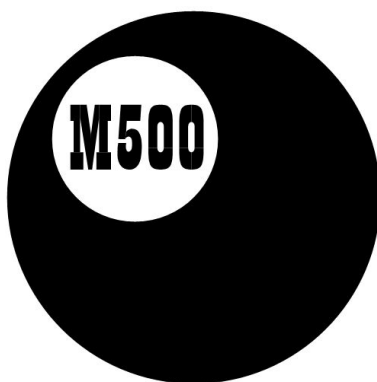


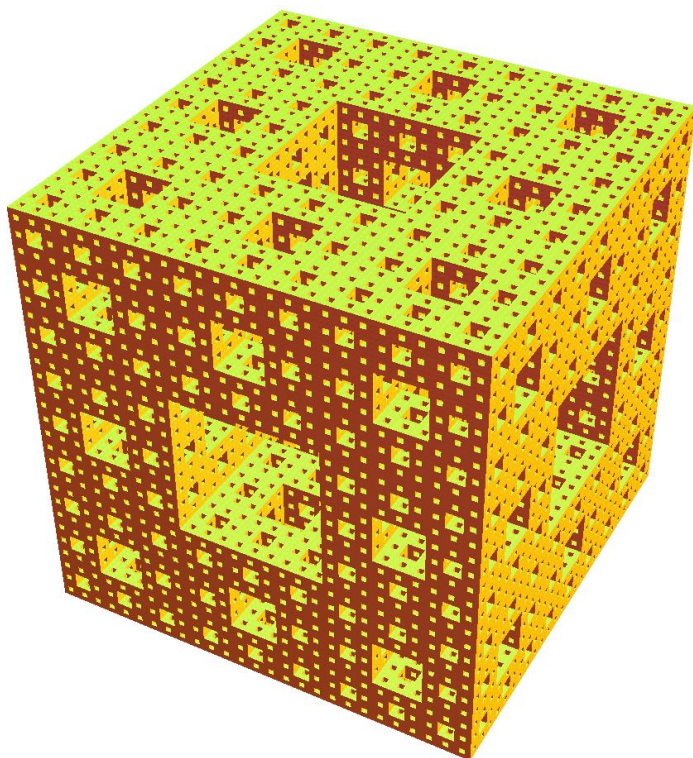
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# Systems of mutually annihilating idempotents

**J. M. Selig**

Some time ago I needed to compute the exponential and other power series of a particular type of square matrix. Since square matrices satisfy polynomial equations it must be possible to express an infinite power series of a square matrix as a finite series in powers of the matrix. The literature is full of numerical methods to compute the exponential of a general matrix, but the matrices I was interested in were not general, and I was after a symbolic result. Eventually I learned from Garret Sobczyk how to express a square matrix as a sum of mutually annihilating idempotents. The technique seems to be ‘well known’, in the sense that people at conferences who I try to talk to about these things all nod sagely as if these ideas are familiar but don’t seem too interested. The trouble is that I can’t find any references to these ideas. Google isn’t any help! Of course this may just indicate that it is well known but under a different name. The ideas here would seem to apply to all sorts of commutative rings but books on linear algebra don’t mention this powerful and fascinating technique.

## 1 Idempotents

An idempotent in a ring is an element  $P$  which satisfies the relation  $P^2 = P$ . Certainly the identity matrix and the zero matrix satisfy this equation. What we are seeking here is a set of idempotent matrices,  $P_1, P_2, \dots, P_n$  which satisfy the equations

$$P_1 + P_2 + P_3 + \dots + P_n = I$$

and

$$\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \dots + \alpha_n P_n = X,$$

where  $X$  is the matrix we are considering and the  $\alpha_i$  are complex coefficients.

Suppose the characteristic polynomial of  $X$  factorizes as

$$\chi(\lambda) = (\lambda - d_1)(\lambda - d_2) \dots (\lambda - d_n),$$

where the  $d_i$  are the distinct complex eigenvalues of the matrix  $X$ . Consider the partial fraction expansion of  $1/\chi(\lambda)$ ,

$$\frac{1}{(\lambda - d_1)(\lambda - d_2) \dots (\lambda - d_n)} = \frac{q_1}{\lambda - d_1} + \frac{q_2}{\lambda - d_2} + \dots + \frac{q_n}{\lambda - d_n}.$$

The coefficients  $q_i$  can be found by comparing the numerators,

$$\begin{aligned} 1 = & q_1(\lambda - d_2)(\lambda - d_3) \cdots (\lambda - d_n) \\ & + q_2(\lambda - d_1)(\lambda - d_3) \cdots (\lambda - d_n) + q_3(\lambda - d_1)(\lambda - d_2) \cdots (\lambda - d_n) \\ & + \cdots + q_n(\lambda - d_1) \cdots (\lambda - d_{n-1}). \end{aligned}$$

In practice the  $q_i$  can be evaluated by the ‘cover-up rule’ or by solving the linear equations found by equating powers of  $\lambda$  on both sides of the equation. The first of these is

$$0 = q_1 + q_2 + q_3 + \cdots + q_n.$$

From the coefficient of  $\lambda^{n-1}$  we get

$$\begin{aligned} 0 = & q_1(d_2 + d_3 + \cdots + d_n) + q_2(d_1 + d_3 + \cdots + d_n) \\ & + \cdots + q_n(d_1 + d_2 + \cdots + d_{n-1}). \end{aligned}$$

The last equation is

$$1 = (-1)^{n-1} \left( q_1(d_2 d_3 \cdots d_n) + q_2(d_1 d_3 \cdots d_n) + \cdots + q_n(d_1 d_2 \cdots d_{n-1}) \right).$$

Now the idempotent matrices are given by

$$\begin{aligned} P_1 &= q_1(X - d_2 I)(X - d_3 I) \cdots (X - d_n I), \\ P_2 &= q_2(X - d_1 I)(X - d_3 I) \cdots (X - d_n I), \\ P_3 &= q_3(X - d_1 I)(X - d_2 I) \cdots (X - d_n I), \\ &\vdots \\ P_n &= q_n(X - d_1 I)(X - d_2 I) \cdots (X - d_{n-1} I). \end{aligned}$$

Since each of these matrices is missing a different factor from the characteristic polynomial it is easy to see that, for any pair of different matrices,  $P_i P_j = 0$ , when  $i \neq j$ . This shows that the matrices are mutually annihilating.

Multiplying out these matrices will give exactly the right-hand sides of the equations we found for the partial fraction expansion but with powers of  $\lambda$  replaced by powers of  $X$ . Hence we have the relation

$$I = P_1 + P_2 + P_3 + \cdots + P_n.$$

It is easy now to see that the  $P_i$  matrices are idempotents. Multiplying the equation above by  $P_i$  gives

$$P_i = P_i P_1 + P_i P_2 + P_i P_3 + \cdots + P_i P_n$$

but since the matrices are mutually annihilating the only non-zero term on the right will be  $P_i^2$  and thus  $P_i^2 = P_i$  for all  $i$ .

