below	See below
$-252X + 4830$	See below	See below

Units for $X^3 + 252X - 420$:

$$\begin{aligned} U_1 &= 2273263731341199594507304530175 5X^2 \\ &\quad + 374833260363940498722980612289 15X \\ &\quad + 579043000247571351020632059893 0341, \\ U_2 &= 294940994195070X^2 + 552909613600485X - 1713564367923239. \end{aligned}$$

Units for $X^3 + 756X + 1302$:

$$\begin{aligned}U_1 = {} & 1564034767124792884363895158617679915549988050910223899816097167752430667418644494740935245171154044112391860033146368X^2 \\ & -26831699156012870854923757560550252535753564767045905738014330501974235152863648143886983944223339786322206047855920 96X \\ & +118701337906118071405540889950761271535087972472983794378307098488448358148236956211147489161129360286886633329638861 2097,\end{aligned}$$

$$\begin{aligned}U_2 = {} & 6625746156754485703052429093491329881367240636156843760 64X^2 \\ & -345198064804615123619363130118929496746289936239126788487 36X \\ & -611702516007752668504586072497703313180502426794433490534 39.\end{aligned}$$

Units for $X^3 - 252X + 4830$:

$$\begin{aligned}U_1 = {} & 36581043675994078434930411122274486568534023550626689755548412095902182087151127939146203030795967024953840826709771275726262292125248647933180349590865806561311763656831727381705485060303401456905770X^2 \\ & -796323060125596514796332521812202138520384743581585214729630163885010819701077016115383845629537825394585047411853352267147340240392698305760280965595946673058395241542059146299533985216814453120236530X \\ & +811652297562910851226215502900946705033339116775505069400967237452132078972844598886559913394917006744753711356925327785331557465863901377818902754830318676259167343257503844670294513477581221237855364 1,\end{aligned}$$

$$\begin{aligned}U_2 = {} & -22321353469948476759855686869424819805488046612297621425273896337507493411945740714609313082277847 0X^2 \\ & +31766008916687831308932598780883967698485447315111490965595855855949372862870062313482522648279987 30X \\ & +17492653205826813588937800149098439669346567553196842237761775976629947406350578639940855094077326796 1.\end{aligned}$$

Units for $X^3 + 1049$ (see page 9):

$$\begin{aligned}U_1 = {} & 38930231244877070269125836499570655168635857474X^2 \\ & -39555979774458968503923588981901128839344663335 6X \\ & +40191786328608224897512996930431910885539128456 33,\end{aligned}$$

$$\begin{aligned}U_2 = {} & -181933606610136594883 06X^2 \\ & -207758450005866925314776X \\ & -232686109116432580407767.\end{aligned}$$

Polynomial$-X^3$	μ	ν	$H, n, H/h$
215	-5.99072641489509	35.8888029781218	42, 1, 2
$X^2 + 502X - 852$	2.68209397538128	506.511534117395	63, 1, 3
$X^2 - 358X + 2588$	-21.2808116841991	116.153757622544	21, 1, 1

Polynomial$-X^3$	U_1	U_2
215	$X^2 - 6X + 36$	$X + 6$
$X^2 + 502X - 852$	See below	See below
$X^2 - 358X + 2588$	See below	$-31138017063039X^2$ $-364877496598581X$ $+7328221300240795$

Units for $X^3 + X^2 + 502X - 852$:

$$U_1 = 18695266576979968589283984382003535814170132709995631898920X^2 + 50142461854264969306663487950936167059342502229695602513700X + 94693681546397889574350606106313069772281565948669021851428 21,$$

$$U_2 = 6042126649897280457514365480X^2 + 14357506456570038082826643900X - 41246510798446197319960187219$$

Units for $X^3 + X^2 - 358X + 2588$:

$$U_1 = 2852059528383008466720985389X^2 - 6069414173564447655809954 7069X + 3312774311848661394702310620 07$$

$$U_2 = -31138017063039X^2 - 364877496598581X + 7328221300240795$$

Polynomial$-X^3$	μ	ν	$H, n, H/h$
1719	-11.9791303926648	143.499564964465	102,1,3
$6876X + 573$	-0.083333249170746	6876.00694443042	13056,1,384

Polynomial$-X^3$	U_1	U_2
1719	$2300X^2 - 27552X + 330049$	$4X^2 + 48X + 1$
$6876X + 573$	$144X^2 - 12X + 990145$	$12X + 1$

Polynomial$-X^3$	μ	ν	$H, n, H/h$
$X^2+15X+6$	0.593461092033354	14.7587349757241	2
$23X+4$	−0.173685239711393	23.0301665624936	2
X^2+9X+1	0.887643829446456	8.90026773850791	3, 1
X^2+5X-1	1.19128244006093	5.22787141193659	3, 1
$X^2+17X+11$	0.344233492366654	16.7742632049003	3
$23X+1$	−0.0434746882925931	23.0018900485221	4
$X^2+10X+6$	0.385442572868276	9.76312340411104	4
$X^2+17X+12$	0.285543813739088	16.7959914558256	5
263	−6.40695857718556	41.0491182097716	5
$X^2+15X+1$	0.933054555759407	14.937536248264	6
$X^2+23X+1$	0.956442843996641	22.9583400698357	6
$7X+14$	−1.50906396767636	9.27727405853912	6, 1
$11X+20$	−1.50702970575668	13.2711385340331	7
$X^2+16X+10$	0.365804499477742	15.7680084323604	7
$X^2+16X+15$	0.0592239545254933	15.9442835222641	8
$X^2+15X+14$	0.0629801999916619	14.9409863055993	8
$X^2+16X+30$	−0.736274440302871	17.2783744917462	9
$X^2+22X+21$	0.0436402535579122	21.9582642181727	10
$X^2+16X+6$	0.619392086616828	15.7642544703467	11
43	−3.50339806038672	12.2737979695215	12
$14X+21$	−1.33141686498496	15.7726708683664	12
1049	−10.1607358881703	103.240553789151	13
267	−6.43927669563891	41.4642843629983	15,1
$X^2+106X+47$	0.555568532794011	105.753087861837	16
$X^2+101X+28$	0.722220519056714	100.79938195909	18, 1
91	−4.49794144527541	20.2314772451263	18, 2, 2
667	−8.73726037221036	76.3397188117975	19
$X^2+25X-26$	1.96651844921074	26.9006763618755	24, 1
514	−8.01040313265657	64.1665583476742	28, -, 2
$X^2+111X+40$	0.638889082560776	110.769290177255	40
217	−6.00924500691737	36.1110255531613	54, 2, 2
614	−8.49942325959926	72.2401957458169	54, 1
813	−9.33319160782525	87.1084655883797	162, 1
2198	−13.0019720874089	169.05127816176	324, 2
4291	−16.2499802761102	264.06185897397	576, 2
8827	−20.666695571702	427.112305853406	1296, 3

