M500 22

AN INTUITIVE ARGUMENT TO CONSTRUCT THE LEAST UPPER BOUND

1. One of the most fascinating and fundamental properties of the real numbers is the following:

   Let $S \subset \mathbb{R}$, $S \neq \varnothing$, be bounded from above. Then S has a least upper bound. Thus, if $\forall s \in S$, $\exists t$ such that $s \leq t$, t an upper bound of S, then $\exists t_0$ such that $t_0 \leq t$.

2. Without this completeness property most of calculus and everything depending on it would collapse to an empty set.

3. Neither M231 nor Spivak (that part on which M231 is based) prove this property. It would be nice if we could prove this lovely fact about reals. But clearly any proof must depend upon the definition of a real number. And M231 does not define a real number in the same way as M202 (Halmos) defines a natural number.

4. However, most of us have an intuitive idea of what a real number is. Let us, therefore, make an intuitive definition of a real number; r is a real number iff r is an infinite decimal. Thus $r = s$ iff both r and s have the same infinite decimal representation.

5. In the argument that follows we need the following facts:

   (A) If M is a non-empty subset of natural numbers, then M has a least member (so-called well ordering principle).

   (B) If a non-empty set of negative integers, K, is bounded from below, then the smallest integer is contained in K.

   (C) If $a < b$, $\forall a, b \in \mathbb{R}$, then $ka < kb$ for any positive real k.

6. Now the demonstration: We are given $S \neq \varnothing$. Let T be the set of all upper bounds of S. Thus $t \in T \Rightarrow t \geq s$, $\forall s \in S$. We have to show that T has a least member, i.e. $\exists\ t_0$ such that $t_0 \leq t$, $\forall t \in T$.

7. We shall consider the three cases:

   (i) Each $t \in T$ is non-negative, i.e. $t \leq 0$.

   (ii) Each $t \in T$ is positive, i.e. $t > 0$.

   (iii) Each $t \in T$ is negative, i.e. $t < 0$.

8. Case (i): It is clear that $0 \in T$ and for every other $t_0 < t$ and hence 0 is the least member of T.

Case (ii): (a) First note that each t which is an infinite decimal has an integral part made up of all the digits left of the decimal point.

(b) Consider the mapping $f : t \to 10^k t\ \forall t \in T$ for some suitable k. Call the image set under f $10^k T$.

(c) The purpose of this map f is to transform each $t \in T$ such that it has a non-zero integral part.

(d) Suppose, therefore, that we have found a suitable k such that each $10^k t$ has a non-zero integral part.

(e) Now consider the integral part of each $10^k t$. It is plain t the set of all integral parts is a set of positive integers, say P, which by property (A) has a least member. Call it L.

(f) It may, however, happen that in T there is more than one member such that the first k digits of their decimal representations are the same. Thus, L may not be unique.

(g) Let $T_1 = \{t: f(t) = L\}$. Thus T_1 contains all those t which have the same first k digits, and, the integer L is the smallest integer in P (see (e) above).

(h) Now apply the mapping f to each t in T_1, and repeat the process until we find a unique t_0 such that $f(t_0)$ has the smallest integral part. The fact that such a t_0 exists follows from

(1) property (A) above, and

(2) f can be applied infinitely many times.

(i) Having found the smallest integer which is the integral part of some $t \in T$, all that needs to be done is to apply the reverse map f^{-1} which will map the set $10^k T$ to T by multiplying each $10^k T$ by 10^{-k}. It is clear that $t_0 \leq t, \forall t \in T$. Recall that maps f and f^{-1} preserve order.

Case (iii): Similar argument as above together with the property (B) would help us to produce the smallest negative $t_0 \in T$.

9. Combining the three cases not only have we demonstrated the existence of the least member of T but actually constructed one.

Unfortunately, the argument is not only intuitive but highly wordy.

Could some reader suggest if the argument can be made rigorous and attractive?

Datta Gumaste

SEQUENCES

Will you allow a just-post-foundation, new subscriber to answer some of Tony Brooks' comments on number sequences (M500/19 p.10)?

Intelligence tests that ask for the next numbers in a sequence may be designed to select conformists. In many jobs originality is not a virtue.

However, Problem 18.4 did ask for a rule for generating further terms, as well as the next two numbers. So, if you produce a suitable rule, any solution is ‘correct’. I certainly envy the intelligence that can invent several ways of continuing the given sequences. I had trouble finding one rule for (a) 1, 2, 4, 8, 18, 52, 206, 1080, 6994, ...; then, obsessed with finite differences, failed to recognise any pattern in (b) 3, 3, 5, 4, 4, 3, 5, 5, 4, 3, 6, 6, ...

But why is one solution better than any others? (i.e. why study maths?). I suggest that mathematicians like to simplify and generalise, so a simple, general rule is preferable to a complex conditional one. An elegant theorem gives some aesthetic satisfaction.

By the way, can we claim to be students of ‘occult justification’ now? Sounds much more exciting than maths, doesn't it?