Next let us take the equation

$$I = P_1 + P_2 + P_3 + \cdots + P_n$$

and multiply it by  $X$ ,

$$X = X P_1 + X P_2 + X P_3 + \cdots + X P_n.$$

This can be written as

$$\begin{aligned} X &= (X - d_1 I + d_1 I) P_1 + (X - d_2 I + d_2 I) P_2 \\ &\quad + (X - d_3 I + d_3 I) P_3 + \cdots + (X - d_n I + d_n I) P_n. \end{aligned}$$

Cancelling the copy of the characteristic polynomial in each term gives

$$X = d_1 P_1 + d_2 P_2 + d_3 P_3 + \cdots + d_n P_n.$$

The point about this representation is that it is now simple to compute powers of the matrix

$$X^k = d_1^k P_1 + d_2^k P_2 + d_3^k P_3 + \cdots + d_n^k P_n$$

since all cross terms disappear. Hence to evaluate a power series in  $X$ , such as the exponential, we have

$$\begin{aligned} e^X &= I + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \cdots \\ &= e^{d_1} P_1 + e^{d_2} P_2 + e^{d_3} P_3 + \cdots + e^{d_n} P_n. \end{aligned}$$

Expanding the idempotent matrices  $P_i$  will then produce a polynomial in  $X$  whose coefficients are analytic functions of the eigenvalues of the matrix  $X$ .

However, this only applies to matrices with distinct eigenvalues; in many cases the matrices we are interested in have repeated eigenvalues. To deal with this situation we need to extend our matrix decomposition to include nilpotents.

## 2 Idempotents and nilpotents

First notice that if the matrix has a repeated eigenvalue then it is possible that it satisfies a polynomial relation with lower degree than the characteristic polynomial. This happens if the Jordan normal form of the matrix has two blocks with the same eigenvalue, see [1, Chapter 11] for example. The lowest degree polynomial satisfied by a matrix is called the minimal polynomial of the matrix. For our purposes we only need to use the minimal polynomial of the matrix to perform the decomposition.

If the minimum polynomial of a matrix has a repeated factor then we can't find an expansion into mutually annihilating idempotents as above. But if we allow nilpotent matrices then we can expand these ideas and decompose the matrix into a sum of idempotents and nilpotents. A nilpotent matrix is just one that satisfies the relation  $N^k = 0$  for some integer power  $k$  greater than 1.

To simplify things a little here let us look at an example. Notice that this situation can only arise if the Jordan decomposition of the matrix has a non-trivial Jordan block, so consider the matrix

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic equation of this matrix is

$$(\lambda - 1)(\lambda - 2)^2 = 0.$$

The partial fraction expansion of the characteristic polynomial is

$$\frac{1}{(\lambda - 1)(\lambda - 2)^2} = \frac{1}{\lambda - 1} + \frac{3 - \lambda}{(\lambda - 2)^2}.$$

So the idempotents are

$$P_1 = (X - 2I)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 = (X - I)(3I - X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly  $P_1 + P_2 = I$ , as required. Suppose we try to emulate the construction of the previous section. Multiply this last equation by  $X$ ,

$$X = XP_1 + XP_2 = (X - I + I)P_1 + (X - 2I + 2I)P_2 = P_1 + 2P_2 + (X - 2I)P_2.$$

Now clearly the product  $N = (X - 2I)P_2$  is nilpotent,  $N^2 = 0$  since this contains all the factors of the minimal polynomial of  $X$ . Moreover, we have the relations  $P_1N = 0$  and  $P_2N = (X - 2I)P_2^2 = (X - 2I)P_2 = N$ . However, the expression for  $N$  has the same degree as the minimal polynomial; so it can be reduced:

$$N = (X - 2I)(X - I)(3I - X) = (X - 2I)(X - I)(2I + I - X) = (X - I)(X - 2I).$$

The matrix expression for the nilpotent is then

$$N = (X - I)(X - 2I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

To recapitulate, the relation for  $X$  is

$$X = P_1 + 2P_2 + N$$

and from here it is simple to find the powers of the matrix  $X$ ,

$$X^2 = P_1 + 4P_2 + 4N, \quad X^3 = P_1 + 8P_2 + 12N, \quad X^4 = P_1 + 16P_2 + 32N, \dots$$

and in general,

$$X^k = P_1 + 2^k P_2 + k2^{k-1} N.$$

Thus, for example, the exponential of this matrix is

$$e^X = eP_1 + e^2P_2 + e^2N = \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix}.$$

What happens if the matrix has a thrice repeated eigenvalue? Let's look at the matrix

$$Y = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The characteristic equation of the matrix is  $(\lambda - 2)(\lambda - 3)^3 = 0$  and the partial fraction expansion is

$$\frac{1}{(\lambda - 2)(\lambda - 3)^3} = \frac{-1}{(\lambda - 2)} + \frac{\lambda^2 - 7\lambda + 13}{(\lambda - 3)^3}.$$

The two idempotents are then

$$P_1 = (3I - Y)^3, \quad \text{and} \quad P_2 = (Y^2 - 7Y + 13I)(Y - 2I).$$

So

$$Y = 2P_1 + 3P_2 + N,$$

where

$$\begin{aligned} N &= (Y - 3I)P_2 = (Y - 3I)(Y^2 - 7Y + 13I)(Y - 2I) \\ &= (Y - 3I)((Y - 3I)^2 - Y + 4I)(Y - 2I) \\ &= -(Y - 2I)(Y - 3I)(Y - 4I). \end{aligned}$$

Clearly  $N^3 = 0$  but  $N^2 = (Y - 3I)^2 P_2 \neq 0$ . So also  $NP_2 = N$  and  $N^2 P_2 = N^2$ . Of course we also have  $NP_1 = N^2 P_1 = 0$ . The first few powers of  $Y$  are

$$Y = 2P_1 + 3P_2 + N, \quad Y^2 = 4P_1 + 9P_2 + 6N + N^2, \quad Y^3 = 8P_1 + 27P_2 + 27N + 9N^2.$$

In general we have

$$Y^k = 2^k P_1 + 3^k P_2 + k3^{k-1} N + \frac{1}{2}k(k-1)3^{k-2} N^2.$$

The exponential of this matrix is then

$$e^Y = e^2 P_1 + e^3 P_2 + e^3 N + \frac{1}{2}e^3 N^2 = \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & e^3 & e^3 & \frac{1}{2}e^3 \\ 0 & 0 & e^3 & e^3 \\ 0 & 0 & 0 & e^3 \end{pmatrix}.$$

Using the Jordan normal form it is fairly simple to see what is going on, but the methods work in the same way for matrices not in this canonical form. Some more details can be found in Garret's book [2].

## References

- [1] P. M. Cohn, *Classic Algebra*, 3rd edition, Wiley, Chichester, 2000.
- [2] G. Sobczyk, *New Foundations in Mathematics: The Geometric Concept of Number*, Birkhäuser, Basel, 2012.

# Rational fundamental constants

## Tony Forbes

Several years after I excitedly wrote about these matters in **M500 260** under the same title, I see that the Comité International des Poids et Mesures has acted. At the 26th Conférence Générale des Poids et Mesures on 16 November 2018 the Comité created the following rational numbers, which will come into effect on 20 May 2019.

