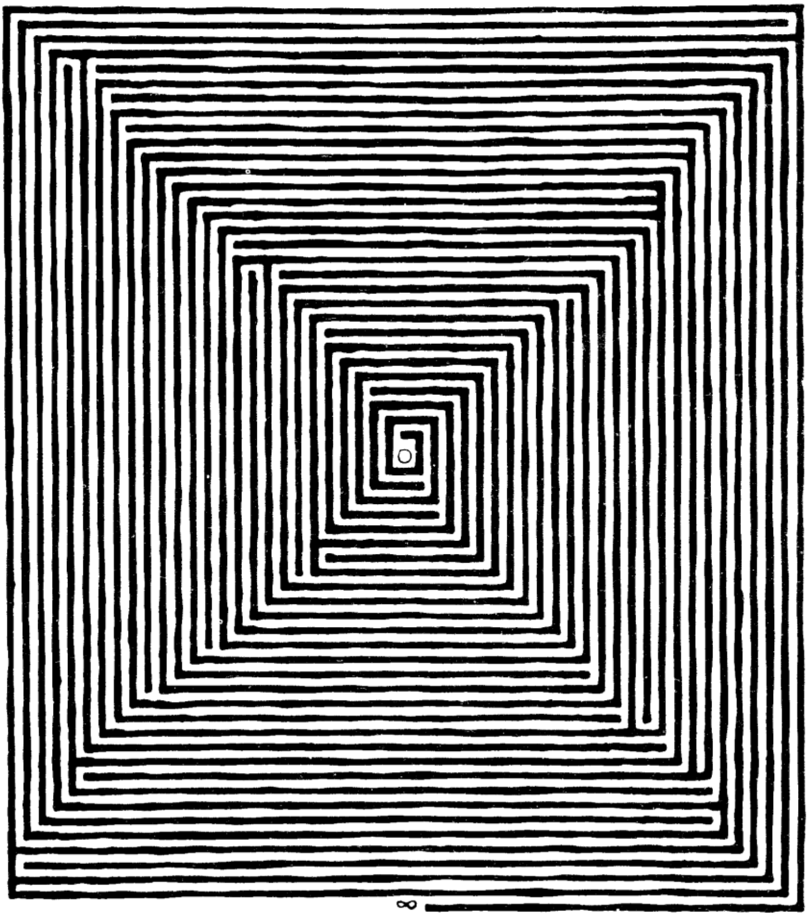

M500 47

M500 is a student-owned, student-operated magazine for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for public discussion of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names and addresses, with telephone numbers and details of past and present courses of voluntary members, by means of which private contacts can be made to share OU and general mathematical interests - or to form self-help groups by telephone or correspondence.

MATES is a special list of MOUTHS members who have explicitly volunteered for their MOUTHS details to be distributed to members in closed institutions such as prisons and special hospitals.

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The cover design is taken, with permission, from "Greg Bright's Maze Book", published in 1973 by Latimer New Dimensions, London. The author says: 'The words "Start" and "End" are limited to the English language. I wanted to replace them with symbols equivalent in scope to the maze itself.' Start at zero and end at infinity.

THE PELL EQUATION

KRYSIA BRODA

$x^2 - 2y^2 = -1$. Solve in positive integers.

I found the first couple of solutions, (1,1) and (5,7) by slog and argued from there as follows.

We have $(a + b/2)\ (a - b/2) = -1$ where (a,b) is a solution. Hence $(a + b\sqrt{2})$ is a divisor of 1 in the field $\mathbb{Q}(\sqrt{2})$. Thus $(a + b\sqrt{2})^2$ is also a divisor of 1 and also $(a+b\sqrt{2})^r$ for any r. Since (a,b) were integers, then so will be (x,y) where $(x + \sqrt{2}y) = (a + b\sqrt{2})^r$.

For example, taking the solution (1,1): $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$, and $9 - 8 = +1$. $(1 + 2\sqrt{2})^3 = 7 + 5\sqrt{2}$ and we know $49 - 50 = -1$. We find that $(l + \sqrt{2})^r$ generates alternately solutions to $x^2 - 2y^2 = -1$ and $x^2 - 2y^2 = +1$. In general $x^2 - Ny^2 = 1$ has a solution and $x^2 - Ny^2 = -1$ sometimes has one.

By chance I happened to be reading Davenport's *The Higher Arithmetic* and found the smallest solution to $x^2 - Ny^2 = \pm 1$ derived using continued fractions. I give an outline of the proof. (If you are interested, the book is well worth buying.)

Notation. I shall write the continued fraction expansion of

$$\alpha = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdots}}} \qquad \text{as } q_0; q_1, q_2, q_3, \ldots ^\dagger.$$

It can be shown that all solutions of an equation such as $x^2 = N$, where N is not a perfect square have infinite periodic expansions. That is, we have $q_0; q_1, q_2, q_3, \ldots, q_{n-1}, q_n, 2q_0, q_1, q_2, \ldots$.

Also $\sqrt{N} = \frac{\alpha_{n-1}A_n + A_{n-1}}{\alpha_{n-1}B_n + B_{n-1}}$ (1), where $\frac{A_n}{B_n}$ is the value of the expansion obtained by ignoring α_{n+1}. ie. $q_0 + 1/(q_1 + 1/(q_2 + \ldots + 1/q_n) \ldots)$. This is known as a convergent.

Similarly $A_{n-1}/B_{n-1} = q_0 + 1/(q_1 + 1/(q_2 + \ldots + 1/q_{n-1}) \ldots)$.

(1) is obtained by manipulation which I will leave out. We also need the relation

$$A_m B_{m-1} - B_m A_{m-1} = (-1)^{m-1}. \qquad (2)$$

One can verify the recurrence relations

$$A_m = q_m A_{m-1} + A_{m-2}, \quad \text{and} \quad B_m = q_m B_{m-1} + B_{m-2} \qquad (3)$$

and then

$$A_m B_{m-1} - B_m A_{m-1} = (q_m A_{m-1} + A_{m-2})B_{m-1} - (q_m B_{m-1} + B_{m-2})A_{m-1} = A_{m-2}B_{m-1} - B_{m-2}A_{m-1}.$$

If we let $A_m B_{m-1} - B_m A_{m-1} = \Delta_m$ then $\Delta_m = -\Delta_{m-1} = +\Delta_{m-2} = \ldots = \Delta_1(-1)^{m-1}$.

