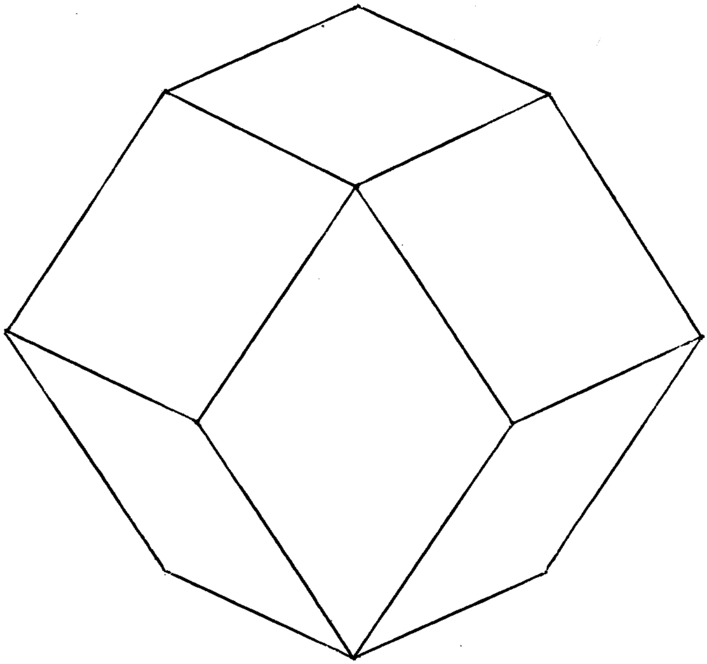

M500 52

M500 is a student-owned, student-operated magazine for anyone interested in Open University mathematics. It is designed to alleviate student academic isolation by providing a forum for public discussion of the mathematical interests of members.

Articles and solutions are not necessarily correct, but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names, addresses, telephones and courses of voluntary members, by means of which private contacts can be made to share mathematical interests - or to form self-help groups by telephone or correspondence.

MATES is a special subset of MOUTHS members who have explicitly volunteered to correspond with members in closed institutions such as prisons and special hospitals.

The views and mathematical abilities expressed in M500 are those of the authors and do not necessarily represent those of either the Editor or the Open University.

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The cover design is by L. S. Johnson who writes about it on page 17 in this issue.
The figure is a regular rhomboidal dodecahedron.

12 faces. Face angle = 2 arctan $1/\sqrt{2}$ = 70.5288°.
24 edges. Angle between adjacent faces = 60°,
14 vertices.

AN INFINITE SERIES WHICH DIVERGES VERY VERY SLOWLY TONY FORBES

If you draw the graph of $x \to \frac{1}{x}$ from $x = 1$ to ∞ and then superimpose on it a descending staircase whose height is $\frac{1}{n}$ for $x \in [n, n+1)$, n=1, 2, ..., you should be able to convince yourself that

(i) $\log N = \int_1^N \frac{dx}{x} < \sum_{n=1}^N \frac{1}{n} < \int_1^N \frac{dx}{x} + 1 = \log N + 1.$

This proves that $\sum_{n=1}^\infty 1/n$ diverges, but it does so very slowly. For example suppose you are a worm crawling at a steady 1 cm/sec along a rope initially 1 km long which suddenly expands by 1 km every second. Then (see *Chez Angelique* problem 13) to get to the other end you will require the time in seconds given by the smallest N for which $\sum_{n=1}^N \frac{1}{n} \geq 100000$. By (i) this is at least 10^{43421} years.