Planck's constant  $h$  is exactly  $6.62607015 \times 10^{-34}$  joule seconds,  
 the electron charge  $e$  is exactly  $1.602176634 \times 10^{-19}$  coulombs,  
 Boltzmann's constant  $k$  is exactly  $1.380649 \times 10^{-23}$  joules per kelvin,  
 Avogadro's constant  $N_A$  is exactly  $6.02214076 \times 10^{23}$  mole<sup>-1</sup>.

The speed of light remains unchanged at  $c = 299792458$  metres per second (exactly). Thus, together with the second, the time for 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of caesium-133, we will have definitions of the metre, kilogram, coulomb, kelvin and mole that will hopefully last for ever. The International Prototype Kilogram will be retired, and all calculations involving  $c$ ,  $h$ ,  $e$ ,  $k$  and  $N_A$  will be completely error-free. If you have not already done so, now would be a good time to update your textbooks.

The two hyperfine levels in the definition of the second refer to the two ways of combining the spin of the outermost electron with the spin of the Cs-133 nucleus. A neutral caesium atom in its ground state has electron configuration [Xe]6s<sup>1</sup> with the spins of the 6s electron and the nucleus either antiparallel or parallel. The energy difference is  $9192631770 \times 6.62607015 \times 10^{-34}$  J, which corresponds to a wavelength of 21413747/656616555 metres, near the long end of the microwave X band.

As far as I am aware, there are no plans to redefine  $\pi$ , although it is possible that at some time in the future it, too, might be rationalized (see Mike Grannell's article in **M500 273**). Anyway, assuming  $\pi$  retains its usual value, 3.1415926..., the permeability of free space would have to change from  $4\pi \times 10^{-7}$  henries per metre to something else, presumably

$$\mu_0 = \frac{2h\alpha}{e^2c} = \frac{7888178750000000000000\alpha}{4580703784999263461548761} \text{ kg m/coulomb}^2,$$

where  $\alpha \approx 0.007297352566$  is the fine structure constant. The uncertainty in the value of  $\alpha$  is about 0.25 parts per billion; so the difference between the new  $\mu_0$  and  $\pi/2500000$  H/m should be detectable.

## Solution 281.6 – Primitive Pythagorean triples

A *Pythagorean triple* is three non-negative integers  $(a, b, c)$  such that  $a^2 + b^2 = c^2$ , and *primitive* means  $\gcd(a, b, c) = 1$ .

Show that a primitive Pythagorean triple contains exactly one multiple of 3, exactly one multiple of 4 and exactly one multiple of 5. For example, in the triple  $(119, 7080, 7081)$  the unique multiples of 3, 4 and 5 are respectively 7080, 7080 and 7080.

### Stuart Walmsley

In what follows, use is made of Fermat's little theorem, which states that for an integer  $j$  which is co-prime to a prime  $p$ ,

$$j^{p-1} \equiv 1 \pmod{p}. \quad (1)$$

As a corollary, for two such integers  $j$  and  $k$ ,

$$(j^{p-1} - k^{p-1}) \equiv (1 - 1) \equiv 0 \pmod{p}; \quad (2)$$

that is, the difference is a multiple of  $p$ .

One of the standard parametrizations of a primitive Pythagorean triple uses two integers,  $u$  and  $v$  which are co-prime, one being odd and one being even, and  $u$  greater than  $v$ . Explicitly:

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

First consider  $b$ . Since one of  $u, v$  is even,  $b = 2uv$  is a multiple of 4.

Next consider  $ab = 2uv(u^2 - v^2)$ . Since the Pythagorean triple is primitive,  $u$  and  $v$  cannot both be multiples of 3. If one of them is a multiple of 3, then  $ab$  is a multiple of 3. The only other possibility is that neither is a multiple of 3 and hence both are co-prime to 3. Hence by (2),  $(u^2 - v^2)$  is a multiple of 3 and therefore  $ab$  is a multiple of 3. Hence it is shown that  $ab$  is a multiple of 3.

Next consider  $abc = 2uv(u^2 - v^2)(u^2 + v^2) = 2uv(u^4 - v^4)$ . The argument used for  $ab$  shows that  $abc$  is a multiple of 5.

To summarize,  $b$  is a multiple of 4,  $ab$  is a multiple of 3,  $abc$  is a multiple of 5. Since  $a, b$  and  $c$  are co-prime, a primitive Pythagorean triple contains exactly one multiple of 3, exactly one multiple of 4 and exactly one multiple of 5:  $b$  is a multiple of 4,  $a$  or  $b$  is a multiple of 3,  $a$  or  $b$  or  $c$  is a multiple of 5.

## Dave Wild

The proof will be by contradiction and will use modular arithmetic.

Modulo 3. Assume that none of  $a$ ,  $b$ , and  $c$  are divisible by 3. Then  $a, b, c \equiv \pm 1$  and  $a^2, b^2, c^2 \equiv 1$ . As  $1 + 1 \equiv 1$  is not true it follows that 3 divides at least one of  $a$ ,  $b$ , and  $c$ . For primitive triples  $\gcd(a, b, c) = 1$ ; so exactly one of  $a$ ,  $b$ , or  $c$  is divisible by 3.

Modulo 5. Assume that none of  $a$ ,  $b$ , and  $c$  are divisible by 5. Then  $a, b, c \equiv \pm 1, \pm 2$  and  $a^2, b^2, c^2 \equiv 1$  or  $4$ . The equations  $(1 \text{ or } 4) + (1 \text{ or } 4) \equiv (1 \text{ or } 4)$  are not true however we chose the values 1 or 4; so, as previously, we can deduce exactly one of  $a$ ,  $b$ , or  $c$  is divisible by 5.

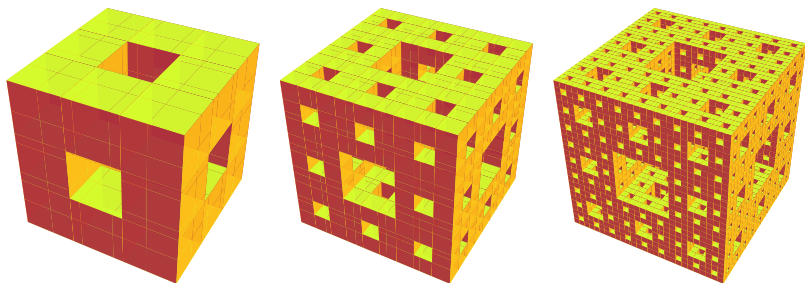
Modulo 8. Things do not work out if we use modulo 4. Assume that none of  $a$ ,  $b$ , and  $c$  are divisible by 4. Then  $a, b, c \equiv \pm 1, \pm 2, \pm 3$ , and  $a^2, b^2, c^2 \equiv 1$  or  $4$ . This is similar to the modulo 5 case, and we can deduce exactly one of  $a$ ,  $b$ , or  $c$  is divisible by 4.

Therefore a primitive Pythagorean triple contains exactly one multiple of 3, 4, and 5. Since  $(3, 4, 5)$  is a primitive triple there are no larger values to be found.