Marjorie Brew

COMPUTING IS MATHS

- *OR IS MATHS COMPUTING*?

Mike Stanton in his letter entitled ‘Computing Is Not Mathematics’ immediately qualifies his position by referring only to commercial computing (M500/19). Even the weaker assertion ‘commercial computing is not mathematical’ is quite untrue. The position is that computing has traditionally been divided into the fields of ‘commercial’, ‘scientific’ and ‘systems’, with some other areas, like information retrieval, being squeezed into one or the other according to taste. It is often forgotten (particularly among some commercial programmers) that underlying all computing is the substructure of Theoretical computing. Commercial programmers are rarely called upon to face this fact, since most of them rely heavily on the expertise of others who work in software houses or for computer manufacturers.

By Theoretical computing I do not mean just the Theory of Computation, as introduced in M202, but the whole body of work involving the theoretical analysis of computational ideas. The major deficiency in M251, apart from errors of fact which have probably been corrected since I took the course, is that there is insufficient theoretical support for the loosely connected set of techniques presented. Such support is, oddly enough, present in part of M202, but in such a form that its applicability to computing is not always immediately apparent. I would suggest the production of ‘course linking’ units in cases like this. Such units would consist of a series of brief notes indicating links, together with more references than are usual in maths units, so that anyone who is so inclined can follow up the leads indicated.

Now to the question posed above - is maths computing? Maths is computing in two different senses. The first sense is that the extent to which any mathematician can solve problems is ultimately determined by the amount of manipulation which he can accomplish. Since computers have come into common use the amount of manipulation which can be done has increased dramatically, for two reasons. One reason (the obvious one) is that once the solution to a problem is formulated in computational terms, the processing by machine is very rapid. The second reason is that the use of computers has forced the development of methods of problem formulation which constitute a new mathematical notation. As with all improvements in notation, there has been an increase in the facility with which problems can be solved.

The second sense in which maths is computing is connected with the foundations of maths. A very important problem, called in English "the decision problem for quantification theory", but usually referred to as the "Entacheidungsproblem" is the central issue here. This problem may be simplified in the following way:

Consider all meaningful expressions that can be written down using the ordinary symbols of symbolic logic: ~ (not), & (and), ∨ (or), ⇒ (if...then), (∀) (universal quantifier), (∃) (existential quantifier), and suitable symbols for variables, relations and functions. Some of these well-formed expressions will be valid, that is, true no matter how the mathematical symbols within them are interpreted. The Entscheidungsproblem is to give an algorithm by which you can tell whether or not such a well-formed formula is indeed valid. Hilbert and his followers did a lot of work on this problem, and research on certain aspects of the problem continues. The problem is regarded as important because the branches of maths can all be developed within this formulation of symbolic logic. For example, it is possible to write down a set of axioms in this notation, such that all of classical mathematics is logically derivable from those axioms (see Gődel, 1940). The idea of logical derivability can be reformulated in terms of validity. Then if there were an algorithm for checking whether formulas are valid or not, it could be used to check whether statements are provable in various branches of maths. All problems in classical maths are therefore essentially computational. The Entscheidungs-problem was shown to be unsolvable by Turing and Church in 1936.

References:

1. A. Church, 1936. "A note on the Entscheidungsproblem". *Jnl of Symbolic Logic*, vol. 1, pp 40-41, 101-102.

2. K. Gődel, 1940. "The consistency of the Continuum Hypothesis". *Annals of Mathematics Studies*, Princeton.

3. A.M. Turing, 1936-1937. "On computable numbers, with an application to the Entscheidungsproblem". *Proc. London Math Society*, Series 2, vol. 42, pp 230-256, with correction vol. 43, pp. 544-546, 1937.

Andrew Arblaster

STAFF-MOUTH APPEAL

My phone *hasn't started ringing*, even with 28 students on my roll this year. *No* MOUTHS have *ever* contacted me. So ?

Peter Hartley (STAFF-MOUTH)

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ALGEBRAIC SYSTEMS

– A NOTE IN REPLY

Datta Gumaste (M500/18) specified a Semi-group (S, o) as a non-empty set and a binary operation which satisfies:

(a) o is closed,

(b) o is associative,

and enquired whether it was possible to construct a Monoid (M, o, e) from the semi-group, so it had the additional property of containing an identity-element, and the construction was carried out in the same way as the set of integers is constructed from the natural numbers.

Also could groups be constructed from the class of Semi-groups or from the class of Monoids.

The answer to both questions is in the affirmative. However, before proceeding further there is an important remark which should be digested.

REMARK

It is of common occurrence to find in text-books (and some OU courses) when demonstrating the ‘Countability’ of the rational positive numbers that these are set out in arrays:

1/1	1/2	1/3	1/4	1/5	...
2/1	2/2	2/3	2/4	.	
3/1	3/2	3/3	.	.	
4/1	4/2	.	.	.	
5/1	.	.	.	.	

it is then stated that a one-to-one mapping onto this array from the set ℕ shows that the set of rationals is countable. It is then pointed out that ‘unfortunately’ each rational number appears more than once (e.g. 1/2 appears as 2/4, 3/6, etc.) and therefore the ‘repeats’ should be ignored, bypassed or forgotten.