$\Delta_1 = A_1 B_0 - B_1 A_0$, $\frac{A_0}{B_0} = \frac{q_0}{1}$ and $\frac{A_1}{B_1} = q_0 + \frac{1}{q_1} = \frac{q_0 q_1 + 1}{q_1}$. ➔

So $\Delta_1 = (q_0q_1 + 1)\cdot 1 - q_1q_0 = 1$, hence (2) is proved.

With these formulæ we can now show that the convergent A_n/B_n for the expansion for $\sqrt{N}$ gives a solution of the equation $x^2 - y^2N = (-1)^{n-1}$. $\alpha_{n+1} = 2q_0 + \frac{1}{q_1 + \dots} = \sqrt{N} + q_0$ (the expansion for $\sqrt{N}$ is periodic). So $\sqrt{N}(\sqrt{N} + q_0)B_n + \sqrt{N}B_{n-1} = (\sqrt{N} + q_0)A_n + A_{n-1}$ from (1). $\sqrt{N}$ is irrational and all the other terms are integers,

so $NB_n = q_0A_n + A_{n-1}$

$q_0B_n + B_{n-1} = A_n$.

$\therefore\ A_{n-1} = NB_n - q_0A_n$ and $B_{n-1} = A_n - q_0B_n$.

$\therefore\ A_n(A_n - q_0B_n) - B_n(NB_n - q_0A_n) = (-1)^{n-1}$ from (2)

and $A_n^2 - NB_n^2 = (-1)^{n-1}$.

Hence A_nB_n provide a solution. If n is odd we have a solution of $x^2 - Ny^2 = 1$ and if n is even a solution to $x^2 - Ny^2 = -1$. To find a solution of $x^2 - Ny^2 = 1$ for even n we can repeat the same argument for A_{2n+1}, B_{2n+1}. ($\sqrt{N} = q_0; q_1, q_2, \dots, q_n, 2q_0, q_1, \dots, q_n, 2q_0, \dots$) where the second q_n is labelled q_{2n+1} if we label the q's consecutively.Then we have

$$\sqrt{N} = \frac{\alpha_{2n+1}A_{2n+1} + A_{2n}}{\alpha_{2n+1}B_{2n+1} + B_{2n}} \quad \text{and} \quad \alpha_{2n+1} = 2q_0 + \frac{1}{q_1 + \cdots} = \sqrt{N} + q_0.$$

We find that $A_{2n+1}^2 - NB_{2n+1}^2 = 1$. For example $\sqrt{14} = 3; 1, 2, 1, 6, 1, 2, 1, 6, \dots, n = 3$ (odd) so we expect a solution of $x^2 - 14y^2 = 1$. $A_n/B_n = 3 + 1/(1+1/(2+1/1)) = 15/4$ and $15^2 - 14\cdot 16 = 1$. (Note that we could have used the recurrence relation (3):

$$\frac{A_0}{B_0} = \frac{3}{1}, \frac{A_1}{B_1} = \frac{4}{1}, \frac{A_2}{B_2} = \frac{2a_1 + a_n}{2B + B} = \frac{11}{3}, \frac{A_3}{B_3} = \frac{A_2 + A_1}{B + B} = \frac{15}{4}.)$$

Apparently it can be proved that the solution so obtained is the smallest possible. Also there seems no way to decide, for given N, whether n is odd or even.

Now what about $x^2 - Ny^2 = \pm M$? For $N = 7$ we have $\sqrt{7} = 2; 1,1,1,4$ and convergents $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \dots$. For $M = 1$, (8, 3) is a solution; for $M = 2$, (3,1); M=3 gives (5,2). For $N = 13$ we get a similar pattern. That is, the numbers for which there are solutions seem to be related to the convergents. Can anyone tell me how?

* * * * * * * *

(† For those who don't know what a continued fraction expansion of a number is I give the following example: We are going to consider infinite expansions for irrationals. Take $\sqrt{14}$ for example. $\sqrt{14} = a + 1/u$ where $u>1$. u is irrational for if it were not then $\sqrt{14}$ would not be. We have $a = 3$. Repeat ➔

the process, $u = b + 1/v$ ($v > 1$). Now from the above equation $\sqrt{14} = a + 1/u$ we have $u = (\sqrt{14} + 3)/5$, and the integral part is 1. So $u = 1 + 1/v$. $v = 1/(u - 1) = 1/(\sqrt{14} - 2) \div 5$ $= 5/(\sqrt{14} - 2) = 5/(\sqrt{14} + 2)/10 = 2 + 1/w$. And so on.

Thus $\sqrt{14} = 3 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{w}}}$; and we can write $\sqrt{14}$ as 3; 1, 2, ... where I have not gone beyond evaluating q_2.

SPECIAL ISSUE 1978 RICHARD SHREEVE

Because of a new interest in our Special Issue the Mathematics Faculty has asked the members of MOUTHS and M500 to produce yet another for next year.

Last year copies went to all our members and to all M100, M201 and M202 students who attended Summer School. This coming year the plan is to also distribute them to students taking other Mathematics courses via the meetings in the Regions at Conditional Registration time.

As you are no doubt aware, it was touch and go whether we would have a Special Issue at all last year, and this next issue will have an even earlier dead-line, so it is vitally important that all of you make a contribution and you do it now while you have some time.

Marion Stubbs has resigned from involvement in any of these fringe activities now and wishes only to be Publisher, so I have stepped in and am trying to live up to her reputation.

Please send your contributions to me, Richard Shreeve, at my home. Of course anything sent to Marion or Eddie Kent will be considered for the Special Issue unless marked to the contrary.

There are three student editors and three staff editors to help me prepare the material which you send so don't worry about me hacking your letters to pieces - they won't let me.

...

"Tunc Gaberbocchum potuisti, nate, necare?
    Bemiscens puer! ad brachia nostra veni.
Oh! frabuisce dies! iterumque caloque calaque
    Lateus eo" ut chortlet chortla superba senex.