Polynomial$-X^3$	U_1	U_2
$X^2+15X+6$	$204866X^2+121580X+3023563$	$-98X^2+184X+91$
$23X+4$	$310393768X^2-53910816X+7148420177$	$-1128X^2-16864X-2895$
X^2+9X+1	X^2+X+9	$-X$
X^2+5X-1	X^2+X+5	X
$X^2+17X+11$	$2844X^2+979X+47706$	$13X^2+7X-1$
$23X+1$	X^2+23	$-X$
$X^2+10X+6$	$13X^2+5X+127$	$-X^2+X+1$
$X^2+17X+12$	$49X^2+14X+823$	$-7X-5$
263	$1809605356825147200X^2-1159406656223180961 4X+74282704205351372009$	$-241895804X^2-865225474X+4386145681$
$X^2+15X+1$	X^2+X+15	$-X$
$X^2+23X+1$	X^2+X+23	$-X$
$7X+14$	$4X^2-6X+37$	$-2X-3$
$11X+20$	$4X^2-6X+53$	$-2X-3$
$X^2+16X+10$	$82X^2+30X+1293$	$-2X^2-6X-3$
$X^2+16X+15$	X^2+16	$X+1$
$X^2+15X+14$	X^2+15	$X+1$
$X^2+16X+30$	$273X^2-201X+4717$	$-3X^2-15X-17$
$X^2+22X+21$	X^2+22	$X+1$
$X^2+16X+6$	$21X^2+13X+331$	X^2+3X+1
43	$4X^2-14X+49$	$-2X-7$
$14X+21$	$9X^2-12X+142$	$3X+4$
1049	See page 6	See page 6
267	$37741839630X^2-243030148380X+1564938370801$	$-19230X^2+100080X+1441801$
$X^2+106X+47$	$81X^2+45X+8566$	$9X+4$
$X^2+101X+28$	$324X^2+234X+32659$	$-18X-5$
91	$4X^2-18X+81$	$2X+9$
667	$48310693141008X^2-422103104734944X+3688024729987585$	$297456X^2+7875552X+46103041$
$X^2+25X-26$	$X^2+2X+27$	$-X+1$
514	$9228X^2-73920X+592129$	$-12X^2-96X+1$
$X^2+111X+40$	$1296X^2+828X+143557$	$36X+13$
217	$X^2-6X+36$	$-X-6$
614	$4X^2-34X+289$	$2X+17$
813	$9X^2-84X+784$	$3X+28$
2198	$X^2-13X+169$	$-X-13$
4291	$16X^2-260X+4225$	$4X+65$
8827	$9X^2-186X+3844$	$-3X-62$

Solution 281.4 – Pythagorean triple generator

Suppose $x^2 + y^2 = z^2$, where x, y and z are integers. Let

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{1}$$

Show that X, Y and Z are also integers and that they too satisfy $X^2 + Y^2 = Z^2$. For example, if you start with the vector $(1, 0, 1)$ and then repeatedly multiply by the matrix, you get

$$(3, 4, 5),\ (21, 20, 29),\ (119, 120, 169),\ (697, 696, 985),\ \ldots.$$

Thus (1) preserves Pythagorean tripleness. Can you find other 3×3 matrices that have the same property?

Tommy Moorhouse

Matrix description of Pythagorean triples We take a column vector V defined as

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $x^2 + y^2 = z^2$. It is easy to see that this condition is equivalent to $v^{\mathrm{T}} B v = 0$, where T denotes the transpose (and so is a row vector) and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We will call vectors with integer components satisfying $x^2 + y^2 = z^2$ 'Pythagorean vectors'.

Now, suppose that there is a 3×3 matrix W with integer components and transpose W^{T} such that $W^{\mathrm{T}} B W = B$. Then if $v_1 = Wv$,

$$v_1^{\mathrm{T}} B v_1 = v^{\mathrm{T}} W^{\mathrm{T}} B W v = v^{\mathrm{T}} B v = 0$$

so that v_1 satisfies the same equation as v and is Pythagorean.

