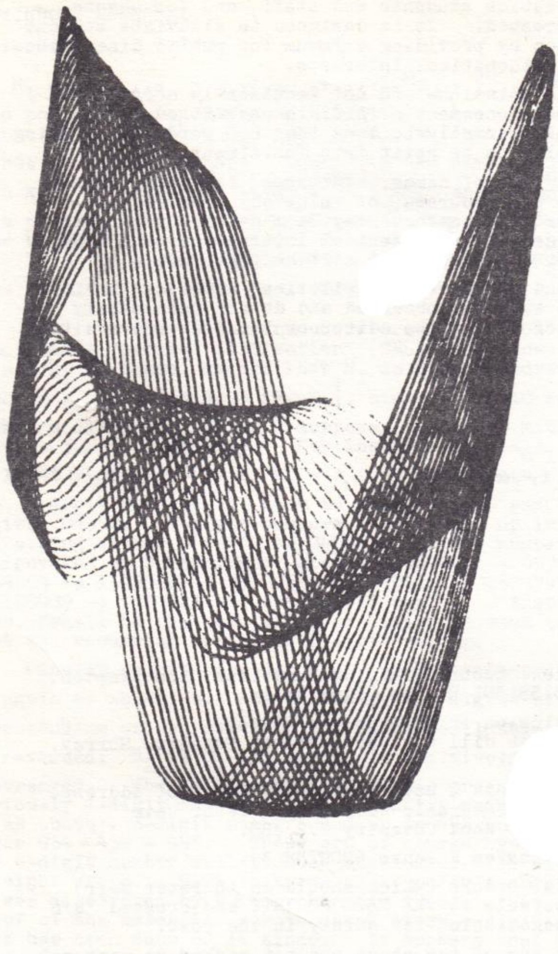

M500 35

M500 is a student operated and owned magazine for Open University Mathematics students and staff, and for anyone else who is interested. It is designed to alleviate student academic isolation by providing a forum for public discussion of individuals' mathematical interests.

Articles and solutions are not necessarily correct but invite criticism and comment. Articles submitted for publication should normally be less than 600 words. Anything longer ought really to be split into instalments.

MOUTHS is a list of names, addresses, telephone numbers and previous and past courses, of voluntary members, by means of which private contacts may be made by any who wish to share OU and general mathematical interests or who wish to form telephone or correspondence self-help groups.

The views and mathematical abilities expressed in M500 are those of the authors concerned and do not necessarily represent those of either the editor or the Open University.

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CONTINUOUS REVERSAL

Eddie Kent

Let $N_1 = a_0a_1...a_n = a_n+10a_{n-1}+...+10na_0 = \sum_{k=0}^{n} 10ka_{n-k}$ be a positive integer with $n + 1$ digits. Associated with the number is another, $R_1 = a_n...a_1a_0$, which has the digits of N_1 but in reverse order. Specify a new number $N_2= N_1 + R_1$ and with $R_2 = N_2$ reversed; then $N_3 = N_2 + R_2$, and so on. Eventually this process produces a palindrome number, one which reads the same backwards as forwards. Or does it? For instance, if $N_1 = 7893$, $R_1 = 3987$; $N_2 = 7893 + 3987 = 11880$; $R2 = 8811$, $N_3 = 20691$, $N_4 = 19602$, $N_5 = 40293$, $N_6 = 29204$, $N_7 = 79497$.

Call the above "continuous reversed addition"; define "continuous reversed subtraction" (CRS) as follows. Specify N_1 and R_1 as above, except that is N_1 is not palindrome (and hence $n \geq 1$). Now $N_2 = |N_1 - R_1|$, etc. (Should really have picked a different letter I suppose, but never mind.) Various results jump out quickly. For instance $N_1 = a_0a_1$ leads to the cycle 45 09 81 63 27 45 ... ; $N_1 = a_0a_1a_2$ produces a cycle around 495, 4-digit numbers all seem to produce 0450 and cycle; 5-digits, 04950. 7-digit numbers usually end up in a cycle containing either 0049500 or 0499950 but the number 9899010 (discovered by D R Kaprekar) goes to 9789021 → 8579142 → 6159384 → 1319868 → 7369263 → 3739626 → 2529747 → 4949505 → 109989 → 9789021 and cycles off again. "What", Kaprekar asks, "shall be the conditions of digits in such numbers?" That is, recurring series of 7-digit numbers.

Finally we can consider CRSO, or CRS with ordering. N_1 is again as above, but now rearrange the digits of N_1 in nonascending order from left to right; call this O_1. R_1 is N_1 reversed; N_2 is $O_1 - R_1$ (which is positive); O_2 is N_2 rearranged. What will happen when this process is continued? Obviously 1-digit numbers vanish, 2-digit numbers cycle around 45 as above; 3-digit numbers end up as 495 and get no further since 954 – 459 = 495. These are of course hypotheses; but any 4-digit number will produce the number 6174 within a maximum of eight steps. 6174 is known as Kaprekar's constant since it was discovered by D. R. Kaprekar (in 1946). His method of proof of the assertion was to check every 4-digit number, but work has been done on it since. It appears that no other size of number leads to a constant - all others end up in cycles of various lengths. For instance there are three cycles associated with 5-digit numbers.

One example of CRSO applied to a 4-digit number:

$N_1 = 1976$; $O_1 = 9761$; $R_1 = 1679$.

$N_2 = 9761 - 1679 = 8082$; $O_2 = 8820$; $R_2 = 0288$.

$N_3 = 8532$; $O_3 = N_3$; $R_3 = 2358$.

$N_4 = 6174$.

ALGEBRA IN THE FIELD

Rosemary Bailey

I always thought I was a Pure (if not pure) Mathematician but since January I have been working on a very practical problem and have been amazed at how much of my knowledge of abstract algebra I have actually been called upon to use. I thought this fact might interest those sceptics among you who believe that Pure Maths has no practical value, so I'll try to give some account of what it is that I do.

Briefly, my work is to design experiments. The design should as far as possible maximise the amount of relevant comparative information which can be obtained as a result of the experiment while minimising the cost of the experiment and any inherent bias it may have. I give some simple examples of the type of experiment in which we are interested.