Hora aderat briligi. Nunc et Slythaeia Tova
    Plurima gyrabant gymbolitare vabo;
Et Borogovorum mimzebant undique formae,
    Momiferique omnes exgrabuere Rathi.

Which ends *Jabberwocky rendered into Latin Elegiacs* by the late Mr Hassard Dodgson, a Master in the Court of Common Pleas.

LETTERS

From Steve Ainley John Reade's piece in M500 45 is fascinating (Intersecting Diagonals of Polygons). I have dug out some papers from some years back on the question "How many triangles are there altogether in the figure resulting from joining all vertices of a regular n-gon?" This depends on the loss of triangles from 'coincs'. (The word is short for coincidence but pronounced 'ko-ink'. It means a place where three diagonals coincide.) Coincs arise at three (at least) levels:

Level 0: none when n is odd;

level 1: arise when n is even, $\neq 0 \pmod 6$;

level 2: arise when $n \equiv 0 \pmod 6$;

level 3: arise when $n \equiv 0 \pmod 6$ and $n \equiv 0$ (mod 5 or mod 7).

All coincs arise from integral solutions of an equation like that of John's lemma 1,

$$\sin A \bullet \sin (B+D) \bullet \sin (C+D) = \sin D \bullet \sin (E+A) \bullet \sin (F+A)$$

where $A + B + C + D + E + F = \pi$; the level 1 coincs when A,B,C are equal to D,E and F; the level 2 ones from general formulae occurring because $\sin \pi/6 = ½$; and the level 3 ones from such peculiarities as (eg) $\sin\frac{\pi}{15}\sin\frac{8\pi}{21} = \sin\frac{\pi}{14}\sin\frac{3\pi}{14}$.

In the hope you might pass this on to John Reade (to whom, if he is interested in this "other side of the penny" I would be glad to send my (reasonably legible) write up of how far I got (in the hope I mean that he might finish it off properly)) I enclose a couple of stamps.

Ed - John finished his proof off in 46. *Steve's letter has been sent to him*.

From Mike Newman Having just completed what should be my last Summer School I am conscious of a strong feeling of panic that my opportunities for mathematical intercourse are receding. My hope is that through M500, MOUTHS and future mathematics Weekends I may be able to repeat the kind of inspirational lift that Summer School has given me.

From J D Proctor. Trivia. Doodling on HP45 pocket calculator which has reverse polish notation:

If k is any number (some values are no good)

k cos cos cos cos ... converges very quickly - why?
k squared ln squared ln ... oscillates around but not far from zero.
k sin cos tan sin ... converges triply. What else?

I will not join M500 - it's above my head. But should you publish the above please send me a copy.

Ed - How will he see any replies wihout joining?

From Jim Phillipson OK! You win: I'll join!
I've been thinking about what I'll do next year with no OU course and the prospect is horrifying! M500 might just keep the withdrawal symptoms under control.

From Peter Hartley In a flash of post-Weekend enthusiasm I attach a short note on Richard Shreeve's article on linear difference equations.

I again enjoyed teaching at the Weekend but the following Monday found me even more sleepy than previous years. Old age or children waking me up during the night? or both?

Keep up the good work. Well fairly good anyway!

Ed - Richand Shreeve's article was in M500 44/1, *Peter's is elsewhere in this issue.*

From Arthur Thomson I am still puzzled about RELATIVE TRUTH! Michael McAree says it kept him amused for some time, so it did me, but not with seeking the many numerical answers that have appeared in M500. If I have missed seeing the 'correct' solution to the original problem printed in M500 I'd better crawl back into my hole. If not, at the risk of boring, I will have to explain. We were originally asked the problem 39.4 RELATIVE TRUTH: a cynic suggested to me that VERSATILITY is 1001 times better than VERACITY. On this reckoning what is the value of RELATIVITY?

Was it just my twisted crossword-type mentality that led me to look for a solution of the form "RELATIVITY is x times better than Y, $x \in$ (probably x=10a + 1) and $Y \in$ set of words in the English language, preferably having some connection with relativity?
In fact I spent a fair bit of time on this, and thought I had found the solution with "RELATIVITY is 1001 times better than REALITY". Only this does seem to need Y = 10. Perhaps that doesn't matter.

(Numerically this gives RELATIVITY = 8821(T)(I)2(I)(T)(10) with T and I any numbers such that I – T = 1.)

However there do appear to be possibilities for reasonably witty solutions in terms of appropriate wprds. Eg thinking of Einstein something like "RELATIVITY = CREATIVITY + a million ****" would have been nice, but I could not find a suitable four letter word for this! Perhaps someone else can.

When solutions started appearing as strings of meaningless numbers I thought you must have been keeping a "correct" solution up your sleeve. Apparently not!

Ed - Is there a suitable four letter word for this?

From Bill Midgley There was some discussion at the Coventry Weekend concerning the idea of becoming affiliated to OUSA. As far as I could gather it seemed that the opinion of the membership was to be sought. As there is but limited time before we have to make a decision if we are to derive financial advantage next year I thought I would open the debate.

I should be understood that I am fairly new to the OU and to M500 and probably do not know significant facts which could affect ➔

the decision. After all, for a year I believed that OUSA was an organisation devoted to gathering in small groups to make sandwiches and drink coffee. I still do not know whether OUSA is associated in any way with NUS and IUS.

However, the thing that struck me at the time of the discussion was that we were all members of OUSA anyway, and that what we seemed to be talking about was whether we ought to become affiliated to ourselves. If that helps us in any way I cannot see anything wrong in it.

Now for a complete change of subject. I think that about the daftest thing you could try to do would be to run Eureka problems in M500 - so I'll set you one. (It won't mean anything to people who were not at the Chez Angelique sessions.)

Situation: There were ten men sharing one umbrella and they didn't get wet.

To help you I will give you two possible questions and the answers:

1. Big umbrella?
   No.
2. Small men?
   No.

Finally I should like to thank and congratulate all involved in arranging a very rewarding Weekend.