It is not pedantic to disagree with this method as unsatisfactory; as will be seen, it is quite crucial to the answer given to Datta Gumaste, and also to the associated problem: “is it possible to count a finite set of absolutely identical objects?”

Therefore, instead of ignoring the repeated elements, the *union* of all similar elements will be taken.

(e.g. call

$1/1 \cup 2/2 \cup 3/3$ ... the rational number 1

$2/1 \cup 4/2 \cup 6/3$ ... the rational number 2

and so on.)

CONSTRUCTION OF MONOID PROM SEMI-GROUP

Let *S* be a set which is not empty, and let o be some binary operation which is associative and let the set be closed under o.

Consider *P*(*S*) which is defined as "the set of all one-to-one mappings onto itself". Even if the ordering of the elements of *S* is not known (e.g. they could be identical objects) still *P*(*S*) will be the same as the total number of mappings.

Suppose one element of *S* is fixed, then the one-to-one mappings from *S* onto itself form an automorphism of *S*. It is obvious that *P*(*S*) consists of all the automorphisms of *S*. These mappings can be regarded as permutations of the elements of the set, and it is customary to write a permutation as the 'product of its cycles', e.g. a set of three elements would have (1, 2, 3) as an identity permutation, and (l) (2 3), (l) (3 2) as its (equivalent) automorphisms of the element 1.

Now since our elements may be indistinguishable let us define equivalence classes as follows:

"any permutation with the same number of cycles is equivalent".

Thus the *union* of all one-cycle permutations is the *identity* in *M*. It is written thus: {(1)}. Now write the union of all two-cycles as {(1), {(1)}} or {2}; the union of all three-cycles as {1, {2}}, or {3} and so on by the principle of recursion for all the possible cycles.

Thus we have our Monoid. The elements consist of arrangements of the elements of the Semi-group, one of which is the identity. The same binary operation will still hold between the elements of the Monoid, and will still be closed and associative. The Monoid was constructed in the way it is possible to construct the integers from the natural numbers, i.e. by counting.

Note: 1. The equivalence of P(S) to the total number of possible permutations of identical objects is used in course ST285 when discussing Boltzman distribution of atoms in a gas.

Regarding the formation of groups – the Monoid is a group since it has an identity and is *closed*. Closure implies inverses. Groups can be formed from *S* by making *S* into a Monoid!

P. Newton

MATH-QUOTE - Ron Davidson

Mathematics is in the most general sense the science of relationships.

Carl Friedrich Gauss

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WHY IDEALS?

Bob Davies raises some interesting points in his discussion of $\mathbb{R}$ modulo 6 (Problem 21.3). Topologically $\mathbb{R}$ modulo 6 (that is, $\mathbb{R}$ modulo $6\mathbb{J}$, where $\mathbb{J}$ denotes the integers) is a circle, and so its direct product with itself is certainly a torus. There are some nice pictures of this in M202 Unit 15. However, the difficulty comes when you try to draw the graph of $y = (x - 1)(x - 2)$ on this torus. In posing this problem one is assuming a ‘reasonable’ addition and multiplication on $\mathbb{R}$ modulo $6\mathbb{J}$ (perhaps that $\mathbb{R}$ modulo $6\mathbb{J}$ is a ring?) and unfortunately there is no reasonable multiplication on $\mathbb{R}$ modulo $6\mathbb{J}$. To take the example given by Bob Davies, we can define $1\frac{1}{2} = 7\frac{1}{2}$ (mod 6) but then what is $\frac{1}{2} \times 1\frac{1}{2}$ to be? $\frac{3}{4}$ and $3\frac{3}{4}$ are possible candidates, but they do not differ by an integral multiple of 6 and so are not the “same” mod 6.

The trouble lies in the way $6\mathbb{J}$ sits inside $\mathbb{R}$. We want to define $a = b$ mod 6 if $a = b + 6n$ for some $n \in \mathbb{J}$. Any sensible definition of “=” must lead to an equivalence relation, that is we require the relation “= mod 6” to be reflexive, symmetric and transitive. These conditions lead in turn to the requirements that $6\mathbb{J}$ contain 0, contain the negatives of its elements, and be closed under addition; in other words $6\mathbb{J}$ must be an additive subgroup of $\mathbb{R}$, as indeed it is.