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## Problem 287.1 – Menger sponge

This is like Problem 274.2 – Holey cube. The  $n$ th iteration of the  $1 \times 1 \times 1$  Menger sponge can be thought of as a collection of  $3^{-n} \times 3^{-n} \times 3^{-n}$  cubes. How many little cubes are there, and how many of their faces are exposed?




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## Problem 287.2 – Magic cube

Arrange the numbers  $1, 2, \dots, 27$  in a  $3 \times 3 \times 3$  cube such that each row, column and whatever the corresponding structure in the third dimension is called sums to 42.

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# Black hole explorers

## Tommy Moorhouse

**Introduction** This article is intended to show the pitfalls of taking the textbooks too literally when considering motion in General Relativity. It explores a paradox, its resolution, and some practical applications. Hopefully anyone with a basic exposure to special relativity (adding velocities) and differentiation will be able to see what is going on.

**An ambitious mission** An advanced civilization has an ambitious plan to explore its local black hole. From an orbiting space station the crew will release a probe which will fall freely to a distance  $r_e$  from the singularity, taking measurements. A powerful spring will then separate the probe into two halves of equal mass, which will fly apart in opposite directions. The first will fall into the black hole, while the second, the data capsule, will travel radially outwards, just reaching the orbit of the waiting space station.

The scientists have done their calculations, based on their copy of S. Chandrasekhar's famous textbook [Chandrasekhar] (Douglas Adams could possibly have explained how it came into their possession), and some believe that it will be possible to have the separation occur inside the event horizon and still recover the data capsule. Here's how their reasoning works. We have set  $c = 1$  and  $G = 1$  to reduce notational clutter.

**The infalling probe** Radial motion in the Schwarzschild space-time is easy to describe. Writing

$$g(r) = \left(1 - \frac{r_S}{r}\right)$$

the Lagrangian for radial motion of a massive particle may be written

$$2L = g\dot{t}^2 - g^{-1}\dot{r}^2 = 1.$$

The dots denote differentiation with respect to a parameter on the world line and  $r_S$  is the Schwarzschild radius of the black hole. To see what the parameter might be we consider the momenta associated with the Lagrangian. First, the (specific) energy is

$$p_t = \frac{dL}{dt} = g\dot{t} \equiv e.$$

The radial momentum is

$$p_r = g^{-1}\dot{r}.$$

The energy-momentum equation

$$g_{\mu\nu}p^\mu p^\nu = m^2$$

gives

$$g^{-1}e^2 - g^{-1}\dot{r}^2 = m^2$$

which means we should take  $\dot{r} = mdr/d\tau$  where  $\tau$  is the proper time of the particle. Then  $e = E/m$ , the ‘specific energy’ (energy per unit mass) of the particle. In this situation an object falling from rest to a distance  $r$  from a distance  $r_0$  will attain a speed (as determined by its proper time  $\tau$ ) of

$$\left(\frac{dr}{d\tau}\right)^2 = r_S \left(\frac{1}{r} - \frac{1}{r_0}\right).$$

You might want to derive this result for yourself. The alien scientists believe that if the probe is released from a distance  $r_0$  then its speed as it crosses the event horizon at  $r = r_S$  will be

$$\sqrt{\left(1 - \frac{r_S}{r_0}\right)} < 1 \quad (\text{i.e. } < c).$$

**Separation** The powerful spring is released when  $r = r_e$ . The probe separates into two halves of equal mass. In the probe’s instantaneous rest frame (the ‘centre of mass frame’ or COM) the pieces fly apart with equal and opposite momenta. Suppose the mass of one of the separated halves is  $\mu$ , and the speed in the COM frame is  $v$ . Let  $\gamma = 1/\sqrt{1-v^2}$ . Then the total energy of the separating halves is conserved:

$$2\gamma\mu = m.$$

Note that the momentum of the halves is  $\pm\gamma v\mu$ . Solving for  $v$  we find

$$v = \sqrt{1 - \left(\frac{2\mu}{m}\right)^2}.$$

Clearly  $2\mu < m$  if the halves separate. We add the separation speed to the instantaneous inward radial speed (which we call  $-|V|$ ) at the moment of separation to get the outward speed in a distant stationary observer’s rest frame:

$$\frac{v - |V|}{1 - v|V|} = v_+.$$

Now suppose that a mass equal to that of our separated module were to be released from the space station. It would reach  $r_e$  with speed

$$|V|^2 = r_S \left( \frac{1}{r_e} - \frac{1}{r_0} \right)$$

as above. If we instead launch the same mass outward from  $r_e$  with this speed then it will just reach the space station, so we take  $v_+ = |V|$  to find

$$\frac{v - |V|}{1 - v|V|} = |V|$$

so that

$$v = \frac{2|V|}{1 + |V|^2}.$$

As long as  $|V| < 1$  we will have  $v < 1$ , which means that the rest mass of the data capsule can be positive. Perhaps the scientists are right—perhaps the probe can send back information from inside the event horizon! Those old textbooks (including [Chandrasekhar]?) could be wrong!

**Not so fast!** There are problems with this analysis, beyond the fact that from the point of view of the space station the probe never reaches the horizon: in the time  $t$  measured by an observer at infinity

$$\left( \frac{dr}{dt} \right)^2 = \left( 1 - \frac{r_S}{r} \right),$$

which tends to zero as the probe approaches the horizon. The real problem is the definition of radial velocity. In fact we should not interpret  $\dot{r}$  as the radial velocity measured in a co-moving inertial frame, because the metric includes the term  $g(r)^{-1}\dot{r}^2$ , which modifies the speed = distance/time relationship.

**Solving the paradox** Show that, in fact, given that the momentum  $p_r = mg^{-1}dr/d\tau$  we must have, in the instantaneously co-moving frame, a radial speed  $v_r$  defined by

$$\gamma_r v_r = g(r)^{-1} \frac{dr}{d\tau}.$$

Expand this using  $\gamma_r = 1/\sqrt{1 - v_r^2}$  to find

$$v_r = \frac{\frac{dr}{d\tau}}{\sqrt{g(r)^2 + \left( \frac{dr}{d\tau} \right)^2}}.$$

Deduce that the probe reaches the speed of light as it crosses the event horizon  $r = r_S$ , regardless of the starting point  $r_0 > r_S$ . Thus the apparent conflict with the standard result is resolved.

The data capsule can still be returned if the separation happens outside the event horizon. Perhaps you can calculate the mass defect  $\Delta m = m - 2\mu$  for the general case (when the capsule just reaches the orbiter). We could consider the ‘ballast’ (the half that falls into the black hole) to be part of the sacrificed mass. My answer is

$$\frac{\Delta m}{m} = \frac{2r_S^2(r_0 - r_e)^2}{r_0^2(r_e - r_S)^2 + 2r_S^2(r_0 - r_e)^2}.$$

**Explore!** For those happy to accept the result, it may still be interesting to see what proportion of the probe’s mass would be lost in certain situations. For example, a probe may enter the Solar System ‘from infinity’ ( $r_0 \rightarrow \infty$ ) to gather data for an alien civilization. What proportion of its mass must be sacrificed if the probe separates at Earth’s orbit and returns to infinity? What about a Sun-grazing probe launched from Earth? Use  $c = 3 \times 10^8 \text{ms}^{-1}$ ,  $G = 6.7 \times 10^{-11} \text{Nm}^2 \text{kg}^{-2}$ ,  $M_\odot = 2 \times 10^{30} \text{kg}$  and take the radius of the Earth’s orbit to be  $1.5 \times 10^{11} \text{m}$ . Then the Schwarzschild radius of the Sun is  $r_S = 2GM_\odot/c^2$ . Some approximations would be justified.