From Nick Fraser Have you ever thought of producing an omnibus edition of the articles printed in the issues of M500? As you know there have been a number of subjects which have caused many people to write. The omnibus could be structured around these. Not forgetting of course all the problems which have been submitted. In fact why can you not do an honoured practice of the BBC and IBA of repeating them in future issues? I joined at issue 28 and so missed out on a great deal..

Meanwhile here is another extract from *A Pun My Soul* by Alan F G Lewis (obtainable from 27 Odds Farm Estate, Wooburn Common, High Wycombe):

We live on a triangular estate
I think we've got the right angle
But you should see
Some of the squares
On the other two sides.

From Peggy Chapman I've three children at conventional universities, one doing finals for MArch, one B Sc and another in second year. All came home to revise, intent on getting results that will be better than mine. We now have two 2-2s and the other out soon. What, I wonder, do I have to do to beat a 2-2 degree? I should love to be "the greatest" but doubt if I will. They will all start pushing me now though.

Must finish and get back to the grindstone; mustn't let the side down.

GAUSS II JEREMY GRAY

On 30th March 1796 Gauss, still only eighteen, solved a problem that had defeated everyone since the Greeks; he discovered a ruler and compass construction for a regular 17 sided figure. He did more, he discovered a systematic theory which explained which regular n-gons would be constructable under this restriction, and which suggested that these were the only such regular polygons, but Gauss did not finish that part off.

Let us briefly review the situation. It is easy to construct a regular three sided polygon (equilateral triangle) and a regular four sided polygon (square). The regular pentagon can also be constructed - it is not so easy, one method was given in Euclid and a simpler one by Luca Pacioli in 1494. The regular hexagon is easy to construct. Because angles can be bisected (under the stated restrictions), from the regular hexagon we may construct successively the regular 12-gon, 24-gon, ... Likewise from the square we may derive the 8-gon, 16-gon, and generally the 2^k-gon (k = 2,3,...). Fom a triangle and a pentagon a regular 15-gon can be derived. (How?) We arrive at the following list, known to the Greeks and never improved upon until 1796: Regular figures of n = 3,4,5,6,8,10,12,15,16,20,24,30,... sides are constructible. But for the missing n (7,9,11,13,14,17,18,19,...) no construction was known and no proof of impossibility had been given. There are certain equivalences with one of the classical problems: that of trisecting an angle. If 60° was trisectable by ruler and compass (it isn't, as every Galois theorist knows) then the 9-gon would be constructible, and so would polygons of $2m–3k–5$ sides (m, k = 1,2,...). Furthermore, the problem of constructing a 7-gon (I shall understand all figures are regular for now) reduces to an angle trlsection. Let us see why this is.

The seven vertices of the 7-gon lie on a circle in the plane. I shall take Cartesian coordinates in the plane with the origin, 0, at the centre of the circle. I shall think of the plane as the complex number plane and let the real axis pass through one vertex, ➔

which is at the complex number 1; the other six vertices are then at w, w^2, ..., w^6 where $w^7 = 1$, so they are all roots of $x^7 = 1$, or $x^7 - 1 = 0$.

The interesting six are the roots not just of $x^7 - 1 = 0$ but, on factorising

$$(x^7 - 1) - (x - 1)\ (x^6 + x^5 + ... + 1) = 0,$$

they are the roots of

$$x^6 + x^5 + ... + 1 = 0. \qquad (*)$$

Write this as

$$x^3 + x^2 + x^{+l} + x^{-1} + x^{-2} + \mathrm{x}^{-3} = 0$$

and introduce $u = x + x^{-1}$ then the equation (∗) becomes

$$u^3 + u^2 - 2u - 1 \ = \ 0.$$

Now solving a cubic equation by ruler and compass is essentially an angle insection problem (recall: $\sin 3\theta = \sin\theta - 4\sin^3\theta$) and to find x from $u = x + x^{-1}$ is a matter of solving a quadratic equation, which is always amenable to ruler and compass methods. So the construction of a 7-gon has indeed been reduced to an angle trisection problem.

But Gauss's problem was: how, if at all, to improve the list of constructible polygons. Trisection was generally regarded as essentially cubic in character, ruler and compass methods as essentially quadratic. The question then is: what prime numbers p are such that the construction of a p-gon can be regarded as essentially quadratic?

The p –1 vertices of a regular p-gon, begun as for the 7-gon with the first vertex at 1, lie at w, w^2, ..., $w^{\mathrm{p}-1}$, the $p - 1$ roots of $x^{p-1} + x^{p-2} + ... + 1 \ = \ 0$. This equation was quite familiar to Gauss by 1796 and he saw that if it is to be reducible to a quadratic by repeated quadratic substitutions (such as $u = x + x^{-1}$) then $p - 1$ must be a power of 2:

$$p - 1 = 2n,$$

say, or

$$p = 2n + 1.$$

But then p must be a so-called Fermat prime (ie, n=2^k) and so

$$p = 2^{2^k} + 1.$$

For k = 2, p =17 and so the 17-gon is constructlble. As a check notice that

when n = 3, p = 9 which is not prime,

when k =3, p = 2^8 +1 = 257 which is prime

and k = 4, p = 2^{16} + 1 = 65 537 which is prime.

So the 257-gon and the 65537-gon are also constructible. However Euler had shown that the next Fermat number, $2^{2^5}+1$ is not prime, and it is not known whether or not there are any more Fermat primes after 65537.

Gauss did not give a literal construction for the 17-gon. Neat ones were subsequently given by other mathematicians and I shall give one at the end of this episode. Obsessional minds have since revealed to mankind how the 257- and the 65537-gon can be drawn, although it is doubtful if the human eye could simultaneously appreciate the regularity and the non-circularity of the latter figure. I believe a regular 17 pointed star has been carved on Gauss's tombstone.

I should like to draw attention to four features of Gauss's discovery.

1. Its novelty: 17 comes as a surprise;
2. Its method: the use of algebra, familiar to Gauss from his involvement with number theory;
3. The role of the complex number plane, handled here with assurance ahead of Wessel's (1797) and Argand's (1806) description of it. What others laboured to perform Gauss frequently took for granted;
4. Its loftiness and austerity: Gauss did not stop to draw the thing.

We shall meet these features again and again in his work.

Our first construction is due to Tietse (1949) and is given in Tord Hall pp 36, 37.