(i) "*Complete Blocks*" We wish to compare 7 different brands of washing powder. 7 washing machines are available and there is sufficient of each powder to do at least 7 washes with it. It is obviously better to use each powder once in each machine than to use each powder 7 times in the same machine as then any effect due to the difference in machines is eliminated when two different powders are compared.

(ii) "*Incomplete Blocks*" As above, but now there is sufficient of each washing powder for only 3 washes. If there is not time to do 21 washes successively in a single machine, how should we best allocate the powders to the machines? Since each pair of powders is to be compared it is better if, for each pair, there is at least one machine in which they are both used. A possible solution is on the right (and if some of you out there are shouting out "projective plane!" you're quite right).

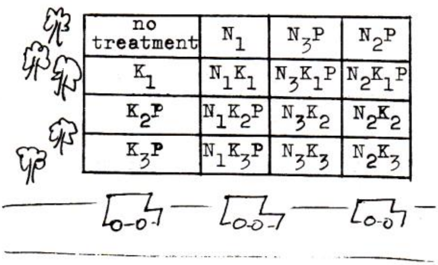
	Powders
Machine A	123
Machine B	145
Machine C	167
Machine D	256
Machine E	247
Machine F	346
Machine G	357

(iii) *Variety Trials* There are 60 new varieties of wheat out of which we want to choose 10 or so good varieties to recommend to farmers. Individual comparisons are less important than in the previous example as we don't mind overlooking a few good varieties so long as we can find sufficient to recommend for use. All the same we can't simply carve up a field into 60 plots and grow one variety on each. What about those varieties that are grown under the trees at one edge, or those on the steep part of the field? The amount of wheat produced on a plot may reflect more on the position of the plot within the field than on the variety of seed used. The variation in soil fertility over the field is less easy to pinpoint than the discrete differences between machines in the previous example, but we deal with it by dividing the field into strips (called "blocks") and treating these blocks as theoretical washing machines! In order to eliminate the effects of these blocks we should place each variety on several plots in different blocks, and try to ensure that most pairs of varieties occur together in the same block at least once. For greater accuracy in the comparisons one should try to ensure that each pair of varieties occurs in the same block approximately as often as every other pair. The design now follows the pattern of example (ii), except that it will be much larger and more complicated.

There is a further difficulty however. Suppose that, to achieve a sufficiently balanced design, we had to use 27 blocks of size 20, so that each variety is grown on 9 plots. The experiment is now so large (540 plots) that it is impossible to harvest it all in one day. All kinds of cosmic disasters, such as thunderstorms, may happen between one day's harvesting and the next, and so we want to ensure that the plots harvested in one day include each variety equally often. As whole blocks will usually be harvested at a time, this means that we must be able to group the blocks into "superblocks" containing each variety equally often.

(iv) "*Factorial Experiments*" In all the previous examples only one item could be tested in a given plot/experimental-run at one time. In contrast, when fertilizers are tested several can be applied to a plot at one time; indeed it is advisable to do this as the optimum fertilizer may be a combination of the basic ingredients.

Suppose that we wish to test the effects of nitrogen (N), potassium (K) and phosphorus (P). There are 4 different quantities of N that we might use on a plot - say none at all, x lb/acre, y lb/acre, and z lb/acre; 4 of K and 2 of P. The "main effect" of N is the effect that different quantities of N would have *in the absence of any* K *or* P *variation*. The "NK interaction" measures how the effect of K varies with each quantity of N. The "NKP interaction" is a measure of that part of the variation of P with the different combinations of N and K which cannot be accounted for by the NP and KP interactions. In order to measure all the interactions we should have to test all possible combinations so 32 plots would be needed. But our field isn't big enough! There is room for only 16 plots. We decide that the NKP interaction is of less importance than the NK, NP and KP interactions, and so try to design the experiment so that each combination of N and K occurs, and similarly for NP and KP. A layout such as that on the left achieves this.



Of course, in practice the particular layout I have drawn is very bad, as the effect of no nitrogen is confused with the effect of shade from trees, and the effect of the largest amount of potassium is confused with the effect of the fumes from the lorries on the motorway. In practice we have to introduce blocks again and try to see that only high-order interactions (such as NKP) are confused with blocks.

Well, how does abstract algebra come into all this? Obviously I can't go into detail but I can mention those topics that I have needed to use. See M500 36.

COINCIDENTAL BIRTHDAYS

Alan G Munford

The article on coincidental birthdays which appeared in M500 33 has recently been brought to my attention by Dr Bryan Orman. I have two comments to make.

My first is that I find the argument quoted from Weaver's book most unconvincing. Weaver argues "... the probability that the third person's birthday differs from that of the first and second is 363/365. These are independent events, so the ...". The probability of the event described above is in fact $(364/365)^2$, 363/365 is a conditional probability. Formally, if A denotes the event that persons 1 and 2 have different birthdays, and B denotes the event that person 3's birthday is different from the other two, then we require $P(A \cap B)$. Note that $P(A \cap B) \neq P(A)P(B)$, since the above events are *not* independent. However we do have

$$(PA \cap B) = P(B \mid A)P(A) = \frac{363}{365} \times \frac{364}{365},$$

which agrees with Weaver's result. Sloppy arguments can sometimes be avoided by using the following systematic approach to problems:

(i) formulate the problem by defining suitable events;

(ii) use the *appropriate* probability axioms;

(iii) perform the calculations.

I have observed that too many students (and teachers it seems), blinded by the goal of obtaining the correct "answer", work through their problems in the order (iii), (ii), (i).

My second comment concerns the assumption that birthdays are distributed uniformly throughout the year. I have a note in preparation in which I show that the uniformity assumption furnishes a minimum for the probability of coincidental birthdays, explaining why we observe coincidental birthdays more often than not in a group of about twenty people, since in general birthdays are not spread evenly throughout the year. This result also enables us to take care of leap year birthdays – simply assume a 366 day year and take the pessimistic view that all days are equally likely.

University of Southampton

An Obscure Writer

Philo, with twelve years study, hath been grieved
to be understood; when will he be believed?

John P W Donne

(Philo wrote several of the course units for the 1596 version of M100.)