Now, is addition properly defined? That is, if $a = b$ mod 6 and $c = d$ mod 6 does $a + c = b + d$ mod 6? In fact yes, for if $a = b + 6n$ and $c = d + 6m$ ($n, m \in \mathbb{J}$), then $(a + c) = (b + d) + 6(n + m)$ and $(n + m) \in \mathbb{J}$. The closure of $6\mathbb{J}$ under addition ensures that addition is properly defined on $\mathbb{R}$ modulo $6\mathbb{J}$. What about multiplication? Does $ac = bd$ mod 6? Well, $ac = (b + 6n)(d + 6m) = bd + 6(nd + mb + nm)$, so $ac = bd$ mod 6 provided $nd + mb + nm$ is an integer. But d, b can be any real numbers, so $nd + mb + nm$ is not an integer in general, and it is not possible to define multiplication sensibly.

Had we looked inside $\mathbb{J}$, rather than $\mathbb{R}$, we would have found that the term $6(nd + mb + nm)$ does lie in $6\mathbb{J}$, and so it is possible to define multiplication in $\mathbb{J}$ modulo $6\mathbb{J}$. In this case:

(1) $6\mathbb{J}$ is an additive subgroup of $\mathbb{J}$

(2) for x in $6\mathbb{J}$ and y in $\mathbb{J}$, xy always lies in $6\mathbb{J}$

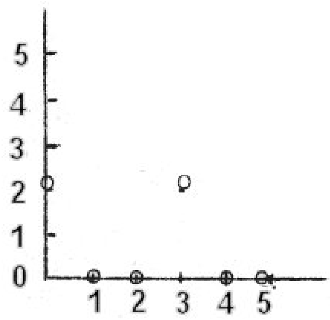
and these two conditions enable us to define addition and multiplication on $\mathbb{J}$ modulo $6\mathbb{J}$, giving us the ring $\mathbb{J}/6\mathbb{J} = \mathbb{J}_6$.

A substructure of a ring which satisfies conditions (l) and (2) is said to be an *ideal*, and we can always produce a sensible new ring by taking the old ring modulo an ideal, as shown in M202 Unit 17.

Incidentally,

(a) R is a field and so has no ideals apart from {0} and itself (see M202 Unit 18);

(b) some quadratics over $\mathbb{J}/6\mathbb{J}$ have “too many roots” precisely because $\mathbb{J}/6\mathbb{J}$ is not a field (see M202 Unit 32);



(c) the only way $y = (x - 1)(x - 2)$ mod 6 can be graphed is a set of distinct points, such as in the figure, but even this is misleading as there is no sensible concept of order (i.e. when is a greater than b?) in $\mathbb{J}/6\mathbb{J}$ (why not?).

Rosemary Bailey (STAFF-MOUTH)

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DEAR ED

I saw your name in one of those news sheets that they send around from time to time. I liked the idea and thought I would give it a go.

As you can appreciate, I am not too clear exactly what is expected of me with regard to MOUTHS or this M500, but I have no doubt that it won't take you long to make things clear.

This is my first try at maths MST281 and I find it rough going. I am what you might call a senior citizen and have plenty of time to spare, but just no-one to argue and fight with. I must have dialogue, to work things out.

Raymond Tiver

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M500 is working already - sanity returns, and my Universe of Discord is becoming a Universe of Discourse again.

I enclose £: $x \rightarrow 2.5$ ($x \in \mathbb{R}$ and $x \not< 1.75$) so that I may become an element of your set. Please buy yourself some flowers with anything left over.

Alexander Macphee

Ed: 75p went into Typewriter Fund, of course. It now stands at £52.60, to which I can only say thanks for so much appreciation. The target is £300 minimum, of course!!

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I am a lonely OU student doing MDT241 and M231, with M100 and E262 already under my belt. First units for MDT241 arrived this morning (10.3.75) and I have yet to receive those for M231. Obviously I shall have timing problems before I can start, as the post from Ethiopia is very erratic, but I wonder if any MOUTHS would be interested in corresponding with me (preferably someone on the same courses) in order to supply encouragement normally found in tutorials and self-help groups.

I arrived in Dessie at the beginning of February for 3 years. My husband works for the Crown Agents helping to build a road aimed at bringing famine relief to remote mountainous areas. I have two children, 1½ and 3½, who are wholly reliant on each other and me for entertainment and company. They are also bent on the interruption of any study I try to do during the day!

Dessie is a large town by Ethiopian standards but there are very few Europeans and no amenities whatsoever. We lose our electricity fairly regularly and the water is cut off two days out of three to preserve the supply. In the in-between days you have to rely on the water in the tanks lasting out. The small rains are over and now it probably won't rain until June or July. In the meantime it will get steadily hotter during the day although the nights are very cool.

Hoping that someone will want to correspond, I shall get back to my units.

Sue Brown

Ed: Air mail forms can be used, or ordinary paper and envelopes @ 8½p for the first ½ oz, and 7p per ½ oz thereafter. Use airmail sticker. Sue's letter apparently took 9 days to arrive here.

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Have just received M500/21 - my first 'regular' copy. It was very welcome since I am suffering (4/3/75) from the set-book problems of M231, sitting surrounded by units which mean nothing without Spivak and which look intriguing. Also the first assignment date is rushing upon me. To compensate a little, I have tried to get ahead with MST282 but have now run out of units for that course. I have just phoned the bookshop and am informed they haven't forgotten me, though.