The curve C_i is a circle with equation as shown.

C_1: $x^2 + y^2 = 1$;

A = (0,1) on OY^+.

C_2: $x^2 + (y - ¼)^2 = {}^{17}/_{16}$;

D = (0,d) at $C_2 \cap OY^+$,

E = (0,e) at $C_2 \cap OY^-$.

C_3: $x^2 + (y - d)^2 = 1 + d^2$;

F = (0,f) at $C_3 \cap OY^-$.

C_4: $x^2 + (y - e)^2 = 1 + e^2$;

G = (0,g) at $C_4 \cap OY^+$.

C_5: $x^2 + (y - h)^2 = (1 - h)^2$;

H = (0,h) midpoint of AF,

J = (i,0) at $C_5 \cap OX^+$,

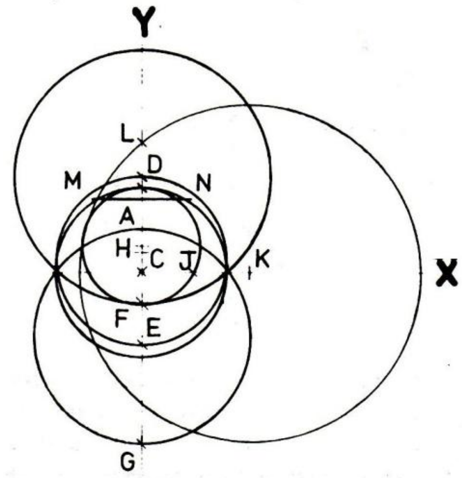
K = (2i,0) .

C_6: $(x - 2i)^2 + y^2 = g^2$;

L = (0, l) at $C_6 \cap OY^+$.

C_7: $y = ¼(l - g)$ meets C_1 at M,N.

M,A,N are three successive vertices of the 17-gon.

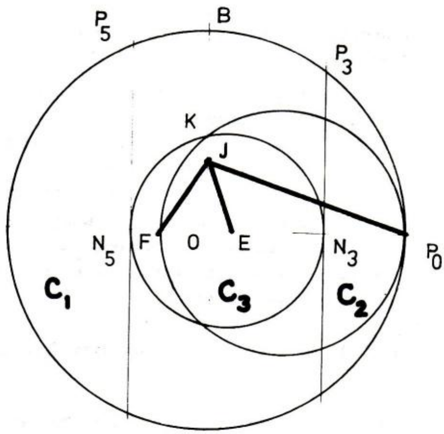


The second construction is due to Richmond (1893) and is given in *Scientific American* (July 1977) .

On a circle C_1 centre O,

$P_0\hat{O}B = \frac{\pi}{2}$; OJ = ¼ OB;

$O\hat{J}E$ = ¼ $O\hat{J}$; $F\hat{J}E = \frac{\pi}{4}$



The circle C_2 has diameter FP_0 and meets OB at K. C_3 has centre E, radius EK, meets OP_0 at N_3,N_5. P_5N_5 and P_3N_3 are perpendicular to OP .

P_3, P_5 are the third and fifth points of the 17-gon.

From Bob Escolme A card Trick.

Holding a few playing cards palmed in the hand and adding them to the pack undetected is not difficult. It is the basis of a card trick which never fails to surprise one's fellow problem drinkers at the local.

It goes like this. Preparation - Before you get to the pub take a pack of 52 playing card (ie leave the jokers behind) and extract any four red suited cards, leaving 26 red and 22 black cards. Place the depleted pack in one pocket and the four extracted cards in another.

Performance - Ask one of your audience which colour, red or black, he prefers. If he says red write on a piece of paper "You have four less than me". If he chooses black write "more" for "less". Fold the paper so as to hide your prediction and place it in front of you. Give the (depleted) pack to someone and ask him to shuffle it, after which he is to take cards from it two at a time. If they are both red (the chosen colour) he is to place them in front of himself; if both are black in front of you; if mixed face down on the palm of your hand. He is to repeat the action until there are no cards left. While he is doing this you can add undetected the four extracted red cards, face down, to the mixed cards which are being placed in your hand. When finished you ask him to count the cards in his red pile and do the same with yours. You then unfold your prediction slip. It will be correct.

Bring all the cards together. You now have a complete pack. You write another prediction: "You have the same number as me". Fold the note as before. Then ask for a colour preference (if you like draw attention to the fact that you made a prediction before a choice was made). Now repeat the previous performance and on this occasion there is no need for you to handle the cards at all. As any M500 reader can see the prediction is bound to come true again.

A proof of this is scarcely necessary since the result is so obvious. Nevertheless when performing you are witholding some of the information from your audience. As a result I have only once had the working of the trick discovered; by one person and then only partially. A mathematics sixth-former who had no business in the pub anyway saw immediately that the second prediction must always come true. By the time he got home he had probably deduced the deceit necessary to work the first part of the trick. By the way don't repeat the trick before the same audience.

And don't explain it to anyone either. Which is what I have just done with you. But, and now for the dreary bit: can you generalise the theorem in any way, and in particular in any useful way (so that the new theorem can be demonstrated with a practical sized pack before a not completely befuddled audience without too great a probability of detection)? One could for example make up a three colour pack (using the backs of three differently designed packs). But what can you do with it? Take them off in threes perhaps but then what?

It's one thing to prove a result required for an assignment; fundamental research is quite another and it is beyond me.

(See, by the way, *Encyclopædia of card magic* edited by Jean Hugard and published in paperback by Faber and Faber.)

PROBLEMS SECTION JEREMY HUMPHRIES

Eddie has traced me to the Black Mountains in Wales where I am snatching a week's holiday. His urgent call for problem section 47 has produced, I'm afraid, only what follows. Luckily I have the latest correspondence with me, though because of exams there's not much of it.

Problems 45.2 *Distant Points*, and 45.3 *Blind Chess* were not attempted. Now you all have nothing to do perhaps somebody can try them.

A lot of this section is devoted to the *Bisector* problem. I found the answers fascinating. Perhaps they should be in the main body of the magazine, but the dividing line is blurred sometimes and as I say, I haven't got much else.

