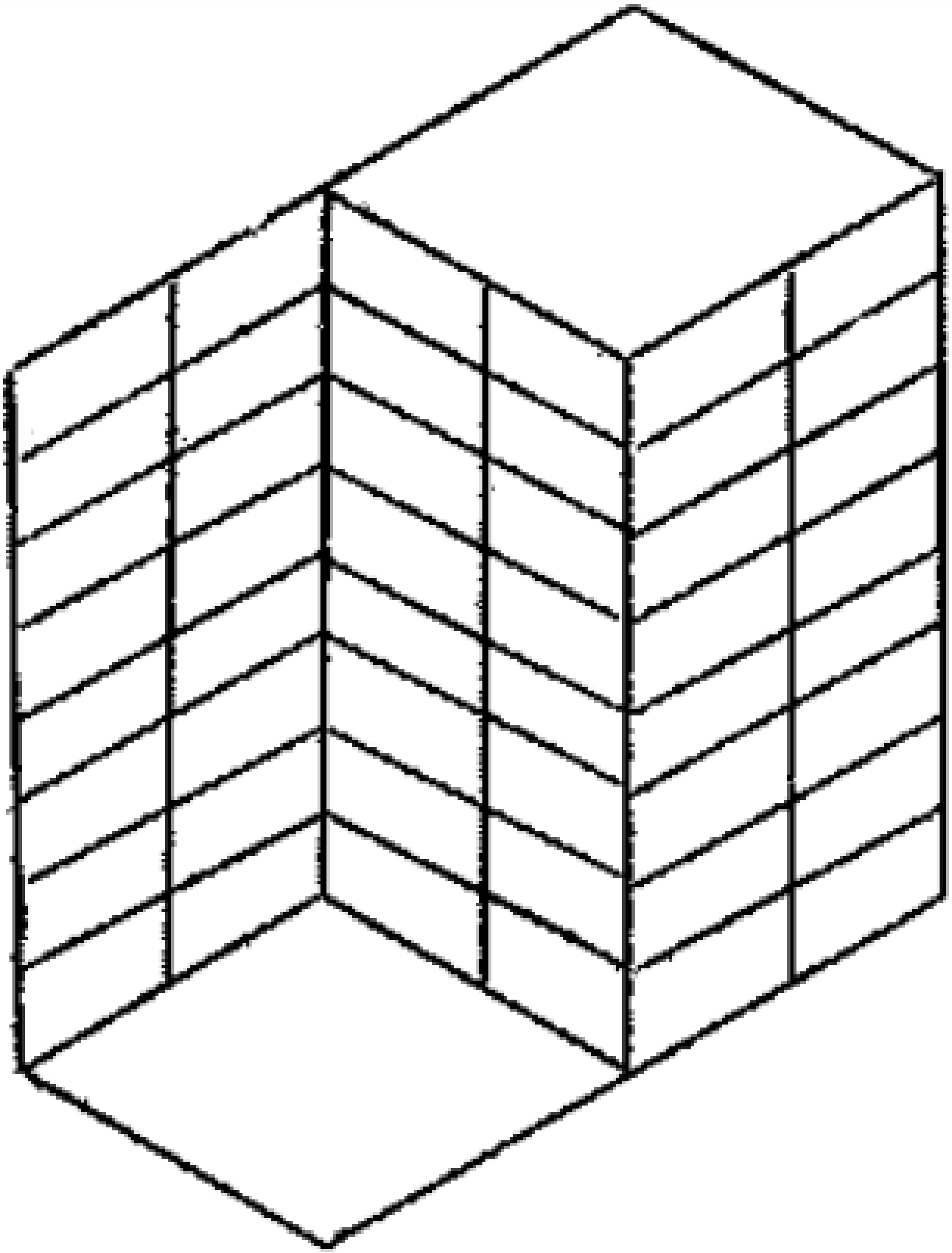

M500 25

M500 is a student-operated and student-owned magazine for Open University mathematics students and staff. It is designed .to alleviate student academic isolation by providing a forum for public discussion of indrviduals' mathematical interests.

Articles and solutions are not necessarily correct, but invite criticism and comment.

MOUTHS is a list of names addresses and telephones, together with previous and present courses of voluntary members, by means of which private contacts may be made by any who wish to form telephone or correspondence selfhelp groups.

The views and mathematical abilities expressed in M500 are those of the authors concerned, and do not necessarily represent those of either the editor or the Open University.

The cover design for this issue is known as "Thiery's figure"'. It has long been very popular with the *gestalt* psychologists, since the clues it presents are contradictory and the image presented is impossible to keep fixed. It is an obvious precursor of the designs Tony Brooks has recently been delighting us with, but it also, as E H Gombrich points out in *Art and Illusion*, contains the quintessence of cubism in that movement's recognition of the existence of impossible worlds. By deliberately withholding easy points of reference painters aimed to shock the viewer into admitting that the enjoyment of a picture requires what on the face of it is a most unlikely 'suspension of disbelief'.