Another very slowly diverging series can be got by considering $\frac{1}{n}\log n$. Again, by drawing graphs and noting that

$$\frac{d \log\log x}{dx} = \frac{1}{x \log x}$$

you can prove that

(ii) $\sum_{n=3}^N \frac{1}{n \log n} = \int_e^N \frac{dx}{x \log x} \pm 1 = \log\log N \pm 1.$

Generalising this construction for any positive integer m and using

$$\frac{d \log^{m+1} x}{dx} = \frac{1}{x \log x \log\ \log x \ldots \log^m x}$$

gives

(iii) $\sum_{n=(e^*m)+1}^N \frac{1}{n \log x \log\ \log x \ldots \log^m (n)} = \int_{e^*m}^N \frac{dx}{x \log x \log\ \log x \ldots \log^m x} \pm 1 = \log^{m+1} N \pm 1.$

Here $\log^k n$ means log log ... log n (with k logs) and I have used the *-notation introduced in M500 46 ($m^*n = m^{m^{m^{\cdots}}}$ taken to n exponents). Also starting the series at $(e^*m)+1$ ensures that $\log^m n > 1$.

Now (iii) implies that for any $m = 1, 2, \ldots$, the series

(iv) $\sum_{n=(e^*m)+1}^\infty \frac{1}{n \log\ n \log\ \log\ n \ldots \log^m\ n}$

diverges and for large m the divergence will be very slow indeed. However I want a series which diverges even more slowly than any of these. Somehow I want to let $m \to \infty$ in (iv) but this raises the problem of where to start the series. If $m \to \infty$ then $e^*m \to \infty$ as well and in the limit the series never gets started at all. One way to get round this difficulty is as follows.

Let us define a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} x & \text{for } 0 < x < e \\ x \log x \log \log x \ldots \log^m x & \text{for } e * m \le x < e * (m+1) \text{ where } m = 1,2,\ldots . \end{cases}$$

Thus $\phi(1) = 1$, $\phi(2) = 2$ and $\phi(3) = 3 \log 3$ because $e \le 3 < e^e$; also $\phi(15) = 15 \log 15$ but $\phi(16) = 16 \log 16 \log \log 16$ since $16 > e^e = 15.1542\ldots$. The $x \log x \log \log x$ form continues up to e^{e^e} and then we have $\phi(3814280) = 3814280 \times \log 3814280 \times \log \log 3814280 \times \log \log \log 3814280 = 157$ million.

Now we can consider the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$. It can be shown that for m = 1, 2, ...,

$$\sum_{n=(e*m)+1}^{(e*(m+1))} \frac{1}{\phi(n)} > \frac{9}{10};$$

hence for large N we have

$$\sum_{n=1}^{N} \frac{1}{\phi(n)} > \frac{1}{1} + \frac{1}{2} + \frac{9}{10} + \frac{9}{10} + \cdots + \frac{9}{10},$$

where the number of $\frac{9}{10}$ths tends to ∞ with N. This means that the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ diverges, but it does so very very slowly, even more slowly than (iv) for any m. And I'll leave it to you to estimate how many terms are required to reach, say, 100000.

DEFINE A FUNCTION BRIAN WOODGATE

Students of analysis will know that the subject is to a large part concerned with the behaviour of functions. We consider characteristics of functions such as Limits, Continuity and Convergence, but let us leave those to that well known double act ε and δ. However, we do need to define the function itself.

The search cannot do better than start with Halmos and Set Theory. He has a good chapter on the subject in *Naive Set Theory* (Van Nostrand, 1960), but he makes life difficult as he develops his subject slowly and so his definitions are not self standing, they depend on previous chapters. I like the Spivak approach {*Calculus*, Benjamin, 1973), ie

> A function is a set of ordered pairs of numbers, such that if the set contains both (*a,b*) and (*a,c*) , then $b = c$.

However the trouble is that we would need to define 'ordered pairs'.

Perhaps the simplest solution is in the Mendelson style (*Introduction to Topology*, Allyn & Bacon, 2nd edn 1971) :

> Let A and B be sets. A correspondence that associates with each $x \in A$ a unique element $y \in B$ is called a function from A to B.

Thought. Does a function have to imply action, ie 'from'? Any suggestions for a short, simple and self sufficient definition?

(The Halmos and Mendelson are set books in M202, Spivak in M231.)

AM 289 AND THE LOG HENRY JONES

AM289, *A History of Mathematics* was one of the most pleasurable studies I have ever made. However, the course contained two fragments of mathematical history which were so treated as to leave me deeply troubled.