A PUBLIC SCHOOL RING

(Our Public School sequence is a sequence $\{v_n\}$ of remainders on dividing u_n by n; where $\{n\}$ are the natural numbers starting with 1, and $\{u_n\}$ is the Fibonacci-type sequence 1 3 4 7 11 ...)

John Reade:

The ring $(m + nA)$ **where** $A = (1 + \sqrt{5})/2$

Note that $A^2 = 1 + A$ so that $R = \{m + nA: m, n \text{ integers}\}$ is certainly a ring. Also $B = 1 - A \in R$ and $AB = -1$ so A is a unit in R, in fact the units of R are precisely all the integral powers a^n of A.

The ring R is the natural setting for a deeper analysis of the sequence $\{u_n\}$. For example one can show that if p is any prime $\neq 2, 5$ then

$$A^p \equiv A \bmod p \text{ if } p = \pm 1 \bmod 5,$$

$$\equiv B \bmod p \text{ if } p = \pm 2 \bmod 5,$$

which gives another proof of the fact that

$$A^p + B^p \equiv 1 \bmod p.$$

It also shows that if $p = 31$, $q = 61$, $r = 271$ then for $n = pqr = 512461$ we have $u_n \equiv 1 \bmod n$, so the answer to Richard Ahrens's first question (does $v_n = 1$ imply n is a prime?) is in the negative.

A RELATED SEQUENCE Max Bramer:

Looking at the sequence $\{w_n\}$ (elements in column 2, $\{u_n\}$, mod 18) the sequence is certainly cyclic with period 24; $w_{6k} = 0$ for k odd only (e.g. $w_{12} = 16$). When k is odd $w_{6k+j} = w_{6k-j}$, (j odd) which is different from the form given in M500 33. For j even, k odd $w_{6k+j} + w_{6k-j} = 18$. $w_n + w_n+_{12} = 18$ appears to be true except when $n = 6k$, k odd. Writing odd k as $2N+1$, a complete set of results for $\{w_n\}$ (all conjectures at present based on a BASIC program for values of n up to 143 - terms in the second column, known as Lucas Numbers, up to 30 digits in length) is: for all $N = 0, 1, ...$

(a) $w_{12N+6} = 0$ and there are no other zero values (these include all N for which $v_n = 0$, except 1);
(b) $w_{12N+6+j} = w_{12N+6-j}$, j odd;
(c) $w_{12N+6+j} = w_{12N+6-j} = 18$ (j even, excluding multiples of 12);
(d) $w_N = w_{N+24}$;
(e) $w_N + w_{N+12} = 18$ ($N \neq 6k$, k odd).

To try to extend these results I next looked at the Fibonacci sequence (1 1 2 3 5 8 ...) in the second column. Perhaps surprisingly the above five results for $\{w_n\}$ still seem to apply provided $12N + 6$ is replaced by $12N$ in (a), (b) and (c) and the following other changes are made: w_1 and w_2 are excluded altogether; k odd is changed to k even in result (e); the comment about v_N is ignored in (a).

Looking at $\{v_n\}$ for the Fibonacci numbers, there are zero values for n = 1 5 12 24 25 36 48 60 72 96 108 120 125 144 168 180 192 ($n \leq 193$). But *not* 84 132 or 156. Can anyone conjecture the next value in this sequence?

SMOKING AT SUMMER SCHOOLS

Steve Osborn

The correspondence so far is based on ignorance. May a veteran, now ex-smoker, offer a touch of reality? (i) For John Wills (M500 33 3): nicotine is a powerful stimulant causing instant release of sugars and adrenalin into the bloodstream. The other narcotic, carbon monoxide, acts as an external stimulus inhibitor. The resulting high internal arousal, coupled with low sensory input, has obvious attractions for the student. (ii) For ditto and Jeremy Humphries (M500 32 11): of course people can work without smoking. What they cannot do is work while suffering narcotic withdrawal symptoms. These are immediate and severe. During the first week of abstinence I could hardly sign my name. To ban smoking at Summer Schools would be cruel and senseless, (iii): For everyone. smoking is a terrible medical and social problem. It will not be solved by cheap facile jibes but by increased understanding of how and why people smoke. Neither will it be solved by legislation - remember Al Capone?

Fred Holroyd

Marion Stubbs says “cigarettes may be killing me, but I am not convinced, since I have solid reasons for saying that people die from diseases said to be caused by smoking when they do not smoke”. Has it occurred to her that this may be (at least partly) because they have to inhale the smoke exhaled by smokers?

Driving and smoking are done for quite different reasons, so there is no direct comparison between the two, as Marion seems to think; but they undoubtedly both cause pollution problems, and it is no argument against the existence of one to point out the existence of the other.

“Like so many contemporary philosophers he especially enjoyed giving helpful advice to people happier than he was.”

Tom Lehrer; “Hen3ry”

ROSEMARY'S SHUFFLE

Max Bramer

I wonder what I can possibly have done to upset Rosemary Bailey, apart from daring to use a computer (sorry, computing machine) to explore an obscure function (Monge's shuffle) and to suggest some surprising conjectures. Whatever would Polya say about that:?

Unfortunately, not only does Rosemary clearly have little or no idea of the point of my article, but her Mathematics has a hole in it.

Of course I did not know when I wrote that each card really must return to its starting point in 2*p* shuffles or less, but was it abundantly obvious to everybody else that no sequence such as 9, 8, 6, 5, 6, 5, ... or even 9, 3, 7, 6, 5, 7, 6, 5, ... could ever occur, *regardless of the definition of f*? If so, please prove - if not, perhaps it was worth mentioning. In fact, however, *any* proof is certain to depend on the fact that Monge's shuffle is a *permutation* of the integers from 1 to 2*p*. Not all mappings have this property by any means; for example $f(x)$ may be greater than 2*p* or $f(x_1)$ may be equal to $f(x_2)$, which would ruin Rosemary's proof. If Rosemary thinks that *f* giving a permutation "can be proved with a little thought" and "in no way depends on the special shuffle *f*", may I suggest she proves it, *without using the definition of f*. Alternatively, a retraction (and perhaps an apology?) may seem, on reflection, to be called for.

Quite obviously the result depends critically on the exact definition of *f* which must be shown (i.e. proved) to give a permutation. Or is this too self-evident to need proof? As most people will realise, calling a mapping "Monge's Shuffle" does not prove it is a shuffle (in the sense of a permutation) either "by definition" or otherwise!