It's good to get back to the comparative sanity of the Maths Faculty after spending last year studying T100. This was my second foundation course, chosen by a process of elimination – didn't fancy any others, so I thought I'd give it a try. Since I'm no technologist (my husband won't even trust me with a screwdriver) and managed to come out with a distinction (nobody more surprised than me!) based, I suspect, more on knowledge gleaned from M100 than T100, I recommend the course to anyone stumped for a choice. Summer School especially was fun – not much academically, but 12 women being pampered by 200 men is an experience not to be missed.

Sue White

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AID FOR VIETNAM MATHEMATICIANS

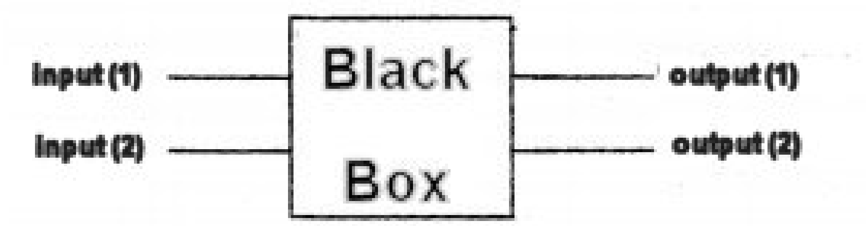
The following appeal is printed in *Historia Mathematica*, vol.2, p.42, February 1975:

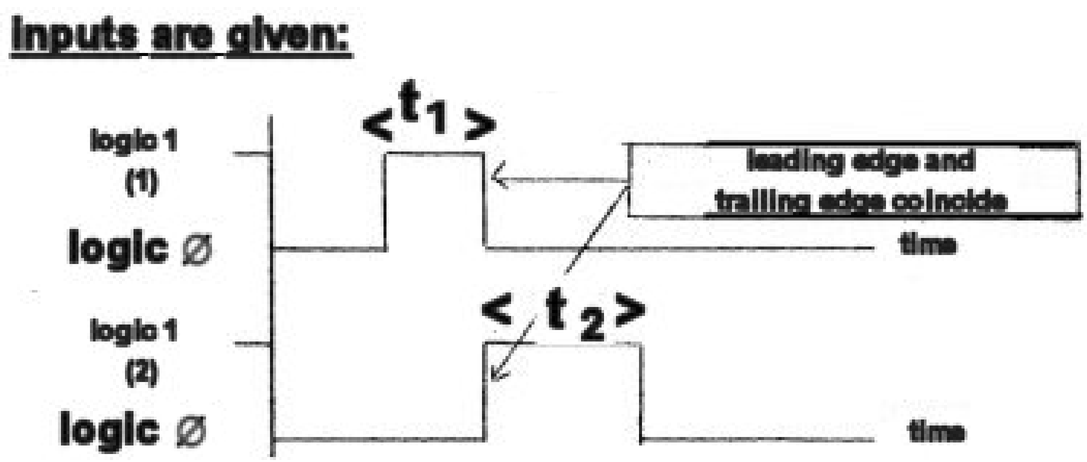
At the International Congress of Mathematicians in Vancouver, August 1974, the Vietnam delegation appealed for material aid from the world mathematical community. Books and journals (in mathematics and all the sciences) may be sent to Mathematical Society of Viet Nam, 39 Tran hung Dao, Hanoi, Democratic Republic of Viet Nam. From countries where direct mail to Hanoi is chancy, books may be sent to Mathematical Society of Viet Nam, c/o Cultural Attaché, Embassy of the Democratic Republic of Viet Nam, 2 rue le Verrier, 75006, Paris, France. Questions about other forms of aid (money for equipment etc.) may be sent to Prof. Le Van Thiem at the same addresses.

PROBLEMS

22.1 *BLACK BOX* - Bob Margolis

(This is a 'straight' question - answers are sought, please.)