Checking the condition Given a matrix W it is easier to check (we assume that W is invertible) that

$$B W^{\mathrm{T}} B = W^{-1}$$

using the fact that $B^2 = 1$ (the unit matrix). In components

$$B \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} B = \begin{pmatrix} a_{11} & a_{21} & -a_{31} \\ a_{12} & a_{22} & -a_{32} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix}.$$

To find W^{-1} we write U_{ik} for the cofactor of the element a_{ij} (that is, the determinant of the 2×2 matrix form by deleting row i and column j from W) and $D = \det W$. In terms of these quantities we have

$$W^{-1} = \frac{1}{D} \begin{pmatrix} U_{11} & -U_{21} & U_{31} \\ -U_{12} & U_{22} & -U_{32} \\ U_{13} & -U_{23} & U_{33} \end{pmatrix}.$$

Now from $BW^{\mathrm{T}}B = W^{-1}$ we have a series of equations linking U_{ij} and a_{ij}:

$$\begin{aligned} U_{11} &= Da_{11}, \\ U_{21} &= -Da_{21}, \\ &\cdots, \\ U_{33} &= Da_{33}. \end{aligned}$$

We can substitute the expressions for U_{ij} into the equation $W^{-1}W = 1$, and we find nine second degree equations for the a_{ij}. For example,

$$a_{11}^2 + a_{12}^2 - a_{13}^2 = 1$$

and

$$a_{11}a_{21} + a_{21}a_{22} - a_{31}a_{23} = 0.$$

Sample solutions In general we see that there are nine equations in nine unknowns, but it turns out that this system is degenerate. We will not try to find a general solution, but in order to see what kind of solutions are possible we assume that W is symmetric and make some choices for the components of W. Choosing $a_{11} = 1$ gives $a_{12} = \pm a_{13}$. We choose the positive sign and work through the rest of the equations to find the result in terms of a single parameter w (writing w for a_{12}):

$$W = \begin{pmatrix} 1 & w & w \\ w & (w^2-2)/2 & w^2/2 \\ w & w^2/2 & 1+w^2/2 \end{pmatrix}.$$

Setting $w = 2$ gives the example in M500 281, while any even w gives a distinct generating matrix, such as

$$W = \begin{pmatrix} 1 & 6 & 6 \\ 6 & 17 & 18 \\ 6 & 18 & 19 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 100 & 100 \\ 100 & 4999 & 5000 \\ 100 & 5000 & 5001 \end{pmatrix}.$$

All these matrices have determinant $D = -1$.

Conclusion We have found a family of transformations that send Pythagorean triples to distinct Pythagorean triples. Each member of the family generates a sequence of Pythagorean vectors. If $W_1 v = W_2 v$ we see that $W_1 = W_2$ so that different Ws give different triples starting from the same seed. No attempt has been made to find the general solution to $W^{\mathrm{T}} B = BW^{-1}$or to find relationships, if any, between the sequences of vectors, and further investigation is possible.

Stuart Walmsley

Integer triples X Y Z and x y z are related by the equations

$$\begin{aligned} X &= x + 2y + 2z, \\ Y &= 2x + y + 2z, \\ Z &= 2x + 2y + 3z. \end{aligned} \tag{1}$$

If (x, y, z) is a Pythagorean triple, show that (X, Y, Z) is also a Pythagorean triple. It is noted that if (x, y, z) is replaced by (kx, ky, kz) then (X, Y, Z) becomes (kX, kY, kZ), so that attention may be confined to primitive Pythagorean triples, that is to say those triples in which x, y and z are mutually co-prime. Furthermore in such a triple $\{x, y\}$ is an odd–even pair of integers and the convention that x is odd and y is even is adhered to as in the familiar example $(3, 4, 5)$.

It is also noticed that

$$\begin{aligned} X - Y &= -x + y, \\ Z - X &= x + z, \\ Z - Y &= y + z \end{aligned}$$

so that the magnitude of the difference between x and y is conserved but the sign is reversed. In this way, in any series of triples, generated using the basic equations, the ratios Z/X and Z/Y are progressively better rational approximations to $\sqrt{2}$.

In number theory, the continued fraction of $\sqrt{2}$ provides a sequence of such rational approximations:

$$\begin{array}{cccccc} t & 1 & 3 & 7 & 17 & \dots \\ v & 1 & 2 & 5 & 12 & \dots, \end{array} \tag{2}$$

each successive doublet (T, V) being generated from its predecessor by

$$T = t + 2v, \qquad V = t + v. \tag{3}$$

Since $t/v \approx \sqrt{2}$, $t^2 - 2v^2 \approx 0$. Then successively

$$t^2 - 2v^2 = -1, \ +1, \ -1, \ +1, \ \dots,$$

suggesting

$$t^2 - 2v^2 = x - y.$$

It is noted that t is always odd and that v is alternately odd and even. An examination of typical primitive Pythagorean triples suggests that $z - y$ is always the square of an odd prime and that $z - x$ is twice the square of an integer. In turn, this suggests $z - y = t^2$, $z - x = 2v^2$. Combining this with the basic relation $x^2 + y^2 = z^2$. leads to

$$x = t^2 + 2tv, \quad y = 2v^2 + 2tv, \quad z = t^2 + 2v^2 + 2tv.$$

Let (X, Y, Z) be the triple generated by the recurrence relations (3). Then

$$\begin{aligned} X &= (t + 2v)^2 + 2(t + 2v)(t + v), \\ Y &= 2(t + v)^2 + 2(t + 2v)(t + v), \\ Z &= (t + 2v)^2 + 2(t + v)^2 + 2(t + 2v)(t + v), \end{aligned}$$

which when rearranged give

$$X = x + 2y + 2z, \quad Y = 2x + y + 2z, \quad Z = 2x + 2y + 3z;$$

that is, the Pythagorean triple generator (1).