My first misgiving arose on reading the narrative on the logarithms of negative numbers. Jean Le Rond d'ALembert and Johann Bernoulli conjectured that real number logarithms of negative numbers exist and that the logarithm of a negative number is equal to that of a numerically equal positive number. Then along came Leonard Euler with 'proofs' that his two great contemporaries were mistaken. Ever since, all academicians, stuck unimaginatively with their primitive exponential notion of logarithms and evidently mesmerized by Euler's genius and authority, have lamentably taken his inadvertently deceptive judgement and naively clung to it to the detriment of their hallowed principle of unification.

Probably the greatest and most persistent philosophical oversight that ever took place in mathematics was in the adoption of the notion of a logarithmic base because, useful as it is, this incidental characteristic creates streaks of scholastic ossification in many branches of mathematics and science. Euler's confidence in the comprehensiveness of his formulæ is baffling. Perhaps it was due to the continental support for the Leibnizian form of the calculus, to science being then in its infancy, or to Euler's use of periodic complex functions to show that every negative number has an infinity of 'imaginary' logarithms. However, the scene has changed so drastically since then that the mathematical hierarchy should, ere now, have had the wit and vision to create a unified logarithmic theory which, in essence, must surely be sublimely elemental.

The crux of the matter is that, for Euler and all his legions of misguided adherents, $\ln e^x$ and its complex number counterpart are the only 'natural' logarithms. The

'Naperian' logarithm is, indeed, the Queen of natural logarithms, but her domain is unbounded and teeming with active citizens, Euler's infinity of logarithms inhabiting just one province, possibly the major one, but a province nevertheless.

D'Alembert and Bernoulli made no mistake, but Euler made a whopper; a statement which, if I were a member of the mathematical elite, I would, in all conscience and as may be fitting, stoutly refute or graciously acknowledge, integrity being to me, as I trust it is to them, more golden than silence. All students will surely agree with this sentiment. Indeed they may well challenge me to justify my statements by producing and effectively using a 'non-existent' d'Alembert–Bernoulli real number logarithm of a negative number. It may be that the Arts Faculty of the Open University could now do this if they were approached.

The second matter to raise questions in my mind concerned the various efforts to provide a sound basis for the calculus which finally gave us the remarkably fertile limit concept. Without such a concept it is highly unlikely that mathematicians could have expressed numbers other than integers and rationals in a meaningful and useful way; nor can one wholly escape from the concept when involved with geometrical tangents and rates of change. However I cannot see why it is necessary to resort to the use of $\delta f(x)/\delta x$ and find the limit as δx tends to 0. Sir Isaac Newton's vanishing increments came in for much criticism; on reflection I sense that his followers were more perspicacious than others thought and that all that was needed was a slight change of emphasis such as in the following definition:

> If $\delta f(x) = f(\delta x) \cdot \delta x$ and if $f(\delta x)$ is reducible to a function of x at δx=0, then that value of $f(\delta x)$ is called the differential derivative of $f(x)$ at x.

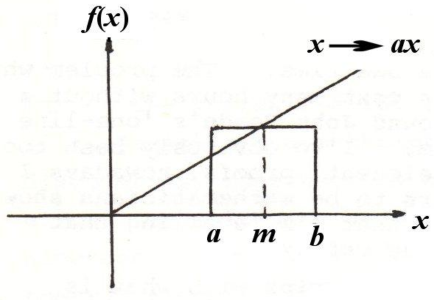
The properties of a differential derivative are deducible from such a definition, thus reversing the usual procedure and rendering many difficult spatial theorems superfluous.