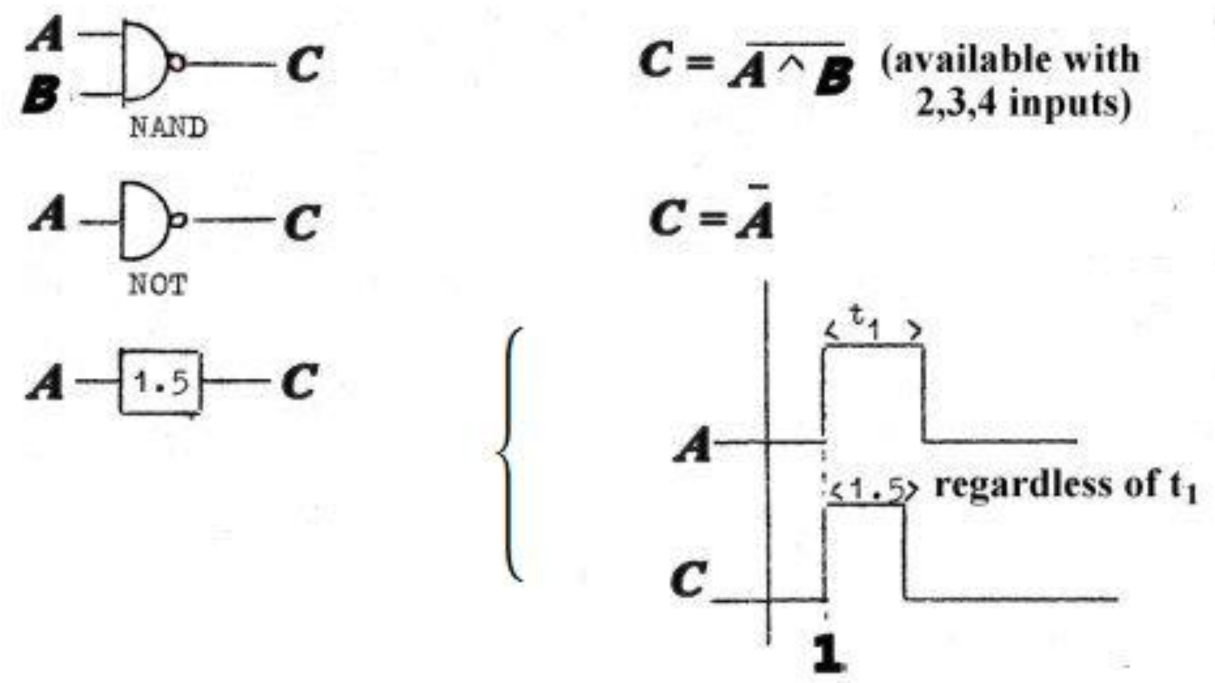


Restrictions: $1 \le t_1, t_2 \le 2$.

Outputs require

$$\left.\begin{array}{l}(1)\ t_1 + t_2 - 1.5 \\ (2)\ t_1 - t_2 + 1.5\end{array}\right\} \text{relative timing unimportant.}$$

PROBLEM: how do you construct the black box, given a supply of the following pieces?



22.2 *QUADRILATERAL* - Harold Moulson

A quadrilateral ABCD has AB = BC; angle ABC = angle BCD = 90°; AD = 2 units and the area = 4 square units. Determine the length of AB.

An obvious answer is 2, i.e. the figure is a square. But are there any other answers?

22.3 *SKETCH GRAPH* - Richard Ahrens

Sketch the graph of the following equation:

$$\begin{aligned}&\left(\sqrt{(x+2)^2+(y-3)^2}+\sqrt{(x-2)^2+(y-3)^2}-4\right)\\ &\quad\times\left(\sqrt{(x+2)^2+y^2}+\sqrt{(x+2)^2+(y-3)^2}-3\right)\\ &\quad\times\left(\sqrt{(x-2)^2+y^2}+\sqrt{(x-2)^2+(y-3)^2}-3\right)\\ &\quad\times\left(y+\sqrt{4-x^2}\right)\left((x+½)^2+(y-3/2)^2-1\right)=0.\end{aligned}$$

22.4 *FIND THE NEXT TERMS* (by N. J. A. Sloane - extract from *Jnl. of Recreational Mathematics*, vol.7 no.2 1974. Baywood Pub. Co. Inc.)

(9) 1, 4, 11, 20, 31, 44, 6l, 100, 121, 144, 171, ...,

(10) 1, 1, 1, 2, 1, 2, 1, 3, 2, 2, 1, 4, 1, 2, 2, 5, ...,

(12) 4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, ... (find the next 15 terms).

For (9) and (10) find the next 2 terms, for (12) the next 15; in all cases find a rule for generating the sequences.

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SOLUTIONS

18.3 *ARITHMOGRAMS* - Steve Murphy

Ed: In typing this solution, I have translated greek letters alpha, beta, gamma etc. by capitals A, B, C, etc. Roman caps A and B in the original have been typed as X, Y.

In M500/19 in his solution to Problem 18.3 Hugh Mclntyre speculated that the operation of forming a pentagon whose vertices were the midpoints of the of a given pentagon would, if repeated, tend to a regular pentagon.

Let $z = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, then $z^5 = 1$ and $z + z^4 + z^3 + z^2 + 1 = 0$.

“U” is the subspace of the vector space $\mathcal{C}^5$, ($\mathcal{C}$ is the field of complex numbers) that has the following 4 vectors as a basis:

$\mathbf{A} = (1, z, z^2, z^3, z^4)$;

$\mathbf{B} = (1, z^{-1}, z^{-2}, z^{-3}, z^{-4})$;

$\mathbf{C} = (1, z^{-2}, z^{-4}, z^{-6}, z^{-8}) = (1, z^{-2}, z^{-4}, z^{-1}, z^{-3})$;

$\mathbf{D} = (1, z^{-3}, z^{-6}, z^{-9}, z^{-12}) = (1, z^{-3}, z^{-1}, z^{-4}, z^{-2})$.