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JUDGING THE JUDGES

An Interesting Relationship: MDT241.

Kendall's Coefficient of Concordance of Unit 7, MDT241 is calculated to show the agreement between *K* judges who rank *N* items - for example, a panel of *K* people testing *N* different brands of instant coffee, *K* television personalities judging *N* girls for beauty, or *K* patients reporting on *N* different sleeping tablets.

Friedman's Two Way Analysis of Variance is presented in Unit 15 as a nonparametric significance test in which *K* treatments are put in order of rank with respect to *N* subjects. The test shows whether the differences between the treatments are greater than would be expected to occur by chance. If we look at this test as a case of *N* subjects each ranking the *K* treatments we have the same situation as in the third example above. The notation is reversed - we are using *N* for *K* and vice versa. It seems there is likely to be considerable similarity between Kendall's Coefficient of Concordance and the Friedman test.

The connection between the two is clearer if we alter the notation. Rewriting the Friedman formula by interchanging *N* and *K*, we have:

FRIEDMAN'S TWO WAY ANALYSIS OF VARIANCE

$$X_r^2 = \frac{12}{KN(N+1)}\left(\sum_{i=1}^{N} R_i^2\right) - 3K(N+1)$$

where K = number of subjects; N = number of treatments; R_i = sum of ranks for treatment i.

KENDALL'S COEFFICIENT OF CONCORDANCE

$$W = \frac{12}{K^2\,(N^3 - N)} \sum_{i=1}^{N}(R_i - \bar{R})^2 \qquad (1)$$

where K = number of judges; N = number of items being ranked; R_i = rank total of ith object; $\bar{R} = \frac{1}{N}\sum_{i=1}^{N} R_i$.

Rearranging (1):

$$W = \frac{12}{K^2\,N(N^2-1)}\sum(R_i - \bar{R})^2 = \frac{12}{K^2\,N(N+1)(N+1)}\sum(R_i - \bar{R})^2. \qquad (2)$$

We note that $\sum R = \frac{1}{2}KN(N+1)$ (from K sets of ranks from 1 to N and

$$\sum_{i=1}^{N}(R_i - \bar{R})^2 = \sum R_i^2 - \frac{(\sum R)^2}{N}$$

(compare $\sum(X - \bar{X})^2$, Unit 3) so

$$\sum_{i=1}^{N}(R_i - \bar{R})^2 = \sum R_i^2 - \frac{K^2N^2(N+1)^2}{4N} = \sum R_i^2 - \frac{K^2N(N+1)^2}{4} = \sum R_i^2 - 3K(N+1)\frac{KN(N+1)}{12}.$$

Hence, from (2)

$$W = \frac{12}{K(N-1)KN(N+1)}\left(\sum R_i^2 - 3K(N+1)\frac{KN(N+1)}{12}\right);$$

and

$$K(N-1)W = \frac{12}{KN(N+1)}\left(\sum_{i=1}^{N} R_i^2\right) - 3K(N+1)$$

Comparing this result with the Friedman formula above, we have

$$X_r^2 = \mathrm{K}(N-1)W$$

where K = number of judges/subjects; N = number of items/treatments being ranked.

This relationship is useful to know because the Friedman calculations are easier without electronic aids than the Kendall formula and it provides a way of testing whether the coefficient, W, shows more agreement between the judges than would have been expected to occur by chance.

Reference: Siegel, Sidney; *Non Parametric Statistics for Behavioural Sciences*, (McGraw–Hill, 1956).

Margaret Corbett

oooooooooo

unprovable facts

Some results. This article is in response to Eddie Kent's on the same subject in M500/20, and concerns the following, which I repeat from the previous article. Define a sequence as follows:

S_1 is an arbitrary positive integer

$S_{U+1} = S_U/2$ if S_U is even, -x-

$S_{U+1} = (3S_{U+1})/2$ if S_U is odd. -y-

The proposition, P, is that there is a U depending on S_1 such that $S_U = 1$.

Source: M100 week long problems, Stirling Summer School, 1973.

I have deliberately ignored the problem since then, due to its addictive qualities, so here is your last chance to skip the rest of this article if you have a family/job &c.

Rl) If you can prove that for all S_1 there is a S_n such that $S_n < S_1$ you are home.

Source: Summer School solution sheet;

proof by induction ('complete induction' (M231) to be precise). An easy SAQ.

R2) If S_1 is not of the form $4k + 3$ then $S_n < S_1$ for some n.

Source: as above;

proof: If S_1 is even $S_2 = S_1/2 < S_1$, by -x- ; if S_1 is odd it is either of the form $4k + \mathrm{l}$ or $4k + 3$. If $S_1 = 4r + \mathrm{l}$ then $S_2 = (3(4r + \mathrm{l}) + \mathrm{l})/2 = 6r + 2$ by -y-; this is even so by -x- $S_3 = 3r + \mathrm{l} < S_1$.

R3) Any odd number can be made even by a finite number of successive applications of -y-.