The set of (t, v) values (2) generate the primitive Pythagorean triples mentioned in the full version of the question:

$$(3, 4, 5), \quad (21, 20, 29), \quad (119, 120, 169), \quad \dots.$$

The convergence to $\sqrt{2}$ depends on the recurrence relations and not on the initial values chosen. For example with $t = 1$, $v = 2$,

$$\begin{array}{cccccc} t & 1 & 5 & 11 & 27 & \dots \\ v & 2 & 3 & 8 & 19 & \dots, \end{array}$$

the value of $x - y$ now alternating between $+7$ and -7. This gives triplets

$$(5, 12, 13), \quad (55, 48, 73), \quad (297, 304, 425), \quad \ldots$$

Another example is $t = 3$, $v = 1$,

t	3	5	13	31	...
v	1	4	9	22	...,

the value of $x - y$ also alternates between $+7$ and -7. This gives triplets

$$(15, 8, 17), \quad (65, 72, 97), \quad (403, 396, 565), \quad \ldots.$$

Are the two series connected?

The transformation (1) may be inverted giving:

$$\begin{aligned} x &= X + 2Y - 2Z, \\ y &= 2X + Y - 2Z, \\ z &= -2X - 2Y + 3Z, \end{aligned} \qquad (4)$$

with corresponding equations for t and v:

$$t = -T + 2V, \qquad v = T - V. \qquad (5)$$

Applying this to $(a, b, c) = (119, 120, 169)$, that is, $T = 7$, $V = 5$,

T	7	3	1	1	-1	3	-7	...
V	5	2	1	0	1	-2	5	....

The same Pythagorean triples are generated in reverse order. However, starting with $T = 11$, $V = 8$,

T	11	5	1	3	-5	13	...
V	8	3	2	-1	4	-9	....

In this way, the two series corresponding to $x - y = \pm 7$ are connected.

More detailed examination (which could form the basis of a more extended communication) shows that in general there are pairs of series with the same value of $x - y$ which may be run into one another by reversing the direction of generation. This suggests that, starting with any primitive Pythagorean triple, new triples may be generated using both (3) and its inverse (5) so that all related triples are found. The triplet $(3, 4, 5)$ is special because its series passes through $(1, 0, 1)$, the two halves being symmetry related.

Ted Gore

It can be shown that (x, y, z) is a Pythagorean triple if for some integers m and n,

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2.$$

Using the matrix shown we get

$$X = 3m^2 + n^2 + 4mn = (2m + n)^2 - m^2,$$

which has the form $M^2 - N^2$ when $M = 2m + n$ and $N = m$,

$$\begin{aligned} Y &= 4m^2 + 2mn = 2(2m + n)m = 2MN, \\ Z &= 5m^2 + n^2 + 4mn = (2m + n)^2 + m^2 = M^2 + N^2. \end{aligned}$$

Thus (X, Y, Z) is also a Pythagorean triple.

If we use the matrix $\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$, we get

$$\begin{aligned} X &= 3n^2 + m^2 + 4mn = (2n + m)^2 - n^2, \\ Y &= 4m^2 + 2mn = 2(2n + m)n, \\ Z &= 5n^2 + m^2 + 4mn = (2n + m)^2 + n^2. \end{aligned}$$

Starting with (3,4,5) we get

$$(15, 8, 17), \ (35, 12, 37), \ (63, 16, 65), \ (99, 20, 101), \ (143, 24, 145).$$

Using the matrix $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$, we get

$$\begin{aligned} X &= 3m^2 + n^2 - 4mn = (2m - n)^2 - m^2, \\ Y &= 4m^2 - 2mn = 2m(2m - n), \\ Z &= 5m^2 + n^2 - 4mn = (2m - n)^2 + m^2. \end{aligned}$$

Starting with (1,0,1) we get

$$(3, 4, 5), \ (5, 12, 13), \ (7, 24, 25), \ (9, 40, 41), \ (11, 60, 61).$$

Clock time and GPS

Sebastian Hayes

Thanks to celebrity physicists such as Brian Cox, a fair section of the general public now knows that corrections based on the Theory of Relativity are factored into the data received from GPS satellites and that, if this were not done, serious errors would very rapidly wreck the accuracy of SAT NAV and other systems that depend on the GPS. So Relativity must be true!

However, a fair proportion of the public has also heard of the Twins' Paradox, i.e. how the airborne twin, Jack the Nimble, making a round trip to Andromeda and back by rocket, ages less than his earthbound brother because, for him, 'time slows down'. Now, the GPS satellites orbiting the Earth are certainly moving at a much greater speed than someone stuck to the surface of the Earth, and so, according to Einstein's Theory of Special Relativity, satellite clock time should lag behind earthbound clock time. Thus, the 'correction' introduced for us earthbound inhabitants should be an increase—because, from our point of view, the satellite clocks always run slow. But, as it happens, the corrections actually made are the reverse of this. So what's going on?

It was not Einstein but the French physicist Langevin who invented the Twins' Paradox; nonetheless, the idea underpinning it goes right back to Einstein's 1905 paper in which the author mentions a 'peculiar consequence' (his words) of the theory.

> If at the points A and B of K there are stationary clocks which, viewed in the stationary system, are synchronous; and if the clock at A is moved with the velocity v along the line AB to B, then on its arrival at B the two clocks no longer synchronize, but the clock moved from A to B lags behind the other which has remained at B by $tv^2/(2c^2)$ (up to magnitudes of fourth or higher order), t being the time occupied in the journey from A to B.

Einstein is here speaking only of constant straight line motion but he immediately goes on to argue that the same principle can be generalized to cover cases of 'motion in a closed curve with constant speed', hence Langevin's idea of a round astronaut trip from and back to the Earth. This feature of SR is known as 'The Clock Hypothesis' or 'The Clock Paradox'.