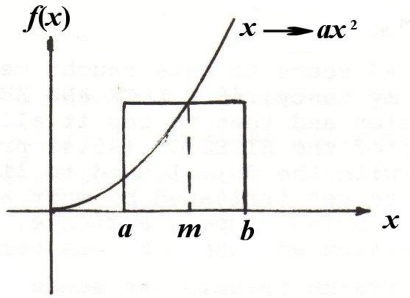
INTEGRATION SID FINCH

Could integration of polynomials be expressed as the sum of N areas? For $\int_a^b x \longmapsto ax;\ x\in\mathbb{R}$ there would be one area. This would be the interval (a,b) times $f(x)$ at the midpoint of a,b. For $\int_a^b x \longmapsto ax^2$ two areas, from the following equation:

$$\int_a^b x \longmapsto x^2 = \left(\frac{b+a}{2}\right)^2 \times \left((b-a) + \left[\frac{\left(\frac{b-a}{2}\right)^2 \quad (b-a)}{3\left(\frac{b+a}{2}\right)^2} \right] \right)$$

and then an additional area for each additional power of x. (See diagrams on page 5.)





ESSENTIAL KNOWLEDGE JEREMY HUMPHRIES

x = 21 669 693 148 613 788 330 547 979 729 286 307 164 015 202 768 699 465 346 081 691 992 338 845 992 696.

$y = x + 1$.

z = 30 645 573 943 232 956 180 057 972 969 833 245 887 630 954 508 753 693 529 117 371 074 705 767 728 665.

A triangle with sides of length x, y and z is right angled.
DOUGLAS ST PAUL BARNARD says "In such a triangle the difference between one of the acute angles and 45° is so small that if two lines diverging at this angle were extended to the outer confines of measurable space, say 10^{11} light years away, it would be necessary to divide an electron into more equal pieces than there are drops of water on the Earth before one of these pieces could be accommodated

between the lines."

It is easy to find such triples. If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is Pythagorean (such that $x^2 + y^2 = z^2$ then so is

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The example given is obtained by successive transformations on (3,4,5). Each triple is more nearly isosceles than its predecessor.

Mathematicians always strive to confuse their audiences; where there is no confusion there is no prestige.

LInderholm.

ONE LINE PROOFS BOB MARGOLIS

M500 49 seems to have caught me at a bad time. The problem which took my fancy, 49.2 LOCK AND KEY has cost many hours without a solution and then to cap it all I found John Reade's one-line proof of the BISECTOR (45.4) problem. I've obviously been too long with the OU; I used to like 'elegant' proofs, nowadays I seem to get irritated by what appears to be mathematicians showing off to a less expert audience. Not that I'm levelling that accusation at John - he was merely the catalyst.

Trying to help our students come to grips with what is acceptable as a proof is a hard problem - so hard that we either avoid it or try to teach by example (as in M231). M202 is wildly inconsistent - the Halmos based units demand a nit-picking-dot-all-the-i.s approach, and yet almost anything goes for Minsky. I suspect that the root cause of all the trouble is that we-what-is-supposed-to-know can't really say what constitutes a proof either. To take a very extreme example, I was once in the position of trying (in an exam) to prove something about a group of odd order which boiled down to showing that it must have at least one non-trivial normal subgroup. In a mixture of despair and hope of fooling/impressing the examiners I finished with

> by a well-known theorem of Feit and Thompson the result follows.

The proof of the 'well-known theorem' occupies 255 pages of close print: I have no idea whether or not, in the course of their proof, Feit and Thompson used the result I was trying to prove. (I passed!)

I'm beginning to lose sight of the message I'm trying to convey. Has anybody got any ideas about proof beyond saying you'll recognise a good one when you see it?

THINK OF A NUMBER EDDIE KENT

In these days of calculators no-one should have any trouble in carrying out your instructions in this attempt to elicit that "wondrous admiration" promised in the old puzzle books.

Have your friend begin by adding together her birth day (day of the month), her month of birth and the last two digits of her year of birth. This will give that important number, her birth number. Then she must

write down her birth day using two digits
repeat these digits to make a four digit number
add to this number her month of birth
multiply by 66
add her *birth number*
multiply by 3
take away her birth number
divide by 2.