Source: self;

proof: If S_1 is odd it is either of the form $4k + \mathrm{l}$ or $4k + 3$. One application of -y- in the first case gives an even number so assume $S_1 = 4r + 3$. Let m be the highest power of 2 in $(4r + 3) + \mathrm{l}$ then $4r + 3 = p2^m - \mathrm{l}$ where p is odd, m successive applications of -y- results in $3mp2m - m - \mathrm{l} = 3mp - 1$, which is even.

Corollary: If P is true for all even S_1 it is true for all S_1.

R4) If P is true for all S_1 of the form $4k + \mathrm{l}$ it is true for all S_1.

Source: self;

proof: Even numbers reduce to odd numbers by halving (-x-). This leaves odd numbers of the form $4k + 3$; the proof is by induction on the binary representation of S_1. Consider $S_1 = x01$ (in binary) where x is any string of zeros and ones. i.e. S_1 is of the form $4k + 1$ and so by the assumption P is true for all S_1 which end in zero followed by a single one. Now assume P to be true for all S_1 which end in zero followed by n ones; i.e. $S_1 = x01...\mathrm{l}$ (n ones). Consider $S_1' = x01...11$ ($n + 1$ ones). S_1 is odd therefore by -y- $S_2' = \frac{3S_1' + 1}{2} = x01...1$ (n ones). (x is not fixed!) Since by the assumption P is true for S_2' it is true for S_1'. Therefore P is true for all S_1 ending in zero followed by any string of ones; i.e. for all odd S_1.

These then are my results. The peculiarity of the last proof indicates just how many hours I have wasted on the problem. Alas, I am doing two credits this year (even though A100 is almost a formality) so I will have to give up any further research. The solution remains as tantalisingly close as ever - you have been warned.

Peter Weir.

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F I C T I O N !

It might interest readers to know that there are one or two good science-fiction stories with maths themes.

The Gold at the Starbow's End, by Frederick Pohl, is one such. In this a team of scientists en route to Alpha Centauri prove Goldbach's Conjecture (using, would you believe, "multiplex parity analysis") and solve the problem of making a hydrogen fusion reactor, but send the results back to Earth in a Gődelized language that defeats IBM.

By the same author is *Shaffery Among the Immortals*, in which the hero, Shaffery, fails to resolve the issues raised by Fermat's Last Theorem.

These stories are in a collection of Pohl's stories published by Gollancz (1973), entitled *The Gold at the Starbow's End*.

Hugh Mclntyre

OXOXOXOXOXOXOXOXOXOXO

A COUPLE OF QUOTES

Picasso: A friend who is writing a book on my sculpture starts it like this: "Picasso told me one day that a straight line is the shortest distance between two points."

Obviously I was very surprised and asked him: "Are you quite sure it was I who found this out?"

Some international computer language from a Nixdorf brochure in German:

Das HardwareSystem ..

Das Software System ...

Das Knowhow der Nixdorf-Teams ...

Floppy-Disk.

(*The Times*)

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INTERPOLATION AND LEAST-SQUARES

Follow-up Article; see M500/23, 24.

Given data (x_k, y_k), $k = \mathrm{O}(\mathrm{l})n$, and a representation

$$y(x) = \sum_{i=0}^{n} a_i\, p_i\,(x)$$

where $\langle p_0, \ldots, p_n \rangle$ is an $(n + 1)$-dimensional (function) space, $y(x)$ is an interpolating function for the data if

$$y(x_k) = y_k\,, \quad k = \mathrm{O}(\mathrm{l})n;$$

i.e.

$$\sum_{i=0}^{n} a_i\, p_i\,(x_k\,) = y_k\,, \quad k = \mathrm{O}(1)n.$$

This is an $(n + 1) \times (n + 1)$ linear system of equations for the $(n + 1)$ coefficients a_i; i.e., in matrix form,

$$PA = Y$$

where

$$p = \begin{bmatrix} p_0\, x_0 & \cdots & p_n\, x_n \\ \vdots & \ddots & \vdots \\ p_0\, x_n & \cdots & p_n\, x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}.$$

If the $x_0, x_1, \ldots, x_n$ are distinct the independence of the $p_0, p_1, \ldots, p_n$ guarantee the non-singularity of P, so

$$A = P^{-1}Y$$

In practice we would solve $PA = Y$ by, e.g., elimination.

Now suppose we have a large number of data points but decide to use a low order representation, e.g. a cubic polynomial to represent 20 data points, then

$$\mathrm{y(x)} = \sum_{i=0}^{m} a_i\, p_i\,(x), \quad m < n.$$

This problem is treated in the least-squares approximation unit of M201. Essentially we are trying to solve

$$PA = Y$$

when there are more equations than unknowns (P is $n \times m$). Since we cannot satisfy all the equations we can choose to minimise

$$\sqrt{\sum_{k=0}^{n} (y_k - \sum_{i=0}^{m} a_i\, p_i\,(x_k\,))^2},$$

which is written $\| Y - PA \|$ (the square norm), and obtain a so-called least-squares curve fit to the data.