The real point of my article was, of course, rather more interesting than this "trivial" matter, namely to state the conjecture that, for example, the whole pack has a cycle length less than or equal to 2*p*. I should like to see this result found (as opposed to proved) using M202 algebra!

I am not too proud to use algebra on occasions, and filled literally dozens of pages with algebraic manipulation without more than a minimal insight into the problem. Computing was the second line of attack and worked well this time, I think. Of course I realise that the very idea that a computer could be of the slightest value is thought crime to some pure mathematicians (not all), but is it necessary to be so hysterical about it? Clearly Rosemary has never heard the old saying that "the real power of computing is insight, not numbers"!

You will notice that I did not actually use the phrase "putting it on a computer" in my article. Has Rosemary invented the phrase just so she could criticise it? Straw men are always the best targets. In future I must remember to use the phrase "high-speed, automatic, stored program, general purpose, electronic digital computer" at all times.

Since Rosemary has graciously left all the hard results for lesser mortals to prove, I suppose the next step is for her to write some original articles of her own which I can then abuse and misrepresent. Alternatively, perhaps we could declare a truce?

SUPERFIELDS Krysia Broda

I must apologise for my ‘solution’ to Datta Gumaste’s problem on Superfields. I realised my answer was not an answer the day after I posted it to M500. The editor must have forgotten my frantic phone call to him to ask him not to publish the article.

The definition of the third operation is not on the elements of a field, for if

$$a = 2^{2^{p}2^{p}},\ a \in \mathbb{R}^{+} - \{(0,2]\},\ p \in \mathbb{R}^{+}$$

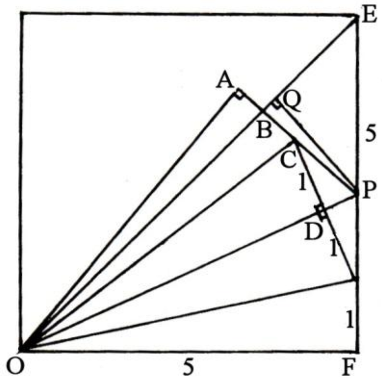
and even if one allows $p \in \mathbb{R}$ and $a = 2^{\pm 2\dot{p}}$ one finds two elements without an inverse - namely 2 and ½.

One can however prove the following:

If (F, +, •, °) is a superfield then (i) (F, +, •) has characteristic 2 and (ii) all finite superfields are of the order 2^k where $2^k - 1$ is prime.

MACHIN’S MOLE-HILL Ansley Fox

The problem was “to prove AB = 5/239 using only the geometry of similar triangles”. So - look for similar triangles!



Produce OD to cut EF at P

Join CP

(ACP is a straight line; cheers!)

From P draw the perpendicular to OB at Q

ABO and QBP are similar triangles.

Notice that QP = QE and you have enough information, using only Pythagoras and proportions, to find the length AB.

Most of the propositions and questions to be found in philosophical works are not false but nonsensical.

Wittgenstein - *Tractatus Loglco Phil* (from Tony Brookes.)

SUPERFIELDS

Tony Forbes

Recall from M500 32 that a Superfield is a structure $(S, 0, +, 1, \bullet, e, \times)$ such that both $S_1 = (S, 0, +, 1, \bullet)$ and $S_2 = (S - \{0\}, 1, \bullet, e, \times)$ are fields. The problem was what superfields are there other than the Mersenne Superfields (where S has 2^p elements and $2^p - 1$ is prime.)

I think I have found all the finite superfields but I can't find any infinite ones at all and I am beginning to suspect that there aren't any.

As for finite superfields, there is only one other than the Mersennes. Its order is 3 and it is constructed from $\mathbb{J}_3$.

To prove this suppose S is a finite superfield and that S_1 and S_2 are the fields used in its construction. Then $o(S_1) = p^m$ and $o(S_2) = q^n$ for some primes p and q and integers m, n; (Herstein lemma 7.6). Therefore

$$q = o(S_2) = o(S_1) - 1 = p^m - 1$$

and the only solutions of this equation are

$$q = 2,\ p = 3,\ m = 1;\ \text{or}$$

$$p = 2,\ q = 2^m - 1.$$

As for infinite fields the only thing I have been able to prove is that S_1 must have characteristic 0 or 2 and S_2 must have characteristic 0. (See Herstein p91.)

The nearest I can get is $(\mathbb{C}, 0, +, 1, \bullet, e, \times)$ where $(\mathbb{C}, 0, +, 1, \bullet)$ is the field of complex numbers, e is the 2.71828... of analysis and $\times$ is defined by $a \times b = \exp(\log a \log b)$ for $a, b \in \mathbb{C} - \{0\}$. But this is not a superfield.

TELEGRAM

To Willem van der Eyken From Miek Warden

GRAAG CONTACT RE TMA04 AM289 STOP KAN NIET BESKITTEN HOE IK DAARAAN MOET BEGINNEN STOP BEL A U B OM CMAS TE TERGELIJKEN OF ZAL IK SCHRIJVEN ? STOP

My slogan for the year: epsilon-delta is the absolute limit.
It only works when you're not watching.

Alan Boultwood

FOUNDATION

Tony Brooks

In the M500 Special, 1976, Datta Gumaste says, referring to M202, “Q: What will be the loss if I don’t take it? A: What will be the loss of a man who lives his life without ever falling in love?” I agree with the sentiments with reference to M202 but I also feel the same about the ideas I have received from Wittgenstein’s later philosophy.

Perhaps one example of the ideas I now have with reference to maths might help. In the first three units of M202 we are presented with set theory which we are told are the ultimate foundations of mathematics. I think this idea is enshrined in Michael Gregory’s diagramme on the back of the 1976 M500 Special. We have logic and sets presented in inner circles, suggesting I think that all else springs from this. Such ideas owe their foundation in Russell’s *Principia Mathematica* which tries to lay mathematics on ‘firm’ *a priori* principles of logic. For example the empty set is defined as the set whose members are those elements which are not equal to themselves. Well since everything is supposed to be equal to itself and cannot be otherwise it clearly has no elements, i.e. it is empty but you cannot just say it has no elements - this is what you are trying to define. From this part Russell is then able to construct for example the integers. Such a construction follows Michael Gregory’s diagram, from apparently irrefutable logic we build sets, integers, and so on outwards.