What about accelerated motion generally? Einstein's SR treatment assumes that a 'clock' inside what, for us earthbound creatures, is an accelerated reference frame such as a rocket does not participate in the acceleration but goes at the same rate as a clock at rest on the Earth. However, 'rocket clock' intervals, viewed from the Earth, are not the same as 'earth-clock' intervals. Let $\Delta t'$ be a 'proper' time interval measured by a clock at rest in a frame Σ', and let Δt be the same interval measured in a different frame

Σ, one relative to which the first frame is in motion. We make the rocket frame Σ' and the Earth frame Σ. According to the Clock Hypothesis,

$$\frac{\Delta t}{\Delta t'} = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \text{or} \quad \Delta t' = \Delta t\sqrt{1 - v^2/c^2}.$$

If the rocket's speed varies, we simply assume that the rocket in Σ' has, at every moment, an instantaneous constant velocity of v_1, v_2, ... relative to the Earth and end up with an integral, where

$$\int dt' = \sqrt{1 - v^2/c^2} \int dt.$$

Thus, according to the Clock Hypothesis, whether the rocket's speed and/or acceleration is constant or not,

$$T'_{\mathrm{R}} = \sqrt{1 - v^2/c^2}\, T_{\mathrm{E}}.$$

The rocket clock thus always 'runs slow' viewed from the Earth since $\sqrt{1 - v^2/c^2} < 1$. There is, as it were, 'less ticking' going on inside the rocket (as we on Earth view things), also fewer heartbeats and muscle contractions if the 'rocket clock' is a human astronaut.

If we expand the term $\sqrt{1 - v^2/c^2}$, we obtain $1 - v^2/(2c^2) + \ldots$ and we can safely neglect higher order terms if c^2 is very much greater than v^2. So $T'_{\mathrm{R}} \approx T_{\mathrm{E}}(1 - v^2/(2c^2))$, i.e. a clock in the rocket/satellite system always 'runs slow' by a factor of $v^2/(2c^2)$ (to second order) compared with a clock at rest in the Earth system, whether the rocket has constant or non-constant velocity relative to the Earth.

So much for Special Relativity considerations. General Relativity, as it happens, usually works in the opposite direction. But, before we leave SR, we can derive the Relativistic version of the Doppler effect since it plays a key role in the eventual argument.

Assume a source of light at rest at the origin O' of a frame Σ' that emits light signals with frequency ν' measured in Σ'. (We can view the signals as successive crests of a continuous light wavefront.) At a moment in time the two origins of Σ' and Σ coincide and from then on Σ' recedes with uniform velocity v relative to Σ. We want to know ν, the observed frequency of the light emitted from Σ' but received at Σ. (Beware the annoying similarity of sign, ν for frequency, and v for velocity.)

We assume that light travels in a vacuum from O' to O with speed c. If identical clocks in the two frames are set at zero when the two origins coincide, a light ray emitted at time t' at O' reaches O in Σ at a later time because we must add on the time taken for the light to traverse the ever increasing distance between the two frames. This 'extra' time is given by x/c, where x is the distance between the two frames when the light ray finally reaches O.

The Doppler effect was originally formulated using the Galilean transformations but SR, as we know, uses the so-called Lorentz transformations where, $y = y'$, $z = z'$ but $x = \gamma(x' + vt')$ and $t = \gamma(t' + vx'/c^2)$. If we assume that light is emitted from the origin O' of Σ', $x' = 0$ always and we obtain $x = \gamma vt'$, $t = \gamma t'$.

If T is the time when a light ray reaches O in Σ, we have

$$T \;=\; t + x/c \;=\; \gamma t' + \gamma vt'/c \;=\; \gamma t'(1 + v/c).$$

If ν' is the frequency of light flashes as measured in O', the period is $1/\nu'$. The next flash will thus occur at time $t' + 1/\nu'$. This flash of light reaches O at a time

$$T + \Delta T \;=\; \gamma t'(1 + v/c) + \gamma(1/\nu')(1 + v/c) \;=\; \gamma(t' + 1/\nu')(1 + v/c).$$

If ΔT is one period in O, i.e. $1/\nu$, $1/\nu = \gamma(1/\nu')(1 + v/c)$. Thus

$$\frac{\nu}{\nu'} \;=\; \frac{1/\gamma}{1 + v/c} \;=\; \frac{\sqrt{1 - v^2/c^2}}{1 + v/c} \;=\; \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}.$$

So,

$$\frac{\nu}{\nu'} \;=\; \nu' \, \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}.$$

Expanding the binomials, we obtain

$$\frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}} \;=\; \sqrt{1 - v/c}\left(1 - \frac{v}{2c} + \dots\right) \;=\; 1 - \frac{v}{c}$$

to first order. Thus $\nu \approx \nu'(1 - v/c)$; i.e. $\nu < \nu'$.

Why is this relevant? Because Einstein's General Theory of Relativity (GR) postulates that it is impossible to distinguish between the effects of being accelerated in a gravitation free region and the effects of being at rest in a strong gravitational field. This is the 'Principle of Equivalence' on which much of GR is based. Although Einstein always seems to have in mind an old-fashioned pendulum clock, any regular repeating sequence of events can function as a clock—and this includes light pulses or successive crests of electromagnetic waves.

Suppose, then, we have a rocket not subject to any gravitational field that has a constant acceleration in a direction we call 'up'. Light pulses are emitted at the 'bottom' and 'top' of the rocket from two identical sources and received at the other end. Rays emitted from the bottom of the rocket have to travel the length of the accelerating rocket in the direction of motion, while pulses emitted from the top and received at the base move contrary to the direction of motion. There will thus be a discrepancy between the emitted frequency and the received frequency of the light signals: signals from the bottom received at the top will be shifted towards the red and those from the top received at the base will be shifted towards the blue end.