Quite recently (since 1950) it has been shown that, although solving the square system

$$P^{T}PA = P^{T}Y$$

this can be an extremely ill-conditioned formulation (M201 07/8) and that a direct attack on $PA = Y$ by methods comparable to elimination can yield the solution in a more stable fashion. More interestingly this ties in with work pioneered by Oliver Penrose's brother Roger, which uses the concept of the pseudo- (or generalised) inverse of a matrix, denoted by P^{+} , and which allows us to write

$$A = P^{+}Y$$

as the theoretical solution of the overdetermined system.

References

1) “A generalised inverse for matrices”, R Penrose, (*Proc. Cambridge Philos. Soc.*, 51 (1955), pp. 406–415).

2) “The least-squares problem and pseudo-inverses”, G Peters and J H Wilkinson, (*Computer Journal*, 13 (1970), pp. 309–316).

Peter Hartley (Staff).

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THE RULES of THE GAME

It seems to me that Richard Ahrens (M500/24) is asking his questions in the wrong order, assuming a commutativity which does not exist, and as a result is muddying rather than clarifying the situation. I would suggest that the order in which his points should be made are

1) Some people come into M100 expecting to learn new problem solving mathematical techniques.

2) Maths has its own way of saying things that are essentially only fully meaningful to those within the magic circle.

3) The OU student is at risk because he does not have ‘easy access’ to tutors.

4) Any rewrite of M100 should allow for points 1 – 3.

Whilst points 1 – 3 *may* be valid (something I want to look at in a moment) point 4 does not necessarily flow from them. Some people do undoubtedly come into M100 expecting to learn new problem solving techniques: among them 5 of the 6 dropouts from my study centre this year (50% of initial registration). If I may say so it would appear that not only were they “unwilling to wade through a lot of wordy material that is quite unlike their expectations”, they weren’t over keen on analysing what the course guide had to say about the M100 course. If they had done so they would have realised that the emphasis is much more on ‘what mathematics is’ rather than ‘what mathematics does’.

I would suggest that there are courses that provide a more nitty gritty and not so much of the faldingoes approach in, say, M251, MDT241, MST281 & 282. The fact that some students are stumbling into M100 and then opting out would seem to suggest not that the course is wrong, but that they are wrong for the course. This would imply a need for effective *preapplication*, (certainly *pre-initial registration*) counselling for all M100 applicants.

There is no doubt that maths does have its inner secrets – I clobber my shins on them with every TMA that assumes that one is happily at home with techniques such as moving into natural logs to find a way round a problem, which are nowhere brought out in the units. The answer must be that as one applies the embrocation one learns from one’s mistakes. Anyone who reckons on doing maths any other way and then complains seems to me to be on equally weak ground as the would be linguist who complains about the difficulty of learning French vocabulary and grammar when ‘all’ he wants to do is read Racine in the original.

I think, certainly at M100 level, that Richard's third point is by far the weakest. Never when I was studying education at University did I have the direct access by telephone to my supervisors that I have now. Perhaps I'm lucky, but the fact that Richard is himself a StaffMOUTH indicates that my tutor certainly isn't unique.

The problem, then, is not one of language but of purpose and, frankly, “the business of communicating mathematically with its students” is a quite inadequate attempt to define the purpose of any course or faculty. I can understand that the staff are somewhat jumpy; certainly Maths doesn't seem as ‘open’ a faculty as any of the others – after all anyone can waffle on about Arts Social Sciences or Educational Studies, as Colleges of Education prove every year (their 5 O-level entry makes them a reasonably ‘open’ subject for comparison). It is, however, worthy of note that the Maths, Science and Languages departments in Colleges of Education tend to be much smaller than the Arts or Social Sciences – unless the college has a reputation for academic success, in which case they increase entry requirements and are no longer open – and far smaller percentages of students within them in my experience of an admittedly small sample achieve advanced main or BEd standard.

What is, I think, important is that M100 is open to those whose motivation coincides with that anticipated by the Course Team. For the others the need is for pre-entry counselling and either an additional more numerical and less conceptual foundation course or, better still I would have thought, provision for those desiring new problem solving mathematical techniques in the post experience sector of courses.

Finally I must take issue with Richard's equating ‘How many different functions g:$B \to B$ are there’ with ‘How many different people Fred are there in your living room’. A much more accurate transliteration would be ‘How many different ways by road from Telford to London are there?’ which strikes me as totally acceptable English.

David Pedlow (M100)

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NON–BOOK OF THE YEAR 1970

Have you ever seen a million? I mean all at once? What about the cube of 81, which happens to be the number of words in the English language, or 46,800 - the number of seconds some wag at Walton Hall laughably suggests as the maximum time you are likely to want to enter in those little boxes on your TMA?

It happens that a book exists, consisting of precisely one million dots. With commendable rigour it is called “One Million” by Hendrik Hertzberg, published in New York (at the Rockefeller Center – of course; by Simon & Schuster in 1970. It is available on loan from Southampton College of Art library, (apply to M Stubbs) where Graphic Design students are liable to do things with dots, from time to time. Graphic Designers are the ones who have the responsibility, among other social duties, for your tea breaks during “World of Sport”, or for your Unit covers, and demand a seemingly infinite supply of what is known in the trade as visual reference.