I do not wish to deny that this is a perfectly good method of building up the structure of maths; what I do deny is that it is the only way. You could take virtually any point on Michael Gregory’s diagram and call it the foundation and then define everything in terms of that starting point. Mathematics is often presented as a building with Logic and Sets as its foundations (and Russell is often presented as having made these foundations firm). This is a totally erroneous concept: it is much better to think of the world of mathematics as like the surface of the earth. Just as you can start at any point of the earth’s surface and explore the rest (there is no unique starting point) so too with maths; you can arbitrarily define any point as the origin and then move outwards (and occasionally you will find some unexplored regions). I think logic and sets are chosen as starting points because it seems they say things that must be true. However the ‘must’ of logic is something of an illusion, a disease deeply ingrained in our way of thinking.

In a very similar way it is just as much an illusion to talk of the foundation of natural science. Physics for example does produce a description of the world, but it must not be thought of as the only unique and absolute one. A good example is Heisenberg’s and Schrödinger’s totally different formulation of quantum mechanics. It makes no sense at all to talk of one of them as the correct description.

The concept behind all of this is that of getting rid of the absolute. We laugh now at mediaeval scholastic debates which argued whether the absolute of Beauty was more perfect than the absolute of Good. Such debates now seem sterile and pointless but arguments for the fixed absolute foundation of maths are in the same class. This century has seen relativity and quantum theory banish the ghosts of absolute space, absolute time and absolute determinism; but we still have a few more ghosts to exorcise.

Russell tried to build up a theory of the analysis of language almost identical to that he developed for mathematics. (This theory is presented in *Logical Atomism*) Wittgenstein does a marvellous job of totally destroying this concept of an absolute analysis in his *Philosophical Investigations*. The same technique could equally well be applied to *Principia Mathematica*. Perhaps Wittgenstein does do this in his *Remarks Oh The Foundations Of Mathematics.*

SOME LETTERS

Lossiemouth - *From Brian Hambling* - My current job is in real-time computing and I would be happy to correspond with anyone interested in the subject. Though I live in the North of Scotland I am a fairly frequent visitor to London and the London area. Anyone interested in beer and a chat when I'm in the area please get in touch.

Many thanks to all contributors to the M500 Special, the most useful and informative booklet I have so far come across. Looking forward to 'doing some maths' next year.

Lancaster - *From Mick Fraser* - Having just moved here from Bromley I thought you might be interested in my reactions. Bromley had two groups of about 12 in M100; in Lancaster there are four students in total. I feel the intruder in the group, not intentional I'm sure but Lancaster is not as cosmopolitan as Bromley. The centre is an adult education centre with a couple of rooms set aside for OU, consequently we have to share the rooms allocated with M201; difficult. At Bromley there was a bar which the group retired to after its rigours, whereas we retire now to a canteen to drink bad coffee and listen to a constantly out-of-tune orchestra (and that takes some doing). The feeling of isolation present in all students with the OU is heightened here; I am at present living in the town itself but am intending to move out into the country in the near future and I fear my feeling of isolation will increase. The answer is to establish contacts with other students, even from other faculties. The course material, interesting and stimulating as it is, is not enough to keep the above feelings at bay.

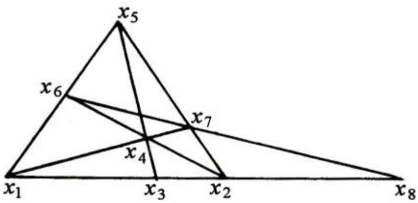
An additional factor that enhanced the above was my chance meeting on a train with a fellow MOUTHS. Strangely he had been to the same exhibition as I had, and works for a competitor's company. Also he lives thirty miles away. So to all you Londoners who complain (LOUSA's newsletter *Open Eye* still arrives at my door) take note of the advantages you have and make use of them.

This situation will probably last for another couple of weeks and retire from whence it came, still while it is around it is a problem of surprising intensity.

POINT CONSTRUCTION (Problem 30.3) From Bob Escolme via Bob Margolis

(Suppose x_1 x_2 x_3 are 3 given points on a line l. x_4 is an arbitrary point not on l and y is any point on the line joining x_4 to x_2. A construction is given for producing a point z on L. Prove that z is independent of the choice of x_4.)

Let V be a vector space of dimension 3 over the field F. Let x_1 x_2 x_3 be 3 independent vectors in *V*, thus spanning *V*.



(Ed - As you can see Mr Ibm hasn't turned up yet with the new typewriter so I have taken the liberty of turning all Bob's Greek letters into English upper case - in the natural way.)

We are given
$$\left.\begin{aligned} x_3 &= Lx_1 + Mx_2 \,;\, L, M \in F \,;\, L + M = 1 \\ x_5 &= Ax_3 + Bx_4 \,;\, A, B \in F \,;\, A + B = 1 \end{aligned}\right\} L, M, A, B \neq 0$$

$$\therefore\ x_5 = ALx_1 + AMx_2 + Bx_4.$$

(Note that the conditions ensure $1 - AL = 0$ and $1 - AM \neq 0$.) The definition of x_6 requires

$$x_6 = Zx_2 + (1 - Z)x_4 \,;\, Z \in F \,;\, Z \neq 0$$

and

$$\begin{aligned} x_6 &= Yx_1 + (1 - Y)x_5 \,;\, Y \in F \,;\, Y \neq 0 \\ &= Yx_1 + (1 - Y)(ALx_1 + AMx_2 + Bx_4) \\ &= (Y + AL - YAL)x_1 + AM(1 - Y)x_2 + B(1 - Y)x_4. \end{aligned}$$

Because $x_1\, x_2\, x_4$ are a basis of *V* we must have

$$Y + AL - YAL = 0 \qquad AM(1 - Y) = Z \qquad B(1 - Y) = 1 - Z$$

from the two expressions for x_6.