If the Principle of Equivalence is correct, exactly the same results would obtain if we are considering the two extremities of a system at rest within a powerful gravitational field. In such a case, we ought to observe a similar discrepancy between the frequencies of light pulses emitted from a spot higher up in the system and then received at the base and the frequencies of light waves emitted alongside one at the bottom of the system. That this is actually the case was demonstrated by Pound and Rebka in a remarkable experiment carried out at Harvard using the Mössbauer effect. For a vertical earthbound system (in this case a lab building at Harvard) with g considered constant the predicted discrepancy in the frequency was $\nu_0 gh/c^2$ and this was confirmed within limits of experimental error. Note that, if we consider the light signals as clocks, the clock higher up will appear to run fast (not slow) compared to the clock at the bottom because the frequencies of light received at the bottom will be blue-shifted.

The question thus is: Does the 'slowing down' of a satellite 'clock' due to the Clock Hypothesis, a strictly SR phenomenon, dominate the 'speeding up' of a satellite 'clock' due to the Principle of Equivalence of GR, or vice-versa? Rosser discusses the issue at length in his excellent book *Introductory Relativity* that I cannot recommend too enthusiastically. I more or less follow his treatment.

When applying the Principle of Equivalence to an Earth/satellite system, we can, of course, no longer assume that g, the average acceleration due to gravity on the surface of the Earth, is constant. We have to replace gh with $\Delta\Phi$, the difference in gravitational potential between the satellite clock and the earthbound one. By convention, we regard the higher position occupied by the satellite as having a greater gravitational potential (less negative) than the lower earthbound one.

We now apply the traditional Newtonian $F = ma$ equation with v^2/r for the centripetal acceleration. Assuming a strictly circular orbit for the satellite and with m = mass of the satellite, M = mass of the Earth, G = gravitational constant, r_0 = radius of the Earth, r = radius of the satellite's orbit, we have

$$\frac{mv^2}{r} = \frac{mGM}{r^2}, \quad v^2 = \frac{GM}{r} = \frac{GM(r_0/r)}{r_0}.$$

Since g, the average acceleration at the Earth's surface, is GM/r_0^2, we obtain for low heights $v^2 = gr_0(r_0/r)$.

We now derive an alternative value for v using the analogy between accelerated motion without a gravitational field and conditions of rest within a gravitational field—in this case the gravitational field of the Earth. If, within our accelerating rocket, a light pulse is emitted from the bottom and received at the top, h being the height of the rocket, light reaches the top having traversed a distance ct. If the acceleration is constant this distance is $h + at^2/2$.

Thus $ct = h+at^2/2$ and $t = h/c+(a/c)t^2/2$. If a/c is extremely small we can reject the second term and write $t \approx h/c$. (This is the most questionable assumption in the argument.) After time t the top of the rocket (with clock 2 at rest inside it) is thus moving with a velocity $v = at \approx ah/c$ and we can use this value of v in the Doppler frequency equation already derived, namely $\nu = \nu'(1 - v/c)$. The discrepancy in the frequency $\Delta\nu/\nu' = -ah/c^2$. And, if we make the acceleration a equal to g, the acceleration due to gravity at the Earth's surface, we have $\Delta\nu/\nu' = -gh/c^2$. Remember that ν' is the 'proper' frequency; i.e. it is the frequency of light pulses measured in a frame where the light source is at rest—whether this frame be the rocket itself or the Earth.

According to the Principle of Equivalence, exactly the same discrepancy will be observed in a stationary system within a gravitational field. Let the orbiting satellite and a laboratory on the Earth be this system and the field be that of the Earth. Our earthbound clock corresponds to the lower one in the rocket and, since it is at rest, it keeps 'proper time'. The satellite clock, on the other hand, corresponds to the clock at the top of the accelerating rocket and, if the Principle of Equivalence holds, the satellite light clock runs fast when viewed from the Earth. The discrepancy is thus positive and applies to any sort of 'clock'. However, we must now replace gh in gh/c^2 by $\Delta\Phi$, the difference in gravitational potential between the clock in the satellite and the earthbound clock. Thus $\Delta\nu/\nu_{\text{Earth}} = \Delta\Phi/c^2$.

We now determine $\Delta\Phi$. If r is the radius of the satellite's orbit and r_0 is the radius of the Earth, we have

$$\Delta\Phi \;=\; \int_{r_0}^{r} \frac{GM}{x^2}dx \;=\; \frac{GM}{r_0}\left(1 - \frac{r_0}{r}\right) \;=\; gr_0\left(1 - \frac{r_0}{r}\right)$$

since $g = GM/r_0^2$. Inserting this value into $\Delta\nu/\nu_{\text{Earth}} = \Delta\Phi/c^2$ we have

$$\frac{\Delta\nu}{\nu_{\text{Earth}}} \;=\; \frac{gr_0(1 - r_0/r)}{c^2}$$

and, since the frequency of light flashes is a perfectly valid means of comparing time generally

$$\frac{\Delta T}{T_{\text{Earth}}} = \frac{gr_0(1 - r_0/r)}{c^2}. \tag{i}$$

This is the strictly GR component of the total discrepancy since it is based on the Principle of Equivalence, which does not come into SR.

Returning to the SR component of the total discrepancy, we had

$$T'_{\text{R}} \approx T_{\text{L}}\left(1 - \frac{v^2}{2c^2}\right), \quad \text{or} \quad T_{\text{satellite}} \approx T_{\text{Earth}}\left(1 - \frac{v^2}{2c^2}\right);$$

i.e. $\Delta T/T_{\text{Earth}} = -v^2/(2c^2)$.