SOLUTION 45.1 OLYMPIAD V $\sum_{j=1}^{q} a_{ij}x_j = 0;\ i = 1,2,\dots,p;\ q = 2p;\ each\ a_{ij} \in (-1,0,1)$. *Prove there exists a solution of the system* ($x_1, x_2, \dots, x_q$) *such that*: a) *all* x_j (j = 1,2,... ,q) *are integers*; b) *there is at least one value of j for which* $x_j \neq 0$; c) $|x_j| \leq q$, (j = 1 ,2, ... ,q) .

From RICHARD AHRENS: we seek non-trivial integer solutions of p homogeneous equations in q (=2p) unknowns, where the equations' coefficients are members of (−1, 0, 1) and the solution ($x_1, x_2, \dots, x_q$) obeys $|x_j| \leq q$ (j = 1,2, ... ,q). I use the pigeonhole principle.

Simply try putting all the q-tuples ($x_1, x_2, \dots, x_q$) satisfying $|x_j| \leq p$, x_j an integer, (j = 1, 2, ..., q) in the left hand side of the p equations. We will obtain a p-tuple of values ($y_1, y_2, \dots, y_p$). Each $x_j \in \{-p, -(p-1), \dots, 0, 1, \dots, p\}$. ie 3 2$p$+l possible values . So we are substituting $(2p+1)^q$ different q-tuples. Now since each coefficient is −1, 0 or 1, the largest numerical value possible for any y_i is pq, so the p-tuple ($y_1, y_2, \dots, y_p$) can take at most $(2pq + 1)^p$ different forms ($y_i \in \{-pq, -pq+1, \dots, 0, 1, \dots, pq\}$). Now $(2p+1)^q = (2p+1)^{2p} = (4p^2+4p+1)^p$ while $(2pq+1)^p = (4p^2+1)^p$. $(2p+1)^q$ is obviously larger than $(2pq+1)^p$, so at least two of the q-tuples must have produced identical p-tuples when substituted. Suppose ($s_1, s_2, \dots, s_q$) and ($t_1, t_2, \dots, t_q$) are different, but give the same result when substituted in the left hand sides. Then ($s_1-t_1, s_2-t_2, \dots, s_q-t_q$) will give all zeros and thus is a solution. It is non-trivial because $(s_1, \dots, s_q) \neq (t_1, \dots, t_q)$. Also since $|s_i| \leq p$ and $|t_i| \leq p$, we have $|s_i - t_i| \leq q$ for all i.

SOLUTION 45.4 BISECTOR *There was money on this one. Consequently it generated more interest than any other problem in* 45. I've sent FRED WHITE some of the solutions and he has kindly sent me the £1. Don't tell Austen.

In figure 1 *consider the points* A,B,C,D,E,F,O. AD⊥BC. O *is on* AD. *Show that* AD *bisects* ∠FDE

THURSTON HEATON, BOB MARGOLIS and STEVE MURPHY sent coordinate geometry solutions. Steve took O as origin, Thurston and Bob took D. Bob had a simplifying idea - call the point O (0,1) rather than (0,x) - which reduced the amount of writing. All three essentially showed that (gradient FD) is –(gradient ED) when AD lies on the y-axis.

Bob says that this method is elementary but somehow unsatisfactory in the 'sledgehammer and nut' sense. Steve wondered what happened if D lay outside BC and found that ∠ADE + ∠ADF = π , ie BD bisects ∠EDF. A nice way of passing a few minutes is to draw these diagrams trying various combinations of D inside or outside BC, O inside or outside AD. Sometimes AD bisects ∠FDE, other times BD bisects it.

Three more solutions came from RICHARD AHRENS, WIM DE JONG and BOB MARGOLIS (II). Here they are:

From RICHARD AHRENS: Recall a problem given by Bob Margolis (see issues 30,33,35,36), from which we found the situation in figure 2. B,D,C are three points on a line. Draw any line through D and choose two points on this line, different from D: O and A. Determine F,E as shown and produce FE to H. The remarkable thing is that H is uniquely determined and depends only on B, D and C, not on the choices made in the construction. H is called the harmonic conjugate of D with respect to B and C.

In figure 1 mark P (as shown). Then E is the harmonic conjugate of P with respect to B and O.

Now find the point S on FD, different from D, which makes BF and OF perpendicular. (Figure 3.)

We now repeat the construction of the harmonic conjugate of P with respect to B and O using S and D in the roles of O and A in figure 2. SO meets BD at T. BS meets DO at V. TV must pass through E. Now TS, VD are altitudes of ΔBTV therefore O is the orthocentre BE is another altitude and ∠BEV is a right angle. Therefore OETD and SVTD are cyclic quadrilaterals. Therefore ∠ADE =∠STV and ∠ADF=∠STV. Therefore ∠ADE = ∠ADF.

From WIM DE JONG: The cross ratio, (W,X,Y,Z) of four collinear points W,X,Y and Z may be defined as (W,X,Y,Z) = (WY/XY)/(WZ/XZ). Clearly (W,X,Y,Z) = 1/(W,X,Z,Y) = (Y,Z,W,X).

If l_1, l_2, l_3, l_4 are coplanar lines lying on a point U and if m is a line intersecting l_1, l_2, l_3, l_4 at W,X,Y,Z we can show (W, X, Y, Z) = $\frac{\sin \text{WUY}}{\sin \text{XUY}} / \frac{\sin \text{WUZ}}{\sin \text{XUZ}}$. Therefore the cross ratio of four collinear points is invariant under central projections. Thus, in figures 4 and 5, (W,X,Y,Z) = (W',X',Y',Z'). The equality holds in figure 5 because $\sin\alpha = \sin(\pi-\alpha), \forall \alpha \in \mathbb{R}$. This invariance is of fundamental importance in projective geometry and was known to Pappus (ca 300) .

Now we consider problem 45.4. Connect F,E in figure 1 and mark the points R and O (so we now have figure 1 as shown in the diagram).