GAUSS VI

JEREMY GRAY

During the years 1796-1801, and intermittently thereafter until 1814, Gauss kept a mathematical diary. In it he recorded his discoveries, observations and ideas in a cryptic Latin that is almost impossible to understand. The diary runs to only nineteen octavo pages, brief notices on number theory, function theory, algebra, and astronomy frequently jostle for attention on the same sheet. They stand, 146 in all, dated, but without proofs, an awesome testimony to Gauss's creative powers. After his death in 1856, the notebook was kept by his family until Paul Stackel heard of it in the summer of 1898 and persuaded them to let it be published the next year. The original nineteen pages then drew forth eighty nine pages of commentary from the editors of Gauss's *Werke* (Klein, Stäckel, and several of the leading mathematicians at Göttingen) which are themselves quite hard going. Here are only a few of the most interesting passages from the *Tagebuch*, starting with the first entry: March 30, 1796, made when Gauss was only eighteen; they are numbered as in the *Werke*. I shall comment as little as possible in this episode that I may quote the more.

1. The principles by which the division of the circle into 17 parts can be made geometrically, discovered. *March* 30, 1796, *Brunswick*.

   This discovery decided Gauss to become a mathematician, as we have already discussed.

2. The quadratic residues are not all numbers less than a given prime - this can be made secure by means of a proof. *April* 8, 1796, *ibid*.

   Here is recorded Gauss's successful first proof of the theorem on quadratic residues (see episodes 3 and 5, and *Disq. Arithmeticæ*, para 131 .) Klein and Bachmann observe that Gauss claimed to have discovered the result by induction from numerous examples in March 1795 but could not prove it until the following year.

4. An extension of the norms of residues to non-prime residues and measures.
   April 29, 1796, *Göttingen*.

   This is the general theorem on quadratic reciprocity, *Disq. Arith*. para 133.

5. Numbers which can be written variously as the sums of two primes.
   May 14, 1796, *Göttingen*.

   This refers to the conjecture Goldbach made in a letter to Euler, 30 June 1742, in which he says that every even number can be written as the sum of two primes. It is still not proved, although the proof may not now be far away, and I hope to return to this in a later episode.

These notes only occupy some of the first page of *Tagebuch* and perhaps give you some idea of the range and profundity of the contents. From now on I shall be more selective.

14\. The sums of factors to infinity = π^2:6 (sum of the numbers). 20 *June*, 1796, *Göttingen*.

This should be read in conjunction with

31\. The number of fractions of which the denominators do not exceed a certain limit compared to the numbers of fractions all of whose numerators or denominators are different, taken to infinity, is 6:π^2 . *No date*.

Of course Gauss is not simply counting these fractions, which form infinite sets, but making estimates of their relative densities - the 'number' of rationals of a given kind. These difficult results might be compared with Euler's $\sum 1/n^2 = \pi^2/6$.

16\. A new proof of the golden theorem all at once from scratch and not at all inelegant.
27 *June*, 1796 ,

The second and much better proof of the 'golden theorem', published in *Disq. Arith*.

para 262.

18. ΕΥΡΗΚΑ! num = Δ + Δ + Δ. 10 *July*, 1796, *Göttingen*.

Εύρηκα, every number is the sum of three triangular numbers. The classification of numbers according to the shapes in which *n* dots can be arranged goes back to the Pythagoreans *c*500BC. Triangular numbers are 1,3,6,10, and so on. This theorem of Gauss was the first significant advance in the subject since Greek times. Subsequently Cauchy showed, in a rare excursion into number theory, that every number is the sum of four squares, five pentagonal numbers, and so on. See Freudenthal's article in the *D.S.B.* on Cauchy.

32. If $\int_0^x dt/\sqrt{(1-t^3)}$ is called $P(x) = z$, and $x = F(z)$ then $F(z) = z - \frac{1}{8}z^4 + \frac{1}{112}z^7 - \frac{1}{1792}z^{10} + \dots$. 9 *September* 1796.