Any set of 5 complex numbers can be represented as a pentagon on an Argand diagram. Because vectors are ordered sets, the vector **A** corresponds to a regular pentagon labelled anticlockwise. **B** corresponds to the same pentagon labelled clockwise. In the same way **C** and **D** correspond to the “same” regular 5-pointed star. However, the 4 vectors are linearly independent. “U” corresponds to the set of pentagons that have the origin as their centroid. So with a suitably chosen origin any pentagon can be represented by

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D} \qquad (a, b, \text{ etc. are complex numbers}).$$

The operation by which the next pentagon is formed is equivalent to a linear transformation (T say) for which:

$$T(\mathbf{A}) = \tfrac{1}{2}(1 + z); \quad T(\mathbf{B}) = \tfrac{1}{2}(1 + z^{-1})\mathbf{B}; \quad T(\mathbf{C}) = \tfrac{1}{2}(1 + z^{-2})\mathbf{C} \quad \text{and} \quad T(\mathbf{D}) = \tfrac{1}{2}(1 + z^{-3}).$$

Now $|1 + z| = |1 + z^{-1}| > |1 + z^{-2}| = |1 + z^{-3}|$.

Thus if a and b are not equal to 0, the sequence of pentagons will tend towards shapes what can be represented by X**A** + Y**B**.

Once in every 5 operations X = a and Y = b.

As a simple counter-example to the original speculation, consider $\mathbf{Q} = \mathbf{A} - \tfrac{1}{2}\mathbf{B}$. The corresponding pentagon can be easily sketched. It is not regular and is convex – yet after 5 operations we get to a pentagon of the same shape as we started with, since:

$$T^5(\mathbf{Q}) = (1/32)(1+ z)^5\mathbf{Q}.$$

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20.4 *DIVISIBLE INTEGERS* - Marjorie Brew

(Given: m is any positive integer, n is any positive integer > 3. Prove (or disprove)

$$X = mn - (m - 1)^{n-1} + (m - l)^n - (m - 2)^{n-2} + (n - 3) \text{ is divisible by } n.)$$

Counter-example: $m = 1$, $n = 4$.

Then $X = 4$, and $(n - 1) = 3$.

4 is not divisible by 3.

21.2 *STEADY UP* - Jeremy Humphries and Eddie Kent (same solution but differently worded).

Assume a square table. Label the legs *A*, *B*, *C*, *D*. We can always get three legs on the floor, since any three points in space specify a triangle. Suppose *A* is off the floor. Rotate the table through 90 degrees, always keeping three or more legs on the floor. Now *A* is on the floor and *D* is off.

At least once during the rotation *A* came into contact with the floor. But then *B*, *C* and *D* were still in contact, otherwise rotation would have been on less than three legs. So all four legs were in contact.

For non-square rectangular table rotate through 180 degrees.

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21.4 *ARRAY PROBLEM* (from Problem 17.3) - Dorothy Craggs

Cartesian co-ordinates of points: two solutions.

Solution A

(0, 3), (0, 8), (1, 6), (1, 9), (2, 1), (2, 3), (3, 5), (3, 6), (4, 2), (4, 10), (5, 0), (5, 10), (6, 0), (6, 8), (7, 4), (7, 5), (8, 7), (8, 9), (9, 1), (9, 4), (l0, 2), (l0, 7).

Solution B

(0, l), (0, 3), (l, 6), (1, 9), (2, 2), (2, 3), (3, 5), (3, 6), (4, 0), (4, 2), (5, 0), (5, 10), (6, 8), (6, 10), (7, 4), (7, 5), (8, 7), (8, 8), (9, 1), (9, 4), (10, 7), (10, 9).

(The problem was: Given an 11×11 array of points, find 22 points such that no 3 of them lie in a straight line in any direction).

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21.5 *FIND THE NEXT TERMS* - Eddie Kent & W. van der Eyken

7. (1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...).

Soln: 199, 322. Rule: $U_n = U_{n-1} + U_{n-2}$ with $U_1 = 1$, $U_2 = 3$.

8. Rule: $U_{n+1} = 2U_{n-1} + U_{n-2}$. Soln: The next number has 14 digits and is 10 650 056 950 807.

(1, 2, 3, 7, 43, 1807, 3263443, ... = given sequence).

13. (1, 2, 3, 4, 9, 27, 5l2, 134217728, ...) - Eddie Kent.

The sequence is equivalent to: $1, 2, 3, 2^2, 3^2, 3^3, 2^9, 2^{27}, ... = 1, 2, 3, 2^2, 3^2, 3^3, 2^{(3^2)}, 2^{(3^3)}, ...$.