The book, however, is really intended as “an aid to comprehension”; it is “a ruler divided into a million parts”. Statistical notes such as the three above are scattered throughout the margins, isolating chosen dots for instant fame. One I particularly like notes gloomily that the odds against a cod dying a natural death are 974,731 to 1. Plenty of space is left for you to add your own significant thoughts. But on the whole you are supposed to riffle slowly through the pages, absorbing the feel of what is meant by a million. Should you wish to contemplate a billion, then of course you must imagine an identical book in place of every dot.

Man, says the author, is a symbol-making animal, and his talent for naming things frequently outstrips his capacity to understand them.

Can anyone disagree?

Marion Stubbs

DIRECT PRODUCT SUBGROUPS

Consider two groups A and B. Suppose we can find a homomorphism: $A \to B$ which is onto. If we can, we can consider the homomorphism, which is a function, as a set of ordered pairs (a, b) where $a \in A$, $b \in B$ and $\varphi(a) = b$. Now consider the set if all possible ordered pairs (a, b) where again $a \in A$, $b \in B$. Our function φ will be a subset of this set. Now the set of all possible ordered pairs is the direct product group construction. I claim that the subset

$$S = \{(a, b) : a \in A, b \in B, \varphi(a) = b\}$$

is in fact a subgroup. Closure follows from $(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ a_2, b_1 \circ b_2)$ which we can rewrite as $(a_1, \varphi(a_1)) \circ (a_2, \varphi(a_2)) = (a_1 \circ a_2, \varphi(a_1 \circ a_2))$ because φ is a homomorphism. For inverses we want $(a, b) \circ (a^{-1}, b^{-1}) = (e_A, e_B)$. Is (a^{-1}, b^{-1}) in our subset? Well, A is a group so a^{-1} is present. Further $\varphi(a^{-1}) = \varphi(a)^{-1}$ so that $(a^{-1}, \varphi(a)^{-1}) = (a^{-1}, b^{-1})$ is in our subset. We therefore have a subgroup.

This is just a start because we can drop the earlier restriction that φ be onto and still achieve a direct product subgroup (f: $a \mapsto e_B\ \forall ea \in A$ is an into homomorphism that will do the trick). We can also look at functions from subgroups of A to subgroups of B. Can we in fact find all direct product subgroups by considering functions between elements as ordered pairs?

(Acknowledgements to thought planted in head by Allan Solomon at Stirling.)

Roger Claxton

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IT'S THE LAST SENTENCE WOT KEEPS ME GOING – Sinbad

From the preface to *Calculus Made Easy*; Silvanus P Thompson, F.R.S.

Considering how many fools can calculate, it is surprizing that it should be thought either a difficult, or a tedious task for any other fool to learn how to master the same tricks. Some calculus tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics – and they are mostly clever fools – seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way. ...

What one fool can do another can.

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M100 *SELF-HELP GROUPS*

After reading M500 24 about the Self-help Group, I was asked to write and tell you about ours. So, here goes:-

We have a Self-help M100 group here in Folkestone consisting of five members only to date but we've all found it an invaluable help. We meet fortnightly or whenever we haven't a tutorial at our Study Centre. Self-help may not be the word we would choose – more like self commiserating at times and definitely self-encouraging. Our group was formed a week after our first tutorial because we all admitted we couldn't "go it alone." Our meetings are humorous, hard work sometimes and it's not just a question of brain-washing – I think brain bashing is nearer the mark. We sit with the assignments before us – brows beetled, foreheads furrowed and brains churning madly. Each will give of his knowledge to help someone else – eager to help, avid to learn from others. Where one is in need of a little extra help (someone's blind spot) another will be happy to give a little extra coaching. A cry goes up – someone has seen the light – (no it's not a revival meeting, nor a warden left over from the blackout). The rest listen, then discuss the points raised but, until each is convinced of the working and/or answer, he will not accept it. We don't doubt his answer, we just want to make his knowledge ours and thereby help ourselves, which is what this group is all about.

We have long ago admitted that we would have given up or fallen by the wayside if it wasn't for our meetings – for the companionship in despair, for the encouragement we will always find from someone. Ours is more like a M100 Samaritans Service! We work so well together that where possible next year and in future years, we will try to do some of the same courses together.

It's hard work, often frustrating but a lot of fun too. As the group consists of brainy chaps, I was afraid my light would show rather dim under the brilliance of their shining lights but they're rather kind to me. We often make jokes in maths language too, which is surely the ultimate or is it a limit or natural sequence or just expansion? The units seem appropriate too – Errors and Accuracy – more of the former and less of the latter. Finite Differences – when we don't all agree on the answer. Inequalities – the others have brains. Logic – no woman should be without it! The units to come – Relations – ours are extremely good within this group and we hope they will remain so. Unit 30 Groups – surely ours is the greatest?