## Reference

[Chandrasekhar] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Classics, 1998.

## Problem 287.3 – Group determinant

Let  $G$  be a finite group of order  $n$ , label its elements  $G_1, G_2, \dots, G_n$ , and associate with each element  $G_i$  a variable  $x_i$ ,  $i = 1, 2, \dots, n$ . The *group determinant* of  $G$  is the determinant of the matrix  $M$  whose element in row  $r$ , column  $c$  is  $x_k$ , where  $r, c = 1, 2, \dots, n$  and  $G_k = G_r G_c^{-1}$ . For example, the diagonal elements of  $M$  will be  $x_e$ , where  $G_e$  is the identity element of the group. (You might want to prove that this definition is sound.)

Now consider  $S_3$ , the group of permutations of  $(1, 2, 3)$ , with elements

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).$$

Construct the group determinant of  $S_3$  and show that it factorizes as  $ABC^2$ , where  $A$ ,  $B$  and  $C$  are expressions in  $x_1, x_2, \dots, x_6$ .

# What makes a theorem a ‘good’ theorem?

## Sebastian Hayes

What is a *theorem*? Something that requires proof as opposed to an *axiom*, a *postulate*, or a *definition*—these, taken together, are the four Euclidian categories. Heath, the best translator of Euclid, speaks of ‘propositions’ rather than ‘theorems’ and in some ways this is better since it has the sense, ‘Hey! What do you think of this?’

A theorem must have generality: it is not the same as a ‘result’. Strictly, to be classed as a theorem an assertion must actually have been proved—otherwise it is a conjecture. But usage is not consistent here: for some reason Fermat’s claims always seem to have been rated as ‘theorems’ although they were often inspired guesses, while we still talk of Goldbach’s Conjecture (‘Every even number greater than 4 is the sum of two odd primes’) even though it is almost certainly true.

On what grounds do we, or should we, consider one theorem superior to another? After some ponderings I came up with the following desiderata. A good theorem should ideally be true, simple, basic, unobvious, illuminating, suggestive, beautiful and readily applicable.

**1. True** Is this essential? Even if untrue a ‘theorem’ can be very worthwhile if it fulfils some of the other categories, in particular if it is suggestive (of new lines of research). Suppose Fermat’s Last Theorem turned out to be false for some very large power; this would hardly matter because it has given rise to such interesting and important mathematics over the centuries.

Conversely, there are theorems which, though false, deserve to be true (e.g. Ramanujan’s formula for the distribution of the primes) while there are in modern mathematics plenty of apparently true theorems that are so nonsensical they certainly deserve to be false (e.g. Banach’s Two Sphere Theorem).

**2. Simple** I mean simple to state not simple to prove. Ironically many of the theorems of pure mathematics that have given the most trouble are the simplest to state (Fermat’s Last Theorem, Four Colour Theorem).

**3. Basic** For example, Angle at centre = twice angle at circumference;  $G.M. \leq A.M.$  The requirement of being basic conflicts with many other criteria.

**4. Unobvious** Is everything obvious? Nothing? G. H. Hardy considered that the term should be banned from mathematics but this approach, typical of a modern author, is impractical. If we did not take certain things as ‘obvious’ we would not be able to live, certainly not think. It is extremely silly to introduce the proposition  $A = A$  into a mathematical system as a theorem as some modern authors do.

In my view,  $1 + 1 = 2$  is best viewed as a definition rather than a theorem—it appears as a theorem in Russell and Whitehead’s *Principia Mathematica*. Certain elementary numerical and above all topological notions (unity, plurality, nearness, farness, on, under &c.) would seem to be hard-wired into our brains, the mathematical equivalent of Chomsky’s Universal Grammar. Such notions/perceptions should not be presented as theorems, i.e. by definition statements that need to be ‘proved’ since this presupposes that there is something even more fundamental on which they rely for their validity. As Ramus wrote, ‘*absurdum naturalium rerum reveritatem per falsas causas demonstrare*’ (‘It is absurd to establish by fantastical reasons the truth of natural things’).

On the other hand, Euclid is absolutely right to introduce  $(a \times b) = (b \times a)$  as a theorem instead of taking it on board as one of the Axioms for Fields as the moderns do. Viewed as a statement about the real world it is by no means obvious and Euclid has to appeal to seven or eight earlier theorems to establish it. A chieftain with control over ten villages, each of which was able to provide seven young men as warriors, might well have been surprised to be told by his shamanic advisor that his strength was no less and no greater than that of a rival who only controlled seven villages each able to provide ten young men.

The combination Basic + Unobvious + True makes for a very powerful—but not necessarily impressive—theorem. An amazing amount of Euclidian geometry can be established on the basis of Ptolemy’s Theorem, e.g. rules for adding sines and cosines, half angle formulae &c., even Pythagoras’s Theorem itself, (see Eli Maor, *Trigonometric Delights*, ch. 6). But the theorem itself, ‘The product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the products of the opposite sides’ seems at first sight mundane and unappealing.

The Taniyama–Shimura Conjecture (‘Every elliptic curve has an equivalent modular form’) was so unobvious as to appear quite fantastic to most mathematicians when it was first proposed. It must in some sense be basic since it was in part by proving this proposition that Wiles established Fermat’s Last Theorem.

**5. Illuminating** A computer, fed with a few axioms, can churn out countless derivations—but how many of them will be worth reading? At the other extreme, we have Fermat who scarcely ever gave any proofs and was often wrong—and yet he had the knack of throwing light on all sorts of areas of Number Theory. Can a theorem be basic without being illuminating? Not sure.

**6. Suggestive** I mean suggestive of further development. Certain theorems close doors rather than open them. Indeed certain theorems are designed to do just this like the theorem which states that there does not exist any algebraic formula capable of producing all the primes. I sort of feel that a destructive theorem shouldn't be rated as highly as a constructive one: we don't put Genghis Khan in the same league as Napoleon or Alexander the Great.

**7. Beautiful** Is beauty in the eye of the beholder? Are there any guidelines? 'Simplicity' and 'Orderliness' are often invoked. Someone said of the Theory of Differential Equations, 'This is not mathematics, it's stamp collecting'. (But the applied mathematician cannot afford to ignore the real world and Nature is not usually tidy.)