The first equation yields $Y = \frac{AL}{AL-1}$ (hence the note above!) and thus $1 - Y = \frac{1}{1-AL}$. The second equation now gives

$$Z = \frac{AM}{1-AL} \quad \text{and} \quad 1 - Z = \frac{1 - AL - AM}{1-AL} = \frac{1-A(L+M)}{1-AL} = \frac{1-A}{1-AL} = \frac{B}{1-AL} \ (L + M = A + B = 1).$$

Thus $x_6 = \frac{AM}{1-AL} x_2 + \frac{B}{1-AL} x_4$. Now $x_7 = Xx_1 + (1 - X)x_4$; $x_7 = Wx_2 + (1 - W)x_5$.

The same type of calculations lead to $x_7 = \frac{AL}{1-AM} x_1 + \frac{B}{1-AM} x_4$. Finally

$$\begin{aligned} x_8 &= Vx_1 + (1 - V)x_2;\ x_8 = Ux_6 + (1 - U)x_7 \\ &= U\left(\frac{AM}{1-AL} x_2 + \frac{B}{1-AL} x_4\right) + (1 - U)\left(\frac{AL}{1-AM} x_1 + \frac{B}{1-AM} x_4\right). \end{aligned}$$

The coefficient of x_4 must be zero: $U\left(\frac{B}{1-AL} - \frac{B}{1-AM}\right) + \frac{B}{1-AM} = 0$ which yields $U = \frac{AL-1}{A(L-M)}$;

$1 - U = \frac{1-AM}{A(L-M)}$. Thus $x_8 = \frac{AL-1}{A(L-M)} \cdot \frac{AM}{1-AL} x_2 + \frac{1-AM}{A(L-M)} \cdot \frac{AL}{1-AM} x_1 = \frac{AL}{A(L-M)} x_1 - \frac{AM}{A(L-M} x_2 = \frac{L}{L-M} x_1$ $- \frac{M}{L-M} x_2$. So x_8 depends only on x_1, x_2 and L, M which determine x_3. i.e. x_3 is independent of x_4, x_5.

Note: If x_1 is represented by (1, 0) and x_2 by (0, 1) then x_3 is (L,M) and x_8 is $\left(\frac{L}{L-M}, \frac{-M}{L-M}\right)$.

Note 2: This note doesn't square with dim $V = 3$ but never mind - Bob M.

Note 3: The same method of solution was offered by Steve Murphy over $\mathbb{R}$ in two dimensions, M500 34.

Note 4: Bob Margolis tackles the general case (which is quite different) in 36. - Ed.

PROCEEDINGS OF THE DIRAC AND KRONECKER DELTA SOCIETY

Philip Newton

Richard Ahrens writes in M500 32 15 that he was puzzled by the impassioned note put forward in connection with the above society and requested that I state what was the second index in the expression $\Sigma(\ ) \cdot \delta_{ij} \cdot \Sigma(\ )$.

Now I sympathise with R Ahrens puzzlement, this is a common affliction amongst tutors. The cause is rather obvious. If you give a TMA to a Technology tutor he counts the number of words submitted and marks accordingly. Your Science tutor marks according to the time taken, whilst the Mathematics tutor counts the number of (mathematical) spelling mistakes. Most tutors seem oblivious to the concepts expressed in the work. (For example see the comments by R Ahrens M500 30 7 on an article submitted by Don Harper.)

The indexes were omitted in an attempt to focus attention on the matter in hand. That was the definition of the deltas and the relative usefulness of our textbooks. For M100 students I demonstrate the triviality of the remarks about the index on the second sigma as follows:

1) By definition δ_{ij} is an $i \times j$ dimension matrix. Therefore it exists.

2) If the numerical value of the index on the second sigma is not the same as the numerical value of the index on the delta then the delta matrix cannot pre-multiply the expression in brackets so we can ignore this case (it is called 'undefined' and the tutor would have written "rubbish" or some other elegant mathematical expression).

3) This leaves only the case where the numerical value of the second sigma is the same as on the delta. In this case we can alter whatever letter has been used on the sigma to a j. This is

guaranteed by either the Schroeder–Bernstein theorem (Halmos page 88) or, for finite numerical value of index, by Peano (5) (Halmos page 47).

Now Mr A how about a brief article on Don's *concept* of the 'Axiomatic approach to Algorithms' showing how it could be tackled?

NOTATION - Marion Stubbs

In M500 32 10 Max Bramer asks which of the definitions of r_x given in my personal précis of M100 Unit 2 (see M500 30) is correct. Perhaps the M101 course team would reply please? In the new Mathematics Preparatory Material, module 7 p. 2 we have

"Relative error: absolute error ÷ exact value. (In practice we often do not know the exact value and take the relative error as absolute error ÷ approximate value instead. The difference is usually small.)"

My précis was produced as a result of reading the Preparatory Material, before which Unit 2 had remained a dark mystery for 5 years despite several re-readings and despite having viewed the tv programme at least 5 times. I incorporated the extra information about r_x since I had previously battled in vain against the first few pages of unit 2, where first one is told that one does not know what the error is and then that one does not know what the exact value is either, after which I had given up in despair. The new Preparatory Material threw light on the whole subject, at least for me with some acquired wisdom of hindsight, and with M231 now behind me. This Material, for the uninitiated, is an experimental set of modules designed to start M100 entrants on the road to success; it was pioneered during this last winter by a select group of 1976 M100 students who volunteered and were lucky enough to be accepted. (I received a set probably because I asked!) I believe that it is intended that the Material, probably duly revised in the light of the pioneers' experience, should be available for new entrants in some form or another, *possibly* on sale in bookshops, in future years. Most of it is completely excellent as an introduction to M100 for those in need of a brush-up courselet.

Apart from the additional definition concerning r_x, namely

$$r_x = \frac{\text{APPROX}-\text{EXACT}}{\text{EXACT}} \qquad \text{as wel1 as} \qquad r_x = \frac{\text{APPROX}-\text{EXACT}}{\text{APPROX}},$$

the rest of my précis was supposed to be unit 2 reduced to a couple of postcards. Admittedly I threw in some attempt to link it all with illicit Leibniz notation, but 1975 M100 will confirm that this "illicit" notation actually had the effrontery to intrude upon their assessment questions, and it has certainly illicitly entered every 2nd-level Mathematics course which I have done, including AM289 wherein one is told brusquely to go and read some unspecified "A-level text" if one does not know Leibniz! I see no reason to exempt M500 from Law-breaking. If Leibniz is illicit then please let 2nd-level Course Teams take note, and M500 *may* dutifully follow.