With F as a centre of projection, (B,P,O,E) = (A,D,O,R); with E as a centre, (A,D,O,R) = (C,Q,O,F); and with D, (C,Q,O,F) = (B,E,O,P). Therefore (B,P,O,E) = (B,E,O,P). Whence (EO/EB)/(PO/PB) = 1. Therefore (sin∠EDO∠sin∠EDB)/(sin∠PDO∠sinPDB) = 1. Therefore (since AD⊥BC) (sin/EDO/cos/EDO) / (sin/PDO/cos/PDO) = 1. Therefore tan∠EDO = tan∠PDO. Therefore ∠ADE = ∠ADF. ➔

The last one is *From* BOB MARGOLIS. He says "It is an odd mixture of Euclidean and projective techniques and I only just believe it myself. It is high falutin' geometry and is therefore written in the appropriate language. We proceed by a series of iemmas. Proofs will only be given where not obvious. (ie ≥ reasonable MSc dissertation.)"

Lemma I In figure 2 the position of H is independent of O (and, indeed, of A).
Proof M500 back numbers. Well known.

Lemma 2 In figure 6: (i) OP/OQ = PR/RQ and OP/OQ = –PS/SQ (taking account of signed lengths). (ii) If R,S are such that both equalities hold then OE, OS are the internal and external angle bisectors at O.
Proof Well-known school Euclidean result.

Lemma 3 In figure 2, (BH/HC)/(BD/DC) = –1.
Proof Well-known result in projective geometry.

Lemma 4 In figure 2, (FH/HE)/(FR/RE) = –1.
Proof Project BDCH onto FH from A. Ratio is preserved.

Lemma 5 FR/RE = –FH/HE and AD⊥DH therefore FR/RE = FD/DE and FH/HE = –FD/DE.
Proof One is needed (!)

Lemma 6 AD bisects ∠FDE.
Proof lemma 5 and lemma 2.

SOLUTION 45.5 HOW MANY WERE THERE AT ST IVES? Across: 1,512; 3,343; 7,171; 10,125; 11,256; 12,240; 16,171; 17,342. Down: 2,16; 4,41; 5,17; 6,216; 7,152; 8,120; 9,369; 13,48; 14,27; 15,64. How many people were in the 'bus when it arrived at St Ives? CHRIS GREEN and MICHAEL GREGORY said 25. Was there no driver and conductor?

PROBLEM 47.1 OLYMPIAD III CONTINUED Soon after M500 45 appeared, STEVE AINLEY sent me 14 cubes of volume 2 in a 2×5×6 rectangular box. Can anyone else do it?

PROBLEM 47.2 ARRAY MICHAEL GREGORY

Consider an array in which each element is characterised by X, ΔX; Y, ΔY. eg: A sequence of elements may be formed by mapping $(x,y) \longmapsto (x+\Delta x,\ y+\Delta y)$ taking remainders modulo 5.

X / ΔX / Y ΔY	1 / 3	2 / 1	3 / 4	4 / 4	5 / 2
1 3					
2 1					
3 2					
4 3					
5 4					

(1,1) ⟼(4,4) ⟼ (3,2) ⟼ (2,3) ⟼ (3,5) ⟼ (2,4) where the sequence stops because the next element (3,2) has already been used.

(i) Which start positions give maximum and minimum number of elements in the sequence?

(ii) Are there other sets of ΔX, ΔY which allow all of the elements of the array to be included in the sequence?

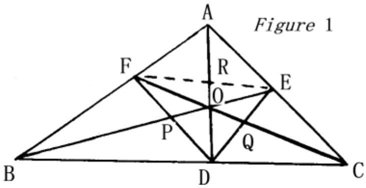


Figure 1

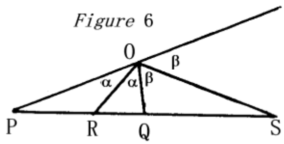


Figure 6

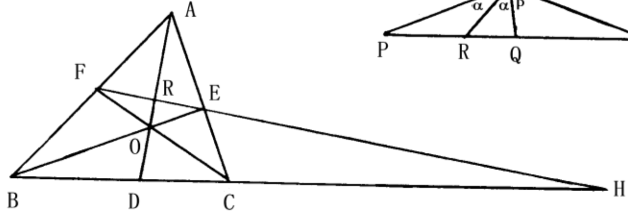


Figure 2

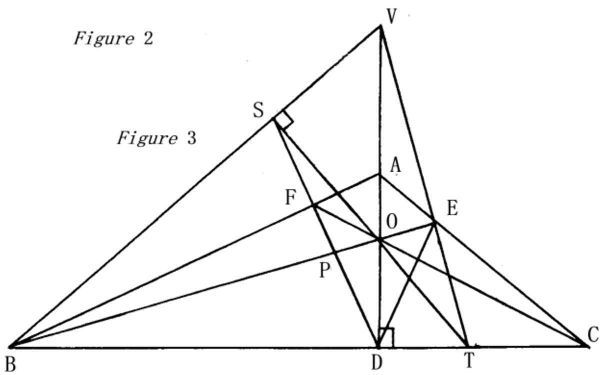


Figure 3

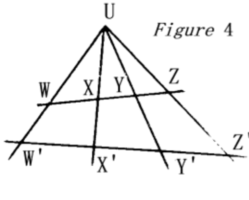


Figure 4

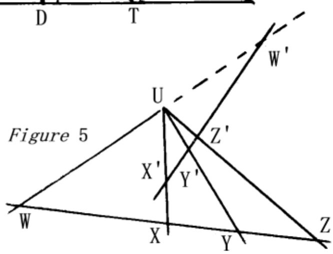


Figure 5

PROBLEM 47,3 FERMAT

Five people made the following statements:

1. Either (a) 3's statement is false and 4's is true, or (b) 2's and 5's are both false.
2. Either (a) 4's is false or 3's is false, or (b) 1's and 5's are both false.
3. 2's is true or 4's and 5's are both true. Moreover either 5's is false or 4's is true.
4. 3's is false or 1's is true.
5. Fermat's Last Theorem is true.

Prove, or disprove, thus, Fermat's Last Theorem.