Simplicity is especially to be applauded when it is least expected. Mathematicians at the time gave up on the search for a sum to the reciprocals of the squares until the incomparable Euler fished out of his hat

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

But richness of texture has its appeal also. If you can get incredibly complicated expressions to boil down to some very simple sum or product, the resulting theorem has a baroque beauty as in so many of Ramanujan's discoveries (see M500 202, pp. 7–8). Symmetry is attractive but not if it is overdone. It is particularly effective, once again, if it appears where one does not expect it as in

$$\tan\left(\frac{1}{2}(n+1)\alpha\right) = \frac{\sin\alpha + \sin 2\alpha + \dots + \sin n\alpha}{\cos\alpha + \cos 2\alpha + \dots + \cos n\alpha}.$$

Baudelaire considered that there must be an element of strangeness in beauty: strangeness is arresting while orderliness reassures. De Moivre's formula and  $e^{i\pi} + 1 = 0$  were surpassing strange when they were first unleashed on the world but it is well nigh impossible to startle a pure mathematician these days. The most unexpected mathematical achievement today would be to discover something whose importance the man in the street can actually understand.

**8. Readily applicable** Alas, it is on this score that so many aspiring mathematical bathing belles get disqualified. Leibnitz's

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

is a pretty enough result but useless for calculating  $\pi$  since it converges so slowly. The same goes for Wilson's crisp  $(p-1)! \equiv -1 \pmod{p}$  as a test for primality.

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## Solution 278.2 – Symbols

There are  $q$  symbols. How many unordered  $n$ -tuples of symbols are there? For instance, when  $q = 3$  and  $n = 2$  the answer is 6, AA, AB, AC, BB, BC, CC.

### Peter Fletcher

Following the example in the question, we can use brute force to construct Table 1.

Table 1: Number of  $n$ -tuples for the first few values of  $n$  and  $q$ .

$nq$	$q$							
	1	2	3	4	5	6	7	...
1	1	2	3	4	5	6	7	
2		3	6	10	15	21	28	...
3			10	20	35	56	?	
4				35	70	?	?	...
$\vdots$		$\vdots$		$\vdots$		$\vdots$		

Obviously, the number of 1-tuples is just  $q$ . Observe that for  $n = 2$ , we have

$$3 = \frac{2 \times 3}{2}, \quad 6 = \frac{3 \times 4}{2}, \quad 10 = \frac{4 \times 5}{2}, \quad \text{etc.}$$

and we can see that the number of unordered 2-tuples is  $\frac{q(q+1)}{2}$ . For  $n = 3$ , we have

$$10 = \frac{3 \times 4 \times 5}{6}, \quad 20 = \frac{4 \times 5 \times 6}{6}, \quad 35 = \frac{5 \times 6 \times 7}{6} \quad \text{and} \quad 56 = \frac{6 \times 7 \times 8}{6}$$

and we can see that the number of unordered 3-tuples is  $\frac{q(q+1)(q+2)}{3!}$ . For  $n = 4$ , we have

$$35 = \frac{4 \times 5 \times 6 \times 7}{24} \quad \text{and} \quad 70 = \frac{5 \times 6 \times 7 \times 8}{24}$$

and we can see that the number of 4-tuples is  $\frac{q(q+1)(q+2)(q+3)}{4!}$ . We conclude that the number of unordered  $n$ -tuples of  $q$  symbols is given by

$$\frac{q(q+1)(q+2) \cdots (q+n-1)}{n!} = \frac{(q+n-1)!}{(q-1)!n!}.$$

## Solution 282.2 – Isosceles triangle

There is an isosceles triangle  $ABC$  with  $BC$  on the  $x$ -axis and  $A$  above it. The side lengths are  $|AB| = |AC| = 191$  and  $|BC| = 60$ . Find all points  $X_k = A + (k, 0)$  such that  $|BX_k|$  is an integer,  $k = 1, 2, \dots$ . Note that  $AX_k$  is parallel to  $BC$ .

### Peter Fletcher

The  $y$  co-ordinate of  $A$  above is clearly  $\sqrt{191^2 - 30^2}$ , which is also the  $y$  co-ordinate of  $X_k$ . If  $B$  is at the origin and  $X_k$  is a distance  $k$  to the right of  $A$ , then the  $x$  co-ordinate of  $X_k$  is  $k + 30$ .

We know that  $BX_k$  is an integer, call it  $n$ . Then since we have a right-angled triangle, it is true that

$$(k + 30)^2 + (191^2 - 30^2) = n^2,$$

or

$$n^2 - (k + 30)^2 = 191^2 - 30^2.$$

This may be written as

$$(n + k + 30)(n - k - 30) = 221 \times 161 = 7 \times 13 \times 17 \times 23.$$

The difference between the factors on the LHS is  $(n + k + 30) - (n - k - 30) = 2k + 60$ .

There are seven ways in which the RHS can be written as a product of two numbers. In each case, the difference between these two numbers must be  $2k + 60$  for a particular value of  $k$ , so subtracting 60 from each difference and dividing by 2 gives this  $k$ . We can tabulate the results.

Two numbers	Difference = $2k + 60$	$k$	$n$
$(7 \times 17) \times (13 \times 23)$	$299 - 119 = 180$	60	209
$(7 \times 13) \times (17 \times 23)$	$391 - 91 = 300$	120	241
$23 \times (7 \times 13 \times 17)$	$1547 - 23 = 1524$	732	785
$17 \times (7 \times 13 \times 23)$	$2093 - 17 = 2076$	1008	1055
$13 \times (7 \times 17 \times 23)$	$2737 - 13 = 2724$	1332	1375
$7 \times (13 \times 17 \times 23)$	$5083 - 7 = 5076$	2508	2545
$1 \times (7 \times 13 \times 17 \times 23)$	$35581 - 1 = 35580$	17760	17791

This table lists all possible values of  $k$  and the resulting values of  $n$ .

We can't have  $(7 \times 23) \times (13 \times 17) = 161 \times 221$  because  $221 - 161 = 60$  and then  $k = 0$ . This is not allowed because the question specifies that  $k$  is a positive integer.

## Ted Gore

Let  $B$  be the point  $(0, 0)$ . Let  $A$  be the point  $(30, p)$ , where

$$p^2 = 191^2 - 30^2 = 35581 = 7 \cdot 13 \cdot 17 \cdot 23.$$

Now  $X_k$  is the point  $(m, p)$ , where  $m = k + 30$ . Let  $q$  be the length of the line  $BX_k$  so that we have  $q^2 = m^2 + p^2$ , or  $(q+m)(q-m) = 35581$ .

There are 8 ways to distribute the prime factors of 35581 into two sets. As an example I let  $7 \cdot 13 \cdot 23 = 2093 = (q+m)$  and  $17 = (q-m)$ . From this we calculate  $q = (2093 + 17)/2 = 1055$ ,  $m = 1038$  and  $k = 1008$ . The complete list of answers is in the table.