Using $v^2 = gr_0(r_0/r)$, a result obtained without using any GR assumptions, we have

$$\frac{\Delta T}{T_{\text{Earth}}} = -\frac{gr_0(r_0/r)}{2c^2}. \tag{ii}$$

Adding (i) and (ii), we obtain

$$\frac{\Delta T}{T_{\text{Earth}}} = \frac{gr_0}{c^2}\left(1 - \frac{3r_0}{2r}\right).$$

Now, r, the radius of the satellite's orbit, is equal to $r_0 + (r - r_0)$; i.e. r is equal to the radius of the Earth + height of satellite above sea level. Thus ΔT is positive if $3r_0/(2r) < 1$, or $r > 3r_0/2$; i.e. if the altitude of the satellite is greater than $r_0/2$. This means the satellite needs to be about 3,200 km above sea level. In such a case the GR contribution predominates and a clock in a satellite runs fast compared with a clock on the surface of the Earth. Alternatively, if the altitude of the satellite is lower than 3,200 km, the SR component predominates and the clock on the satellite 'runs slow' compared with an earthbound clock.

According to Wikipedia, the 24 GPS satellites, carrying atomic clocks, orbit the Earth at a height of 20,000 km and so the GR component easily wins. Incidentally, the satellites do not follow geosynchronous or geostationary orbits but have an orbital speed of 14,000 km/hour and an orbital period of 12 hours. The atomic clocks advance by 45 microseconds per day because of GR gravitational effects and lose 7 microseconds per day because of SR considerations, so that, all in all, satellite time is 38 microseconds ahead of Earth time. Incredibly, if the appropriate corrections were not made, the accumulated 'errors' would put the GPS system out by about 10 kilometres a day!

Solution 180.6 – Two pedestrians

> Two pedestrians, P_1 and P_2, are travelling at right angles to each other, but are obscured by the corner of a building. Each is travelling parallel to the wall nearest to them. Pedestrian P_i is travelling at a constant speed of V_i and the perpendicular distance between P_i and the building is L_i, $i = 1, 2$. If the time they meet occurs at zero, find the time before impact at which they can both see each other across the corner of the building.

Peter Fletcher

Let T be the time before zero at which the two pedestrians see each other. Then the further distance that P_1 has to walk before bumping into P_2 is V_1T and similarly, the further distance that P_2 has to walk before bumping into P_1 is V_2T.

The wall parallel to P_1's path is a constant L_1 from P_1 and the wall parallel to P_2's path is a constant L_2 from P_2 (until each pedestrian passes the corner of the building).

This means that the distance from where P_1 first sees P_2 to the corner of the building is $V_1T - L_2$. Then by similar triangles, we have

$$\frac{V_1T}{V_2T} = \frac{V_1T - L_2}{L_1},$$

which gives

$$T = \frac{V_1L_1 + V_2L_2}{V_1V_2}.$$

Problem 286.1 – Sum to powers

Given n, a positive integer, show that it is possible to find distinct positive integers a, b and c such that $a + b$, $a + c$ and $b + c$ are distinct n-th powers of positive integers.

If this problem looks familiar, it might be because the solution of the $n = 3$ case by providing an actual example could have earned you a £25 discount for the 2019 M500 Winter Weekend.

Problem 286.2 – 7777

Show that for positive integer n, $7777n^2 + 1$ is never an integer square. Or find a counter-example.

Solution 173.1 – Binomial coefficients squared

Show that

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{n/2}\dbinom{n}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Reinhardt Messerschmidt

The identity clearly holds if $n = 0$, so suppose $n \geq 1$ and let

$$N = \{1, 2, \ldots, n\},$$
$$\mathcal{P} = \{(A, B) : A, B \subseteq N, \quad |A| + |B| = n\},$$
$$\mathcal{P}_r = \{(A, B) \in \mathcal{P} : |A| = r\},$$
$$\mathcal{P}' = \{(A, B) \in \mathcal{P} : A \neq B\}, \quad \mathcal{P}'' = \{(A, B) \in \mathcal{P} : A = B\}.$$

We have

$$\binom{n}{r}^2 = \binom{n}{r}\binom{n}{n-r} = |\mathcal{P}_r| = \sum_{(A,B)\in\mathcal{P}_r} 1;$$

therefore

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 = \sum_{r=0}^{n}(-1)^r \sum_{(A,B)\in\mathcal{P}_r} 1$$
$$= \sum_{(A,B)\in\mathcal{P}} (-1)^{|A|} = (\text{first sum}) + (\text{second sum}),$$

where the first sum is over $(A, B) \in \mathcal{P}'$ and the second is over $(A, B) \in \mathcal{P}''$.

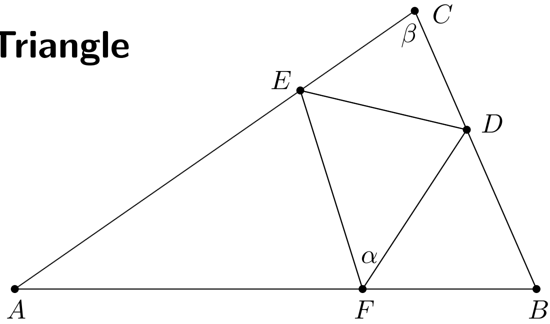
First sum. Let f be the function from $\mathcal{P}'$ into $\mathcal{P}'$ defined as follows: given $(A, B) \in \mathcal{P}'$, let m be the smallest element in $(A - B) \cup (B - A)$, and let $f(A, B)$ be the element of $\mathcal{P}'$ obtained from (A, B) by moving m from A to B if $m \in A - B$, or from B to A if $m \in B - A$. Since $f(f(A, B)) = (A, B)$, it follows that for every $(A, B) \in \mathcal{P}'$ there exists a unique $(A', B') \in \mathcal{P}'$ such that $(A, B) = f(A', B')$ and $(A', B') = f(A, B)$. Furthermore, $|A| = |A'| \pm 1$. This implies that the terms in the first sum can be paired up in such a way that the sum of each pair is zero; therefore the first sum is zero.