You will notice that the sum of the digits between the (n–l)th comma and the nth comma is n. Since this is not enough characterisation to fix the sequence uniquely other criteria must be sought - i.e. the sequence is ascending, etc. The next number would be $3^{(3^3)} = 3^{27}$ and the one after that $2^{(3^{(3^2)})}$. This is not so unlikely as it might sound, especially considering sequence no. 24 (not yet published)!

Ed: Readers may be interested to know that I passed this sequence (among others) to Eddie around Christmas 1974, while awaiting permission from the publishers to reprint them. I would feel happier if the 7th term were 256, so that powers were set out in ascending order. However, I have no better solution to offer.

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20.1 *SQUARE FACTORIAL* (Prove that $m^2 \neq n!$ for $\forall m, n \in \mathbb{Z}^+ - \{1\}$.

Ed: Various readers have enquired about Bertrand’s Postulate, mentioned in the solutions (M500/21). Dorothy Craggs referenced *Encyclopedia Britannica* article: ‘Theory of Numbers’, but hers arrived too late to be printed and so her solution was classified with the staff solutions citing B's P in general, without refs.

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When sending in solutions to problems, apart from keeping them in short (précis) form, please could you cite problem number and title? Since Eddie Kent is now our official voluntary Indexer, he requires titles to problems, while as editor I require number codes. I have to keep looking them up *en passant*, while typing, and this wastes a great deal of time. Also, please make sure that, as per TMAs, your name and MOUTHS number (if applicable) are *on every sheet* sent to 176. Solutions to the Balls of 20.2 are still corning in, and are still being rejected editorially. Sorry!

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OPUS MINICOMPUTER - from the Director of Marketing, Walton Hall

Although we hope to market this instrument we will not be able to do so until mid-1975. We have not finally priced OPUS at present, but we have estimated that it will be between £150 - £200. The OPUS, or Open University System, minicomputer has been designed by two members of the Course Team to be the major component of the home kit for the course TM221: *The Digital Computer*. It represents a significant advance in the design of small cheap computers and is intended primarily for teaching purposes. The specification includes:

8-bit word; 120 word memory, directly addressable, using 8 general registers; interrupt facility; microprogram implemented with read-only memories.

The front panel houses 8 data lamps, 8 data switches and 5 function switches forming the control panel; also included are a useful range of peripheral devices - a keyboard and two 7-segment displays. The keyboard has a status indication and can also set the interrupt. The interrupt and enable flip-flops are directly available as peripherals. Thus both hardware and software can be demonstrated.

Ed: Please note that OPUS is not a “Do-it-Yourself Kit” for electronics enthusiasts but is a real computer. One talks to it in machine language familiar to M251 students. TM221 is intended to be complementary to M251. So far (up to the end of Unit 5) I have not yet needed my non-existent knowledge of electricity or electronics, and thoroughly recommend TM221 to anyone interested in M251.

M500 is a student-operated and student-owned magazine for Open University mathematics students and staff. It is designed to alleviate student academic isolation by providing a forum for public discussion of individuals mathematical interests. Articles and solutions are not necessarily correct, but invite criticism and argument.

MOUTHS is a list of names, addresses, telephones and courses of voluntary members, by means of which private contacts may be made by any who wish to share OU and general mathematical interests or who wish to form telephone or correspondence self-help groups.

The views and mathematical abilities expressed in M500 are those of the authors concerned and do not necessarily represent those of either the editor or the Open University.

M500 is published and edited by Marion Stubbs.

This issue M500/22 was published in April 1975.

Subscription: £1.75 for 10 issues.

EDITORIAL

M500/23 is partly filled. We seem to have plenty of 'articles' - even enough to overflow into 24, probably - and some problems and letters. We need more of the 'chat' part - letters and comments. I am trying to give highest priority to new readers, where feasible, and apologise to old members whose pieces are still in the queue. Maximum for articles is 600 words (approx. 2 sides of finished typescript), or 300 words for letters. If subjects require more space you can split them into instalments.

Please note the *WEEKEND WORK-IN*, at University of Aston in Birmingham, September 5-6-7 (Friday eve to Sunday lunch) for the following courses: M100, M201, M202, M231, M251, M321, M331, M332, MDT241, MST282, SM351 and TM221. (MST281 can come if prepared to join M100). So far 34 have sent firm bookings, out of 64 who expressed initial interest. We need 120 students. This is because 12 staff have to be paid, and thus need an average of 10 students each. Please show some interest! Otherwise this may become a general non-M500 event for non-MOUTHS who answer Sesame ads, whereas it is supposed to be our annual opportunity to meet each other in an atmosphere of intensive academic activity. Full details are available on receipt of a stamped (5½p or 7p) addressed envelope.

M500 readership was 292 on 29th March 1975, still rising. Each Stop Press seems to have different wording; M251 is still using the 1972 publicity requesting sae + 3p stamp - the sample costs 7p actually! No publicity in M331 at all yet, so I am told.

Tell them at tutorials - sae (9 × 4″) + 10p (stamps).

Thanks. M.

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