$m$	$k$	$q$
30	0	191
90	60	209
150	120	241
762	732	785
1038	1008	1055
1362	1332	1375
2538	2508	2545
17790	17760	17791

## Chris Pile

Let  $BP$  be perpendicular to the  $x$ -axis, meeting the line through  $X_k A$  at  $P$ . Then

$$|PB|^2 = 191^2 - 30^2 = 35581.$$

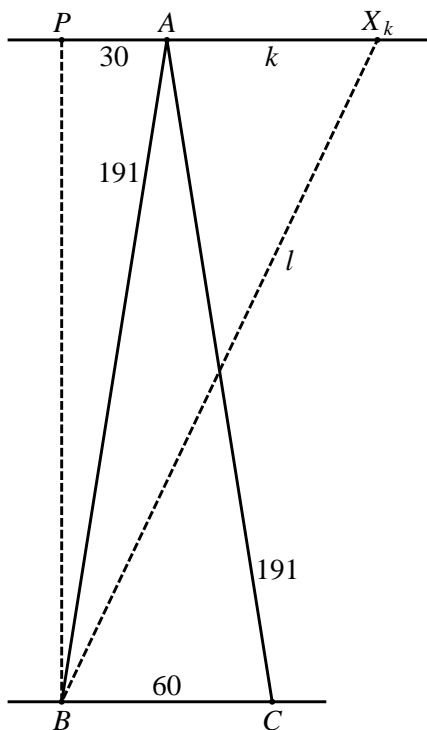
Let  $|BX_k| = \ell$  and  $|PX_k| = d$ . Then

$$k = d - 30 = |AX_k|.$$

We require  $d > 30$  and  $\ell$  to be integers, where

$$\ell^2 - d^2 = 35581.$$

But  $35581 = 7 \cdot 13 \cdot 17 \cdot 23$  and  $(\ell + d)(\ell - d)$ . These are the only pairs of integers that give 35581 when multiplied. [See the table in Ted Gore's solution, above.]



## Solution 225.5 – Pythagorean triangles

Find right-angled triangles with integer sides  $x$ ,  $y$ ,  $z$ ,  $z^2 = x^2 + y^2$ , such that  $z$  and  $x + y$  are squares.

The problem was posed by Fermat to Mersenne in 1643. So it shouldn't give you too much trouble, assuming Fermat did actually have a triangle or two in mind. However, I (TF) do not know whether Mersenne succeeded in finding any solutions.

### Dave Wild

If we look for a triangle with  $\gcd(x, y, z) = 1$ , then we can use the well-known formula which can be used to generate all primitive Pythagorean triples; i.e. triples  $(x, y, z)$  where  $z^2 = x^2 + y^2$ . If  $u$  and  $v$  are integers,  $u > v > 0$ ,  $\gcd(u, v) = 1$  and  $uv$  is even, then

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2$$

is a primitive Pythagorean triple.

In our case we know that  $z$  is a square; so we have  $z = Z^2 = u^2 + v^2$ . So we can generate this new primitive triple  $(u, v, Z)$  in a similar manner. If  $r$  and  $s$  are positive integers,  $r > s$ ,  $\gcd(r, s) = 1$  and  $rs$  is even, then

$$u = \max(r^2 s^2, 2rs), \quad v = \min(r^2 s^2, 2rs).$$

We can now choose any suitable values  $r$  and  $s$  and then calculate  $(x, y, z)$ . We know that  $z$  will be a square; so we just have to check whether  $x + y$  is a square. I tried writing  $x + y = f(r, s)$  to see if it would help me select appropriate values of  $r$  and  $s$  but got nowhere. Out came the computer. I chose  $n = 3, 5, 7, \dots$ , and positive integers  $r, s$  such  $r + s = n$  and  $r > s$ , calculated  $x + y$  and checked if it was a square. I was too lazy to check whether  $\gcd(r, s) = 1$ .

When  $r = 1469$  and  $s = 84$ , we have

$$\begin{aligned} u &= 2150905, & v &= 246792, \\ x &= 4565486027761, \\ y &= 1061652293520, \\ z &= 4687298610289 = 2165017^2, \\ x + y &= 2372159^2. \end{aligned}$$

The values  $u, v, x, y, z$  agree with those given for this problem on the Wikipedia page for 'Pythagorean triple'. It also states there are an infinite

number of such triples. If we multiply  $r$  and  $s$  by a positive integer  $m$ , then the corresponding values of  $u$  and  $v$  will be scaled by a factor of  $m^2$ , and  $x$ ,  $y$  and  $z$  by a factor of  $m^4$ .

Fermat must have had a better way of generating a solution. Are there any other primitive solutions? I checked for  $r + s = n < 100000$  and did not find any. Fermat was a lawyer and judge. If he sentenced someone to a triple life sentence, did that mean the miscreant had to spend the rest of their life computing Pythagorean triples?

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## Problem 287.4 – Two games

### Roger Thompson

The two-player game PickABead uses an even number  $N$  of interlocking beads, numbered 1 to  $N$ . The first player puts the beads together in some order to form a necklace. The second player breaks the necklace in one place, then removes a bead from one end. The players then alternate removing beads from an end of their choice, until there are none left. The winning player is the one with the higher sum of bead numbers. What strategy can the second player use to ensure that the first player never wins?

A second game HighBead is identical to PickABead, except that each player must always remove the end bead with the higher number. What is the minimum  $N$  ( $M$ , say) such that, by suitable construction of the necklace, the first player can always win. Is there a simple way of devising a winning construction for any  $N \geq M$ , ideally one that gives the highest possible winning margin?

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## Problem 287.5 – Four sins

### Tony Forbes

Compute

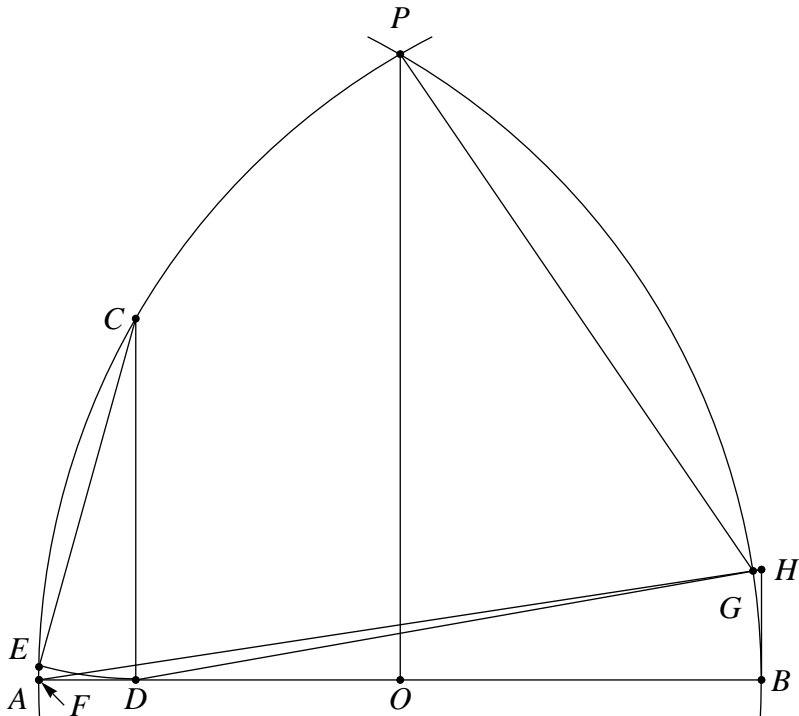
$$\int \frac{\sin x}{\sin(x-1)\sin(x-2)\sin(x-3)} dx$$

as an indefinite integral.