Second sum. If n is odd, then $\mathcal{P}''$ is empty and the second sum is zero. If n is even and $(A, B) \in \mathcal{P}''$, then $|A| = n/2$ and

$$(\text{second sum}) = (-1)^{n/2} \sum_{(A,B)\in\mathcal{P}''} 1 = (-1)^{n/2}\binom{n}{n/2}.$$

Solution 243.4 – Triangle

Here's something that you can easily do on the train to work. Given that $AE = AF$ and $BD = BF$, show that $\beta + 2\alpha = \pi$.



Peter Fletcher

We know that triangles AEF and BDF are both isosceles because $AE = AF$ and $BD = BF$, so let

$$A\hat{E}F = A\hat{F}E = \gamma \quad \text{and} \quad B\hat{F}D = B\hat{D}F = \delta.$$

Then since AB is a straight line, we have at F, $\gamma + \alpha + \delta = \pi$ so that $\gamma + \delta = \pi - \alpha$. Adding up the angles of triangle ABC, we get

$$(\pi - 2\gamma) + (\pi - 2\delta) + \beta = \pi \quad \text{or} \quad 2(\gamma + \delta) = \pi + \beta.$$

Therefore $2(\pi - \alpha) = \pi + \beta$ and $\beta + 2\alpha = \pi$.

Problem 286.3 – Tritium oxide

Imagine you are foolish enough to take a bath in pure tritium oxide, T_2O. Would you be able to get out of the bath and survive? Whilst idly browsing online I (TF) was surprised to find several and diverse answers to this interesting question.

(i) No. The radioactivity of the tritium would kill you, if not whilst in the bath, soon after you got out.

(ii) No. The T_2O would be boiling, heated by the decay of the tritium. You would be cooked.

(iii) No. You would suffer irreversible chemical damage. Radiolysis creates amongst other things high concentrations of T^+ and OT^- ions.

(iv) Yes. Get out quickly and wash yourself thoroughly with natural H_2O. There should be no permanent ill-effects.

Which is correct? Please provide a theoretical solution. *Do not perform any experiments.* For definiteness, assume the bath capacity is $2\,\mathrm{m}^3$, say 2 m long, 1 m wide and 1 m deep. Recall that tritium, hydrogen-3, is radioactive with half-life 12.3 years. A tritium nucleus beta-decays to helium-3 releasing 18.6 keV of which on average 5.7 keV is the kinetic energy of the electron. The rest of the energy is carried off by an electron antineutrino. As the He-3 nucleus is created in its ground state there is no gamma radiation.

M500 Mathematics Revision Weekend 2019

The forty-fifth M500 Revision Weekend will be held at

Kents Hill Park Training and Conference Centre,

Milton Keynes, MK7 6BZ

from Friday 10th to Sunday 12th May 2019.

The standard cost, including accommodation (with *en suite* facilities) and all meals from dinner on Friday evening to lunch on Sunday is £275 for single occupancy, or £240 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £160.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2019 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 10th April 2019. The Late Booking Fee for applications received after 10th April 2019 is £20, with no membership discount applicable.

There is free on-site parking for those travelling by private transport. For full details and an application form, see the Society's web site:

`m500.org.uk`.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

Please note that the venue is not the same as last year.

Problem 286.4 – Evaporation

A solution of sodium chloride contains 90 per cent water. After a while, due to loss by evaporation, the solution contains only 80 per cent water. What percentage of the water has evaporated?

Answers to the quiz on page 24 of M500 **285**: (1) rose (2) rose (3) rows (4) Rose (5) roes (6) rose (7) Ro's (Rosemary's) (8) rows (9) rho's (10) roe's (11) rose (12) Ros (13) rose (14) row's (15) roe's (16) rhos (17) roes (18) Rose (19) rose (20) rose (21) rose (22) roes' (23) roes' (24) rows' (in a stochastic matrix the rows' sums are always 1) (25) rose

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04.00

Problem 286.5 – Factorization

Given a positive integer n, denote by $\phi(n)$ the number of positive integers $m < n$ such that $\gcd(m, n) = 1$. If we know the complete factorization of n, say $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ with positive integers a_1, a_2, ..., a_r and distinct primes p_1, p_2, ..., p_r, we can easily compute

$$\phi(n) = (p_1 - 1)p_1^{a_1 - 1}\,(p_2 - 1)p_2^{a_2 - 1}\,\dots\,(p_r - 1)p_r^{a_r - 1}.$$

Is this process reversible? Given n and $\phi(n)$, is it possible to construct the complete factorization of n without too much difficulty? If it is, try factorizing

$n =$ 1586 02481 31293 11974 04552 75968 73607 71145 55549 11334 22976 68001 07012 76942 47700 99421 02756 52867 12646 06754 07200 60245 86133 22978 29252 68997 66323 73467 11294 88572 28050 70734 96620 00789 92252 73781

given that

$\phi(n) =$ 1586 02481 31293 11974 04552 75968 61357 52500 38117 66204 95814 65893 63834 77320 84117 64608 84450 72128 92453 11845 67688 17202 39729 99520 14675 73124 90380 04713 14884 75595 39954 75897 02033 54901 63971 74000.