That is, if the integral $\int_0^x \frac{dt}{\sqrt{(1-t^3)}}$ is thought of as defining $z = P(x)$ then the inverse function $x = F(z)$ is given by the series above. Compare this with the integral $\int_0^x \frac{dt}{\sqrt{(1-t^2)}} = z$ where z is again some function of x, in fact $z = \arcsin x$. (The function) sin is very attractive - it is single valued and periodic - whereas arcsin, although it is given by a simple integral is not attractive: it is only defined mod 2 or, as it is described in complex function theory, it is infinitely many valued. But what is Gauss doing with $\int_0^x \frac{dt}{\sqrt{(1-t^3)}}$ and with

50.
$$\left.\begin{aligned} \int \sqrt{\sin x}\,dx &= 2\int \frac{y^2\,dy}{\sqrt{(1-y^4)}} \\ \int \sqrt{\tan x}\,dx &= 2\int \frac{dy}{\sqrt[4]{(1-y^4)}} \\ \int \sqrt{\frac{1}{\sin x}}\,dx &= 2\int dy/\sqrt{(1-y^4)} \end{aligned}\right\} y^2 = \begin{pmatrix} \sin \\ \text{or} \\ \cos \end{pmatrix} x.$$

7 *January* 1797.

51. Begun to make a thorough examination of the dependance of the elastic lemniscate on $\int dx/\sqrt{(1-x^4)}$. 8 *January* 1797.

92 Most elegant things concerning the lemniscate, exceeding all expectations, and by methods which seem to open up a whole new field. *July* 1798.

98. The final value of the arithmetico-geometric mean of 1 and $\sqrt{2}$ is π/ϖ calculated to 11 decimal places, which, having been proved, opens up a whole new field of analysis. 30 *May* 1799, *Brunswick*.

105. The theory of transcendental quantities $\int dx/(1-\alpha x^2)(1-\beta x^2)$ led through to the height of universality. 6 *May* 1800.

110. Our theory now immediately applicable to elliptic transcendents. 5 *June* 1800.

and many other similar entries? I shall answer this question in two episodes time when a deep connection with lattices will also appear. Meanwhile some definitions. The lemniscate has equation $2x^2 = r^2 + r^4$, $2y^2 = r^2 - r^4$. The formula for its arc-length is $s = \int_0^r dr/\sqrt{(1-r^4)}$; the arc-length of the ellipse is $\int \sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}d\phi$, whence the name elliptic integral; and $\varpi = 2\int_0^1 dx/\sqrt{(1-x^4)}$. $\int \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\, d\phi$ can be transformed into $\int d\phi/\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}$.) The arithmetico-geometric (a-g) mean of a and b is obtained as follows. Set $a = a_0$, $b = b_0$, $a_1 = a_0+b_0/2$, their arithmetic and $b_1 = \sqrt{a_0 b_0}$, their geometric means, and in general $a_{n+1} = (a_n+b_n)/2$, $b_{n+1} = \sqrt{a_n b_n}$. Then $\lim_n a_n = \lim_n b_n$ (why?) = $M(a,b)$, the a-g mean. EXERCISE: Calculate $M(1, \sqrt{2})$ to 11 decimal places, as Gauss did. (For the introduction of the elliptic integral into mathematics see, eg, AM289 C4.) i should conclude this episode with the entries on Astronomy and Probability but these must wait. Let me end, however, with the only entry still to defy interpretation. In its entirety it runs

43. Vicimus GEGAN. 21 *October* 1796, *Brunswick*.

Correction: Gauss IV, G3 Should read area = $\alpha+\beta+\gamma-\pi$.

LONG DIVISION MARION STUBBS

There are occasions when one needs more digits in the solution of a division problem than one's calculator provides. Good old school arithmetic can be used for this job, using even the simplest calculator possessing only the four basic arithmetic functions (+, –, ×, ÷) and one memory. The algorithm below assumes that you or your child has such a machine. It also assumes that the calculator is so primitive that it does not even have a key to change the sign of a number, as calculators like this are common.