MONGE ‘ S SHUFFLE

Richard Ahrens

David Asche’s conjecture in M500 31 got me looking through my copy of *Elements of Number Theory* by Vinogradov (Dover) for some help. To my surprise I found a problem on shuffling cards that I was able to adapt to the Monge shuffle problem. In fact the problem in Vinogradov provides a technique that can be applied to many shuffling problems. I will illustrate the technique by proving that David’s conjecture is correct and although I am still unable to reconstruct Moage’s formula I can produce a similar result which is probably just as useful.

Remember that the Monge shuffle takes $2p$ cards labelled l, 2 ,..., $2p$ in order and rearranges them as $2p$,..., 6, 4, 2, 1, 3, 5, 7, ..., $2p-1$. David conjectures that if it takes m repetitions of this shuffle to get the cards back to their starting point then

$$(4p + 1)k \pm p = 2^{m-2}\ ; \qquad \text{for some } k.$$

If we multiply by four we get

$$2^m = (4p + 1)4k \pm 4p = (4p + 1)(4k \pm 1) \mp 1$$

which we could write in the equivalent form: $2^m \equiv \pm 1 \pmod{4p+1}$. In fact m (>0) is the smallest power of 2 which is congruent to either +1 or –1 (mod $4p$ +1).

Proof: Contemplate, if you will, the following sequence of integers

$$2p-1\ \ 2p\ \ 0\ \ 2p\ \ \ldots\ \ 2\ \ 1\ \ 1\ \ 2\ \ \ldots\ \ 2p-1\ \ 2p\ \ \mathbf{0}\ \ 2p\ \ 2p-1\ \ldots\ \ 2\ \ 1\ \ 1\ \ 2\ \ \ldots\ \ 2p\ \ 0\ 2p\ \ 2p-1\ \ldots\ .$$

That is, we write the numbers from 1 to $2p$ in order, follow this by a zero, and then write the numbers in reverse order. This block of numbers is then repeated infinitely many times in both directions. For example if $p = 3$ we would have the sequence:- ... 4 5 6 0 6 5 4 3 2 1 1 2 3 4 5 6 **0** 6 5 4 3 2 1 1 2 3 4 ... . Call the bolded position position zero and number positions to the right and left in the usual way.

If you now select from this sequence the numbers in the following positions:-

$$-2^{k+1}p,\ -2^{k+1}p + 1.2^k,\ -2^{k+1}p + 2.2^k,\ -2^{k+1}p + 3.2^k,\ \ldots,\ -2^{k+1}p + (2p-1)^k$$

you will find that these $2p$ numbers give the arrangement of the pack of cards after k shuffles! For example with $p = 3$ and $k = 2$ we should select numbers in positions

–8×3, –3×3+4, –8×3 + 8, –8×3 + 12, –8×3 + 16, –8×3 + 20; i.e. –24, –20, –16, –12, –8, –4.

These numbers are 5,1,4,6,2,3 which is indeed the result of two shuffles on 1,2,3,4,5,6. I will leave it to the reader to convince himself that this trick always works. It requires a slightly messy induction argument but is not really too bad. (Remember that the card in position r is moved by one shuffle to position $p + 1 - \frac{r}{2}$, r even; $p + \frac{r+1}{2}$, r odd.) If you will now accept that the sequence does do what I claim the rest is fairly easy.

Consider the card labelled 1, that began life as the first card in the pack. This card will return to the first position after m shuffles if and only if

$$-2^{m+1}p \equiv 2p \pmod{4p + 1}$$

because the 1's in our sequence occur in positions which are congruent to $2p$ or $-2p$ (mod $4p + l$). Divide by $2p$ (which is relatively prime to $4p + l$) to get $2 \equiv \pm 1$ (mod $4p + l$). In which case all the cards are to found in our sequence at positions $-2p, -2p + l, -2p + 2, -2p + 2p - l$, (mod $4p+l$). i.e. they are all back where they belong. Thus we have proved David's conjecture.

PROBLEMS: (These can go here as the first two depend on the above.)

35.1 *SHUFFLE* - Richard Ahrens

Monge has given a formula which is supposed to give the position of card x_0 after m shuffles. Unfortunately it has been misprinted and no M500 reader has succeeded in restoring the original. However the sequence technique (page 15) can be used to find a relationship between x_0 and x_m (where x_m is the position in the pack of card x_0 after m shuffles).

Prove that $2^{m+1}x_m \equiv 2^m \pm (2x_0 - 1)$ (mod $4p + l$).

(Note: If you wish to use this to calculate x_m for a particular x_0 and m and p it is best to begin by preparing a list of powers of 2 (mod $4p + l$). e.g. if $p = 2$ we have $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv -1$, $2^4 \equiv -2$, $2^5 \equiv -4$, $2^6 \equiv 1 \equiv 2^0$ (mod 9). We could now write 2^{-1} for 2^5, 2^{-2} for 2^4, etc; and then rewrite our equation as $x_m \equiv 2^{-1} \pm 2^{-(m+1)}(2x_0-1)$ (mod $4p + l$). This looks as though we have two values for x but you will find that only one of these lies in the set $\{1, ..., 2p\}$.)

35.2 *RIFFLE* - R Ahrens

I am told that some card sharps are capable of executing perfect riffle shuffles with an ordinary pack of cards. That is, the pack is split exactly in halves and the two halves are interleaved with each other. An "out" riffle leaves the top and bottom cards unchanged - an "in" riffle moves the top card to position 2 and the bottom card to position 51. Ordinary mortals find it easier (but slower) to perform the inverse operations: Inverse of "out": 1,2,3,...,52→1,3,5,...,51,2,4,6,...,52; Inverse of " in": 1,2,3,...,52→ 2,4,6,...,52,1,3,5,...,51.

Find appropriate sequences to analyse these inverse riffles and hence show that 8 "out" riffles will restore the pack to its original order while no less than 52 "in" riffles are needed to do the same thing.

(Hint: 2^{52} is the smallest power of 2 which is congruent to 1 (mod 53).)

And now for something cd:

35.3 *FIND THE NEXT TERMS* - Two more sequences taken from R J A Sloane's article in *Jnl Recr Math* vol 7 no. 2.