Pat Pammenter

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What is crooked cannot be made straight and what is lacking cannot be numbered.

Ecclesiastes I 15.

From Steve Murphy

THE AXIOM OF ARCHIMEDES

What is Archimedes' Axiom? OU courses have

(a) Spivak, page 116, saying of a line "if you place unit segments end-to-end you will eventually get a segment larger than any given segment".

(b) Weir, page 3, saying "there is no real number K which satisfies $n \leq K$ for all integers n".

(c) Unit 1 of M331, page 40, saying "The sequence 1, 2, 3, ... has no real upper bound".

Now one or all of these statements may be what mathematicians today refer to as Archimedes' Axiom but in fact they are not what Archimedes wrote, nor do they mean what he meant.

Indeed Archimedes would not have regarded (c) as requiring expression as an axiom, but would surely have been able to argue thus: the prime numbers are integers; Euclid proved that there is no largest prime; therefore there is no largest integer! which makes (c) a corollary of Euclid's famous theorem.

There is a complete translation in fairly modern notation of Archimedes' extant writings by Sir T L Heath, F.R.S. (*The Works of Archimedes*, 1897, supplemented by the later-discovered *The Method of Archimedes*, 1912, both from Cambridge University Press) but the following quotation from page 19 of *Archimedes* by the same author (Macmillan Co., 1920) Is sufficient:

"Archimedes says that the theorem of Euclid XII, 2, was proved by means of a certain lemma to the effect that, if we have two unequal magnitudes (i.e. lines, surfaces, or solids respectively), the greater exceeds the lesser by such a magnitude as is capable, if added continuously to itself, of exceeding any magnitude of the same kind as the original magnitudes. This assumption is known as the Axiom or Postulate of Archimedes, though, as he states, it was assumed before his time by those who used the method of exhaustion."

In fact Archimedes' contribution was the explicit recognition that a significant property of what we now call the real numbers underlay the assumption. What is this property? It is "that so-called infinitesimal numbers do not exist" (E W Hobson, *Theory of Functions of a Real Variable*, Dover edition, 1957, vol. 1, page 46). Putting it another way - if δx is the difference between two unequal "magnitudes", δx is not a zero-divisor.

Percy Sillitto

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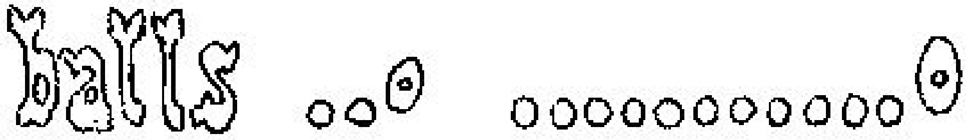
Mathematics is Magic ("The Big Yin" - attrib.)

from William Watson



In M500 23 Richard Ahrens describes a most ingenious solution to the ‘odd ball’ problem. My only comment is that this procedure is interactive, by which I mean that what one does at any stage depends on what has happened before. Here is a solution to the general case which gives a weighing procedure independent of preceding events.

There are ½(3*n* – 3) balls, one of which is odd, and we require to determine in *n* weighings which one it is, and whether it is heavier or lighter than the rest.

We label the balls using column vectors of dimension *n* with entries from the set –1, 0, 1. The point of this is to include the weighing instructions in each label. For example, for the case $n = 3$, a ball labelled $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ would be placed on the right-hand scale for the first weighing, omitted from the second weighing and placed on the left-hand scale for the third weighing. Having chosen a suitable set of labels, we carry out the weighings according to these instructions and so obtain a result vector. Here we use the convention that –1 means ‘tips down on left side’, 0 means ‘balances’ and 1 means ‘tips down on right side’. If our labels are suitably chosen, we should be able to identify the odd ball and decide its relative weight simply by looking at the result vector. For example, if the result vector was $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and there was a ball labelled $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ we would know that it was lighter than the rest.

We now have to choose a set of labels. There are two constraints. One is that we cannot use both the vectors *u* and –*u*, since we must use the sign of the result vector to distinguish heavy and light. The other constraint is that the sum of all the vectors used must be zero, in order that we have the same number of balls on each side of the balance. After a bit of the old ‘Polya’ stuff, I have found an $n \times \frac{1}{2}(3n-3)$ matrix whose columns form a suitable set of labels. This matrix A_n can be described inductively as follows:-

$$A_n = \left[\begin{array}{c|c|c|c} A_{n-1} & A_{n-1} & A_{n-1} & \begin{matrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ -1 & 0 & 1 \end{matrix} \\ \hline 0\,0 \ldots 0 & 1\,1 \ldots 1 & -1\,-1 \ldots -1 & 0\;1\;-1 \end{array}\right]$$

where

$$A_2 = \begin{matrix} -1 & 0 & 1 \\ 0 & 1 & -1 \end{matrix}.$$

Thus, to get A_n from A_{n-1}, we take three copies of A_{n-1} and sit them on a row of 0’s, a row of l’s and a row of –l’s. The last three columns consist of a column of –l’s, a column of 0’s and a column of l’s sitting on 0, 1 and –1 respectively. If we let C_n equal the number of columns in the matrix A_n, you can see that $C_n = 3C_{n-1} + 3$ where $C_2 = 3$. This has the solution

$$C_n = ½(3n – 3)$$

as required. The fact that both our constraints are also satisfied can be easily proved by induction.