Since my inspiration for this program came from a newsletter for Hewlett–Packard highly sophisticated calculators and their owners, I assume that the simplicity of the idea has escaped plenty of other people besides myself, so that it is worth printing. The algorithm is based on the Old School Chant (recently depicted on tv as a Victorian method of teaching arithmetic but it was the way I was taught in the 1940s!) : "We will divide 97 by 67. How many 67s in 97? ONE! 1 x 67 = 67. 67 from 97 = 30. Bring down a 0. How many 67s in 300? FOUR! 4 × 67 = 268. 268 from 300 = 32. Bring down a 0. And so on ad nauseam - and it nearly is ad nauseam with 97/67. But in the end, with patience, $97/67 = 1.\dot{4}4776119402985074626865671641791\dot{0}$ (recurring between the first 4 and the last 0).

Using a primitive calculator for this, with no 'Change Sign' key and no X ↔ Y interchange key, we run into difficulties almost immediately, and resort to 97 – (1 × 67) = 1 × 67 – 97 × –1 , with the –1 stored permanently in our solitary, precious Memory. Otherwise the method is effectively the same as the O.S.C. above.

GLOSSARY

N = the current numerator, which changes (eg 97, 300, 320, ...)
D = the unchanging denominator (eg 67)
INT = the integer part of a displayed number
M = contents of Memory (=–1 in fact).

INITIALISE

0 – 1 =
Store the result in M.
Clear the display

PROGRAM

Step	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
key or number input	N	÷	D	=	WRITE INT	INT	×	D	–	N	×	M	×	10	=	WRITE NEW N	GOTO-STEP-2

NOTES

Step 5 means 'Write down the integer part of the number displayed'. You then enter this integer in step 6 (clearing the display first if your machine needs it).

Step 16 means 'Write down the displayed number which is the new numerator. You will use it next time you reach Step 10'.

Step 17 *does* mean go to Step 2. You already have new N on display so you do not need Step 1 again.

This algorithm assumes that your machine uses algebraic logic, in which (A + B) × C, or C(A + B) is entered and solved as A + B × C=. Some machines use Reverse Polish logic so that the same problem would be entered as CAB+×. This should deal with people who wildly claim that mathematics they happen not to have studied makes furniture dull. I may even one day tell such Reversed Poles what the difference is between merged instructions (quaintly called 'Steps' above) and non-merged. (Clue - if 'non-merged' then Step 14 above would take two Steps, not just one, and the same passim where relevant; whereas if 'merged' it takes one instruction or Step.)

ART LUCY POLAND

From time to time odd derogatory comments about arts courses crop up in M500 (eg Marion's comment in 48.4) and I must admit I chose to do A202 this year as I thought it would be an easy way to finish my degree. How wrong can you be? Although I don't have to think so much as I did on maths courses the amount of work involved is staggering. To take just one week, the Unit on Restoration Europe, the reading amounts to five chapters from one set book, seven from another set book, plus two fairly long articles from the course reader. Not to mention 48 pages of the Unit itself. Unlike most maths courses I've done A202 has no NO TEXT weeks. No due dates are given for assignments, just cut-off dates; and the timetable is tight. For the first TMA the cut-off date is only seven days after the date scheduled for completion of the last relevant unit. Arts courses may make different demands to maths ones but they're certainly not an easy option.

And anyway, isn't maths (some of it at least) also an art?

Incidentally this is an example of the level of arts-degree maths: 'While the supply of food expanded in arithmetical progression (1,2,3,4), population grew in geometrical progression (2,4,6,8)'. A202 Unit 5/6 page 87. Maybe somebody should tell them.

The great advantage about not having to do mathematics this year is that I am far more interested in pursuing it recreationally.

As regards OUSA I am positively against affiliating. OK so the money might be useful but personally I'd rather pay a larger subscription to M500. I am anti-union in any shape or form, including students unions. OUSA's only purpose seems to be one of self-perpetuation - all committee meetings and conferences to arrange what to discuss at the next committee meetings and conferences. It seems they need us more than we need them. M500 has managed without OUSA aid so far. Long may it continue to do so.