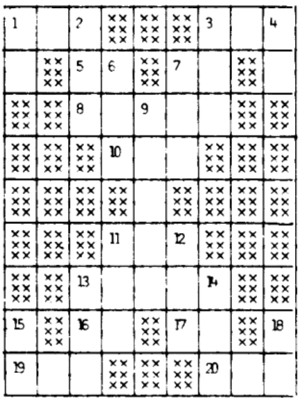
PROBLEM 47.4 SPHERE AND PARTICLE

A sphere of radius *r* has its bottom on a point *P* of a plane. Gravity, γ, is everywhere perpendicular to the plane and holds the sphere on the plane. (I'm trying to avoid unreasonable interpretations.) A frictionless particle slides from the top of the sphere. (I know, if it's on the top it won't slide, but pretend it does.) How long is its path on the sphere? How far from *P* does it first strike the plane?

PROBLEM 47.5 POTTON v BARFORD

This year Potton went to Barford for their annual cricket match and were well satisfied with their total of 213 runs. Great disappointment was caused when their star batsman was dismissed after the third wicket partnership had put on only five runs but Potton felt their total was large enough to carry the day and so it proved. Unfortunately the scorer, unduly elated at his side's success, lost the scorebook on the way home. Can you help by telling him the totals at which each of the Potton wickets fell?

Across : 1 - The ninth wicket fell at this total. 3 - The third wicket fell at this total.



5 - The number of extras in the innings. 7 - An odd number. 8 - The square of the total at which the third wicket fell. 10 - The second wicket fell at this total. 11 - The fourth wicket fell at this total. 13 - The square of the total at which the second wicket fell. 16 - Half the score at which the first wicket fell. 17 - The first wicket fell at this total. 19 - The second wicket fell at this total. 20 - The total the seventh wicket fell at (reversed).

Down: 1 - Twice the number of extras. 2 - The eighth wicket fell at this total. 3 - The seventh wicket fell at this total. h - Eight times the number of extras. 6 - The fourth wicket fell at this total. 7 - The sixth wicket fell at this total. 9 - The square of the total at which the ➔

fifth wicket fell. 11 - This was the score when the fourth wicket partnership had put on 26 runs. 12 - One greater than the total at which the second wicket fell. 13 - The total at which the second wicket fell. 14 - Eight times the total at which the second wicket fell. 15 - The number of extras. 18 - Only three partnerships put on more than this number of runs.

(*Taken from Fun With Figures by L H Clarke. By kind permission of Heinemann Educational Books Ltd.*)

* * * * * * *

I'm making a list of things the BBC do to annoy mathematicians. This is the latest. Early one recent Wednesday evening Radio 4 announced that in Science Now later that night there would be an item on the new uncrackable codes. Not really having grasped this trap-door one way function business with the big primes I set up the tape recorder and waited. There was no such item.

The Fermat problem, 47.3, reminds me that there is a method of forcing things to happen at will. I've forgotten how it goes. I know you need more than one person - either two or three, I think. Someone makes a wish; someone else makes a wish about the wish; and so on. And fulfilment of the first wish is logically forced.

If anyone knows how to do it, tell me and we'll decide whether to visit Monte Carlo or Raquel Welch.

JH

M202 END OF A COURSE JOHN WHITE

I've set out on a journey. Some days the way lies clear ahead. I make good progress and cover many miles. Then the mists come down. I wait. I listen. I must cry for help. If only I could cry for help. There must be help around. I do. Help comes. "These are hard parts to travel alone," comes the advice, "take a friend and companion in future." Suddenly the mist clears. There ahead, direct on my route scenery of such breath-taking beauty. I can only stand, gaze and wonder at the mystery of things.

It has been suggested that an event be organised to mark the demise of course M202. The duration is likely to be a weekend. The time, say, first weekend in February 1979. The first venue is likely to be in the Home Counties or Midlands. The format could be Start Saturday 10am, finish Sunday 5pm, four formal lecture/problem-solving sessions, organised social event on Saturday evening. Cost to be kept to a reasonable level, but not so as to detract from enjoyment.

A minimum number to make it worthwhile? Perhaps 75?

If reaction is favourable from a large enough number of ➔

respondents I am prepared to organise such an event. Please let me have your reaction within three weeks of reading this notice. Suggestions about what to include or leave out are certainly welcome. In particular offers of assistant organiser are welcome.

(John N M White.)

EDITORIAL

At the recent Weekend who should I meet but Wim de Jong who said he had thought it funny to see his own *Congruence Classes* printed in M500 41 12 and credited to John Reade. I remained unrepentant and said it was his own fault for not puttinq his name on it. I always throw covering letters away (because that way I don't have to reply to them and they don't take up filing space) and don't always look to see that the name has been carried forward. Be warned. Incidentally, sorry John; I suppose your writing looks like Win's.

I apologise for my apparent absorption with death. It has been suggested to me that it would be better to notice people before they die. I agree - but have you realised how much we seem to need an excuse to write things down. Fven Jeremy Gray's *Gauss* needed a centenary. Death is certainly a turning point of a kind.

Perhaps some readers have heroes among the mathematicians they would like to commemorate for no other reason than that they would like to. Gauss will probably come to us complete by the time Jeremy has finished: there are others.

I paid 60p some time ago for the first issue of *Vole*, the new magazine devoted to 'the good life' and beer; I thought I would mention it if only to convince myself the money wasn't wasted. The editor is Richard Boston, the man who ruined my local pub with his Campaign for Real Ale - lining the bar with poncy beers and dragging oafs in from all over the country.

Amongst other things the magazine has an article on Horace Saville, the man who invented the world's first concrete false teeth, a charming book review by C O Jones, the Spanish dialectician and a cover which is not continuous but a horizontal translation. Altogether a wild twelve-shillingsworth; but I suppose it's out of print now.

Since starting this edition I have read Marion Stubbs's Publishorial for 46 and now feel called upon to apologise that 47 is not hard on the heels of 46. One reason of course is that I am no longer alone in the production of M500. I had to wait for Jeremy Humphries's Problems and then do his drawings (during the course of which my straight piece of plastic broke so I can never do any more) and also for Jeremy Gray's *Gauss* - and his illustrations were so small I had to get them done professionally (J Wilkinson *fecit*) , hence delay. While I am about it another apology seems to be called for. Page 5, it is stated that Peter Hartley's *Linear Difference Equations* appears herein: not so.

Eddie Kent.