David Asche (Staff Tutor)

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STATISTICAL DIFFICULTIES

Thank you for sending M500 24 - interesting as ever. I was specially interested in Roger Claxton's statistical problem.

Unfortunately I do not understand from his explanation exactly what the structure of the data is in his problem, which makes it likely that any comments will be inapposite. My first suggestion as a reference is Sidney Siegel's *NonParametric Statistics* 1956 McGraw Hill. This is quite an old book but it is very easy to use and has good explanations, though these are not phrased mathematically, and plenty of worked examples. It also contains a magnificent systematic scheme for selection of an appropriate method.

I would like to hear more about this problem.

Now, I have a problem which I have set out as if it were an exercise in quality control, but it happens to be a quite serious real life problem about which I am about to advise my brother. I should be glad to have somebody else’s solution against which I can compare my own. It is as follows:

> A fisherman buys hooks in packets of 100. He intends to spend time and use rare materials in tying artificial flies on these hooks. When he eventually goes fishing he is anxious not to lose fish which he has hooked because the fly is tied on a poorly tempered hook which straightens under the strain or alternatively is brittle and snaps too easily. He decides to test

to destruction a number of hooks randomly selected from the packet and note for each one whether it is of satisfactory or unsatisfactory quality. How many hooks should he test before accepting or rejecting the whole packet?

Margaret Corbett

3.14159265358979323846

ALGEBRAIC SYSTEMS

In M500 22 P Newton states that a monoid is a group. This is not so, unfortunately(!)

In a group, say G, $\forall x \in G$, $\exists y$ such that $xy = e$ and $yx = e$. y is called the inverse of x. This ‘axiom of inverse’ is not satisfied by a monoid.

Whilst every group is a monoid the converse is not true. Consider $(\mathbb{Z}^+, 1, \times)$: $\mathbb{Z}^+$ the set of positive integers, 1 is the multiplicative identity, $\times$ is the usual multiplication – binary operation. Then $\mathbb{Z}^+$ is a monoid but not a group.

There are also infinitely many finite monoids which are not groups. Find! (Hint: consider $\mathbb{J}_n$, the set of integers modulo n.)

Datta Gumaste

F E R M A T   N U M B E R S

Fermat showed that if m is odd and composite then $2m + l$ cannot be prime, because

$$2^{(2k+1)d} + 1 = (2^d + 1)\big((2^d)^{2k} - (2^d)^{2k} - (2^d)^{2k-1} + \dots - 2^d + 1\big).$$

If m contains no odd factors it is of the form 2^n and Fermat remarked that in that case $2^m + l$ is prime. Such numbers are called Fermat numbers: $F_n = 2^{2^n}+1$. $F_0 = 3$; $F_1 = 5$; ... ; F_5 has ten digits.

About 100 years later Euler showed that 641 divides F_5.

Proof: $641 = 625 + 16 = 5^4 + 2^4 = (5 \cdot 2^7) +1$: thus we have, modulo 641, $5 \cdot 2^7 \equiv -1$, $5 . 2^8 \equiv -2$, $5^4 \cdot 2^{32} \equiv 2^4$, $-16 \cdot 2^{32} \equiv l6$; whence $2^{32} \equiv -1$ or $2^{32} + 1 \equiv 0 \pmod{641}$.

Eddie Kent

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Ad(endorsed by Marion Stubbs who has just received CC vol.1 No.4)

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SOLUTIONS

24.1.a - FOUR FOURS.

Express 19 using 4 fours and any symbols.

(i) 19 = 4! – 4 4/4}; Marion Stubbs.

(ii) 19 = [√444] – √4, where [ ] means integer part; Eddie Kent.

24.1.b - Express all the integers up to 100 using 4 fours with a single formula.

Let the number to be expressed be *N* and set up the following equation:

$$N = -\log_{\sqrt{4}}\ \log_{\sqrt{4}} \sqrt{\sqrt{\sqrt{4\times 4}}}.$$

The use of the dotted line for one of the radicals indicates the number, *n*, of successive radicals – not fixed but depending on the value of *N*. It is actually *N* + 2. To illustrate:

If *N* = 3 then *n* = 5. The 5 successive square root extractions of 4 × 4 (which is, of course, 2^4) are:

$$2^2,\ 2^1,\ 2^{½},\ 2^{(½)^2},\ 2^{(½)^3}.$$

The logarithm to the base 2 (that is √4) of the last expression is $(½)^3$, of which in turn the logarithm to the base 2 is –3. This is converted by the minus sign to the required +3. Obviously this procedure can be applied to any positive value of *N*. It can be simply adapted for minus numbers by discarding the leading minus sign given at the head of this solution.