(18) 1 2 4 8 1 6 3 2 6 4 1 2 8 2 ... .
(19) 1 3 5 8 12 18 24 30 36 42 52 60 68 78 ... .
(20) 0 1 1 2 4 7 13 24 44 81 149 274 ... .

Next two terms and a rule wanted.

More problems

35.4 *INFLATED RUGBY* - Max Bramer

A problem taken (and reworded slightly) from another illustrious publication, which shall be anonymous:

Under the old Rugby Union points system, where only multiples of 3 or 5 points could be scored, there were four different scores which could never be achieved: 1, 2, 4 and 7.

Suppose there had been as many as 35 impossible scores, one of them precisely 58, what would the two basic values have been then?

35.5 *RATIONAL TERMINATION* II - Krysia Broda

Find a number system in which *all* rational fractions have finite decimal expansions.

SOLUTIONS

33.2 *THE KINGS MOVE* - Get from one corner to its diagonally opposite on an *n*-dimensional chessboard. (la) $n = 2$, moves parallel to the axes only. Number of different routes; Jeremy Humphries: 3422; Steve Murphy: If N denotes a move to the north and E east each pattern of $n - 1$ Ns and $n - 1$ Es defines a distinct route. Total: $\frac{(2n-2!)}{(n-1)!(n-1)!}$.

(1b) $n = 3$; routes, JH 399072960; SM $21!/(7!)^3$.

(lc) general case; JH & SM: $\frac{(\sum_{i=1}^{n}(|y_i - x_i|))!}{\prod_{i=1}^{n}((|y_i - x_i|)!)}$.

(2a): la with diagonal moves. JH 48639 different routes; SM: With *d* diagonal moves we have to consider a pattern of $n - 1 - d$ each of Ns and Es with *d* Ds, total $2n - 2 - d$. For each allowable *d* we therefore have $(2n - 2 - d)!/d!((n - 1 - d)!)$ possible routes and so the total number of routes is $\sum_{d=0}^{n-1} \frac{(2n-2-d)!}{d!((n-1-d)!)^2}$.

(2b) The number of routes from (x_1, x_2) to (y_1, y_2). SM: If $|x_1 - y_1| = a$ and $|x_2 - y_2| = b$ and $p = \min(a, b)$ then number of routes: $\sum_{d=0}^{p} \frac{(a+b-d)!}{d!(a-d)!(b-d)!}$.

33.3 *NEXT TERMS*

(1) 14916... (squares); (2) 1 3 6 10 15 ... (triangles); (3)1 1 2 3 5 ... (Fibonacci).

(4) 1 2 5 12 29 70 169 408; 985,2378. $u(n) = 2u(n - 1) + u(n - 2)$.

(5) 1 1 3 1 5 3 1 1 9 5 11 3 13 7 15 1; 17, 9, 19. Divide each *n* by the highest power of two possible. Brian Woodgate and W P Evans.

33.4 *THE PROFESSOR* - Brian Woodgate and Datta Gumaste both gave the answer 50 without satisfactory justification. The answer does *not* depend on the assistant knowing the professor's age. It is more subtle than that. Full answer in our next.

33.5 *MAXNIM* - Steve Murphy has submitted a solution to this but in view of Max's SS program we won't print it just yet.

EDITORIAL

In some unpublished letters of Max Bramer's (well, honestly, someone else has to get in occasionally) he makes a couple of points which might well have relevance for us - because after all we are mainly undergraduates. In the first place he feels that he is constantly being attacked for holding views that are alien to him and for saying things that he has never even thought. Well it's a tough cruel world. I have noticed that few of the bits coming in here show evidence of the author having read much of the work he is commenting on. But surely it is a psychological truism that we read more into a work than we take from it. Unless, that is, we analyse it in depth ("research"). When I did logic we were presented with an example of a standard fallacy which had us rolling in the aisles. I now believe it contains the ultimate truth about education: "You can teach me nothing because either I know it already, or else I don't and so cannot understand you."

Which brings me to the other point of Max's where he says (direct quote): "I am simply suggesting rather more use of such phrases as 'It seems to me' or 'this looks like a proof...' But surely this is unnecessary since as undergraduates we should be learning to treat every statement with scepticism, as one is here not to learn Mathematics or Urdu but how to impose a personal order on the chaos out there. That is to say, as people, we should even take the words of academics as hypotheses (although of course they are divinely inspired for the purpose of any particular TMA).

A couple of problems recently have generated the same sequence as was noticed by Richard Ahrens in M500 33. The sequence goes 1,2,5,14,... and was discovered by Euler, who produced the recursive formula $2 \times 6 \times \ldots \times (4n - 10)/(n - 1)!$ In *Scientific American* this June there is an article on these numbers by Martin Gardner. He says they are called Catalan numbers and they are sequence 577 in N J A Sloanes *A Handbook of Integer Sequences*, 1973; a book we have been pillaging at second hand for some of our "find the next terms" problems. In his article Mr Gardner produces many isomorphisms to explain the connection Richard noticed between Euler's problem and the Kings Move problem, and surprisingly many more, including the one for which the sequence was named when it was solved by E C Catalan in 1838. But the thing that grabbed me, and which I shall pass on to you, is a recursive procedure for generating the next term found by Johann Andreas von Segner in the eighteenth century.

Start with an extra 1. Write out the list of numbers you have so far horizontally left to right. Underneath write the same numbers right to left (or backwards). Multiply each number in the sequence by the number below it. Add up all these products. That sum is the next number. Viz:

$$\begin{array}{rrrrrrrrrrrr} & 1 & & 1 & & 2 & & 5 & & 14 & & \\ \times & 14 & & 5 & & 2 & & 1 & & 1 & & \\ \hline & 14 & + & 5 & + & 4 & + & 5 & + & 14 & = & 42 \end{array}$$

Two more brief words about sequences. Word 1: the u sequence which generated our Public School (v) sequence is known as Lucas numbers, I have been told; and Word 2: again from Martin Gardner. Do not confuse the Catalans with the Bell numbers 1,2,5,15,52,203,877. (A 14-line sonnet can have 190899322 - the 14th Bell - different rhyming schemes.) The rest is silence.

Eddie Kent.