You might want to try  $\int (\sin x)/\sin(x-1) dx$  first. And when you have done them both, what about  $\int (\sin x)/(\sin(x-1)\sin(x-2)) dx$ ?

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**Problem 287.6 – 10 degrees**



Look at the diagram. We have  $|OA| = |OB| = |CD| = |CE| = 1$ ,  $|AB| = |AP| = |AG| = |BP| = |BC| = |BE| = 2$ ,  $|OP| = |GP| = \sqrt{3}$ ,  $D$  is vertically below  $C$ ,  $F$  is on  $AB$  and vertically below  $E$  (and very close to  $A$ ). Finally,  $H$  is on  $FG$  extended and vertically above  $B$ . Show that

$$\tan(\angle BDH) = \frac{\sqrt{3}(7 + \sqrt{5})(5 - \sqrt{13})}{8(7\sqrt{3} + 3\sqrt{13} + \sqrt{15} - 11)}.$$

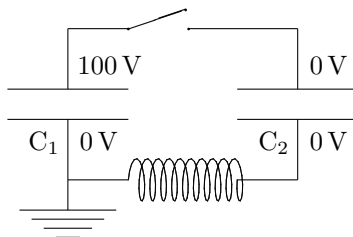
Hence  $\angle BDH \approx 10.00000004407655$  degrees. Since  $\tan(\angle BDH)$  is ruler-and-compasses constructible, this problem gives a nice way of trisecting an angle of 30 degrees, at least approximately. A similar construction appears in Chapter ‘Nonagons, Regular’ of *Mathematical Cranks* by Underwood Dudley.

## Letters

### Re: Problem 285.3 – A coil and two capacitors

My immediate reaction to this (highly theoretical!) problem was that if the components of the circuit (including the coil) have zero resistance, then, after the switch is closed, the current in the whole circuit will continue to oscillate indefinitely. But if it is going to oscillate, then it is likely to generate electromagnetic radiation and over time, lose energy that way. So I don't think it will oscillate indefinitely.

However, the junction between  $C_1$  and the coil is earthed. So that also complicates matters. If we assume that the Earth itself, and the connection to it, both have zero resistance, then the Earth will presumably behave as a giant capacitor.



When I studied for A level physics in 1950–51, we had to learn three types of electrical units. The EM unit of current was 10 amps, the ES unit of voltage was 300 volts, and the ES unit for capacitance was (as far as I remember) the capacity of a sphere of radius 1 cm. When I joined the OU in 1972 (or thereabouts), I found that the whole stupid system had been replaced by SI units. That was a fantastic improvement!

Anyway, if the Earth is a sphere of radius about 6,400 km, then it presumably has a capacity of  $4\pi\epsilon_{\text{Earth}}R_{\text{Earth}}$ , assuming  $\epsilon_{\text{Earth}} = \epsilon_0 \approx 8.854 \cdot 10^{-12}$  farads per metre. That only seems to be about 0.71 millifarads. I would have thought it would be greater than that! Anyway, another capacitor where earth is shown could upset matters.

Referring to the original Problem 256.5 (i.e. Problem 285.3 with the coil replaced by straight wiring) and the solution given involving an infinite current during zero time, which may be an excuse for me to have missed the obvious point that, although you may be able to reduce the resistance of the connecting cables to zero (e.g. by immersing everything in liquid helium), I think that you will not be able to reduce their few attohenries of inductance, and hence reactance, to zero.

The various solutions that M500 members had already given to the problem are, of course, largely mathematical. Even so, I still maintain that inserting a relatively large inductor in the form of a 'coil' will cause the whole circuit to oscillate when the switch is closed. And as there must be some inductance in the connecting wires, the original circuit will also oscillate, but at a very much higher frequency over a much shorter (but not zero!) time. In both cases, energy will be lost in the form of electromagnetic radiation, but not necessarily as heat at infrared frequencies.

**Colin Davies**

## Number words with letters in reverse alphabetic order

I have come across M500 262 (February 2015) 14, where Ken Greatrix asserts that ‘one’ is claimed to be the only number word with its letters in reverse alphabetic order, but he knows at least one other. This refers to equation (1) on page 19. This is quite cryptic and doesn’t really give an answer. After some contemplation, I have ‘pi’. Presumably this is right.

The only number word with letters in alphabetic order is ‘forty’. Another possibility is ‘cent’. Moreover, ‘e’ might be included in both categories and ‘ $e^x$ ’ might be considered as having letters in alphabetic order.

**David Singmaster**

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## M500 Mathematics Revision Weekend 2019

The forty-fifth M500 Revision Weekend will be held at

**Kents Hill Park Training and Conference Centre,**

**Milton Keynes, MK7 6BZ**

**from Friday 10th to Sunday 12th May 2019.**

The standard cost, including accommodation (with *en suite* facilities) and all meals from dinner on Friday evening to lunch on Sunday is £275 for single occupancy, or £240 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £160.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2019 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 10th April 2019. The Late Booking Fee for applications received after 10th April 2019 is £20, with no membership discount applicable.

There is free on-site parking for those travelling by private transport. For full details and an application form, see the Society’s web site:

**m500.org.uk.**

If you have any further questions please email the Revision Weekend Organizer on [weekend@m500.org.uk](mailto:weekend@m500.org.uk). The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

*Please note that the venue is not the same as last year.*

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## Problem 287.7 – Matrix powers

Given  $a_1, b_1, c_1, d_1$ , let  $M$  be a  $2 \times 2$  matrix defined by  $M^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ ,  $n = 1, 2, \dots$ . Show that  $b_n c_1 = b_1 c_n$  for  $n = 1, 2, \dots$ .

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## Advice for authors

We welcome mathematical contributions to M500 at any level from trivia to serious research. The magazine M500 is published six times a year, with publication dates 28 February, 30 April, 30 June, 31 August, 31 October and 31 December. Upon publication, an issue of M500 is distributed to all members of the M500 Society. It may also be given to non-members at the discretion of the Editor or other M500 Society members. We have recently updated our guidance notes, Advice for authors, which may be found on our website at

[m500.org.uk/magazine/](http://m500.org.uk/magazine/).

Key points are as follows.

1. We prefer an informal style but articles should be reasonably well written. We almost always edit submitted material, sometimes quite considerably, especially from authors whose first language is not English. If we have your email address, you will usually have a chance to check your article before it gets published.
  2. The most important advice we can offer is: Please read recent issues of the magazine and please conform to its style. Please also note that M500 is printed on paper using only black ink and that the text block is only 115 mm wide. For this reason,
    - (a) simple articles may be submitted on WORD or equivalent but not using the special equation functions since these are incompatible with our publishing software;
    - (b) substantial articles should ideally be written in LATEX but if this is not possible then further guidance may be found on our website at the address stated above, or by contacting the Editor at [editor@m500.org.uk](mailto:editor@m500.org.uk).
  3. Unwelcome material. Yes, we do seem to get our fair share of things that we cannot possibly use and more detail may be found in the guidance notes.
  4. GDPR. There are new rules on holding personal information and details of our policy with respect to authors information may be found on our website at [m500.org.uk/privacy/](http://m500.org.uk/privacy/).
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