(from *Games & Puzzles* 24, May 1974)

24.2 and 24.3: no solutions offered.

24.4 FIND THE NEXT TERMS (14) 1, 1, 2, 1. 2. 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 3.

Next two terms: 3,3. (Next 8 terms are 3, 3, 4, 2, 3, 3, 4, 30).

Rule: the sequence runs in quadruples: *n*, *n*, *n* + *i*, *i*; *n* = 1, 2, ...; *i* = 1, 2, ..., *n*.

(15) 1, 2, 3, 4, 5, 6, 7, 6, 6, 10, 11, 12, 13, 14, 15, 8, 17, 12, 19, 20, 21, 22, 23, 18, 10, ...

Next two terms: 26, 18.

Rule: $f(x) = \begin{cases} 2ky \text{ if } x = k^{2^y} & \text{for some integer } k,\ x \in \mathbb{Z}^+ \\ 2ky \text{ if } x = ky^2 & \text{for some integer } k. \\ x \text{ otherwise.} \end{cases}$

(16) 1, 8, 11, 69, 88, 96, 101, 111, 181, 609, 619,.... . Next two terms: 689, 888.

Rule: Integers in the sequence are unchanged when read upside down.

Marion Stubbs

PROBLEMS

25.1 *TRAIN CATCHING* - Richard Ahrens (Staff)
A man lost in the desert hears a train whistle due west of him. Although the track is too far away to be seen he knows that it is straight, and runs in a direction somewhere between south and east, (but he does not know its exact course). He realises that his only choice to avoid perishing from thirst is to reach the track before the train has passed. In which direction should he travel to give himself the greatest possible chance of survival? (You are expected to assume that the man and the train both move at constant speeds, the former more slowly than the latter.)

25.2 *CUBIC HYPERCUBE* - Roger Webster, University of Sheffield.
Prove that the product of four consecutive integers is not the cube of an integer.

25.3 *FOOLPROOF SYSTEM* - Eddie Kent
A man starts with an amount of money m. He stakes m/n on a game whose probability is 1/2. Whether he wins or loses he stakes $1/n$ of what he has left on the next play. He continues to do this, staking $1/n$ of what he has on each play of the game for $2q$ tries. At this point he has won q times and lost q times. How much money does he take away?

25.5 *INSTANT ADDITION* - Jeremy Humphries
A Fibonacci trick with which to baffle your friends. Ask your victim to write down two numbers (hidden from you) and from them to generate a Fibonacci sequence of 10 terms. Look at the list and write down the sum of the ten terms in two seconds. How? (Assuming you can't really add ten numbers in two seconds.)

25.3 *THE INHERITORS* - Chuquet, *c*1484 (but attrib. Euler)
A group of heirs divide their heritage as follows: The first takes £a and the nth part of the remainder, the second takes £$2a$ and the nth part of the remainder, and so on, each succeeding heir taking £a more than his predecessor and an nth part of the new remainder. In this manner the heritage is divided into equal parts. How many heirs are there, and how much does each receive?

1827641255216343512729

ERRATUM: The first line of page 6 has gone AWOL.

Please insert: "solution to the above problem can be found (in theory) by".

EDITORIAL

This issue has for the first time been edited by someone other than Marion Stubbs. It seems to me that there is very little difference to be noted except perhaps a certain loss of the humour we have all come to take for granted. The job is surprisingly difficult and I found it very hard not to take everything that was submitted personally. I couldn't bring myself to put Peter Weir's piece in without making alterations, although I did try to ring him first; only, noticing much later that his telephone number had been changed. But after a while I grew reconciled and put in whatever came along even though Percy Slllitto seems to me to be using a circular argument at the beginning of his article and Datta Gumaste is shooting at straw dogs.

On the whole it has been a thoroughly enjoyable experience except for the terrifying ordeal of finding titles and lettering. And it has given me a great lift to know how much so many people care about what is happening at the OU. May I now make this appeal for everyone who reads this issue to contribute something to M500 and try to make Marion's job a little bit easier.

Finally, to fulfil a promise, the lettering for the heading of "The Axiom of Archimedes" was designed by Min Kent, (aged 13).

ooooooo

Now, to get down to serious business, we are desperately short of material for number 26. And even more noticeable, nothing that comes in these days is chatty about OU maths courses in general. Surely some people are miserable or happy with what they have undertaken and would like to tell us. Looking back over past issues of M500 one sees Bob Margolis has submitted several MST282-type problems. But no one has ever sent in a solution to any of them. There must be a reason for this: either they are too difficult or too boring or MST takes up so much time that its students have no leisure to amuse the rest of us. It would be nice to know, if someone could write in and say.

M500/26 will be out in September because August is the holiday month. This does not mean that no-one need write anything for a month but what it does mean is that everyone should write something now (not necessarily before breakfast but certainly today.)

Please sent to Marion Stubbs, as usual.

Eddie Kent

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