M500 3 9 MARION STUBBS

The Legal Deposit Libraries (Bodleian Oxford, University of Cambridge, Edinburgh and Trinity Dublin) now demand copyright deposit of all M500 issues from the first issue of 1977 onwards. Previously only the British Library has legally demanded copyright deposit, and has two complete sets from 29 onwards (the first printed issue).

Unfortunately M500 39, the first issue of 1977, is now totally out of print. Do four members exist who are willing to part with their copies of 39 for the National Archives, please? Send to Marion Stubbs if so.

Obviously M500 has achieved greatness, by the very fact of this demand. The copyright Act 1911 requires deposit of all printed publications in the BL, but they only demand copies for the other Legal Deposit Libraries if the publication is significant enough to warrant it. It would be nice to be able to comply with the backdated demand in full.

I am also trying to supply the BL with a complete set of M500 1-28, of which only one master set now exists. If anyone has odd, unwanted copies from this era please let me know. Eventually the BL will have the only master set, and I will then possess a photocopied set - which is not as good - in the M500 office. I console myself with the thought that this is the price of M500's fame and must bear the sacrifice of my master set bravely!

FACTORIALS EDDIE KENT

Here is a pretty problem: Under what conditions can the following equation hold?

$$m! + 1 = n^2 , n \in \mathbb{N}.$$

It was conjectured by H Brocard in 1876 that it is true only if $m \in \{4,5,7\}$. Albert H Beiler in *Recreations in the Theory of Numbers* says that this has been investigated by computers up to m =1020 without finding another solution.

Wilson's Theorem (this is of no help, but interesting for itself) states

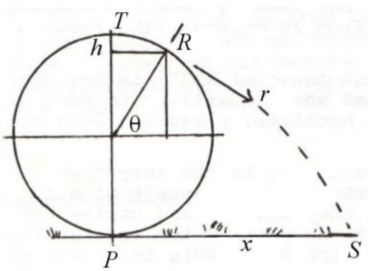
$$n | (n - 1)! + 1 \Leftrightarrow n \text{ prime.}$$

eg: $(13 - 1)! + 1 = 12! + 1 = 479\ 001\ 601 = 13 \times 36846277$.

Joseph S Madachy, in *Mathematics on Vacation* states that there only four integers equal to the sum of the factorials of their digits. Here are three of them: 1! = 1; 2! = 2; 4!+0!+5!+8!+5! = 40585 (provided one accepts the convention that 0!=1. This one was found by Leigh Janes in 1964); and one other, of three digits, which the reader is invited to find. (It should take about ten seconds.)

PROBLEMS CORNER JEREMY HUMPHRIES

First I must make some corrections to my solution to 47.4 SPHERE AND PARTICLE, given in issue 49. That was my attempt to condense Timothy Wilkins's answers. He quickly pointed out that it was not only not what he'd said but also rubbish.



I said that the time for the particle to traverse *TR* is the same as it would take to fall freely a distance *h*; but wouldn't travelling be interesting, were it true? The correct approach is to use vertical and horizontal components of separation velocity

$$\cos\theta\sqrt{2gh} \text{ and } \sin\theta\sqrt{2gh}$$

to calculate the free horizontal travel distance of = 0.716*r*. Adding the horizontal travel distance on the sphere, $r\cos\theta$ gives the value for $x = 1.46r$.

HOWARD PARSONS has sent me a strange fact which younger subscribers may not understand:

£l2.12s.8d = 12128 farthings.

He would like to know if anyone can find anything even more useless.

<u>SOLUTION 48.2 MODULUS</u> *If* $|x|<0.1$ *and* $|y|<0.1$ *what is the probability that* $|x-y|<|xy|$?

I have two different solutions for this - which one is right? Are they both right depending on how the question is interpreted? Can we have some comments?

THURSTON HEATON says: $|xy| \in [0, 0.01)$; $|x-y| \in [0, 0.2)$. Probability that $|x-y| < |xy|$ is the probability that $|x-y| \in [0, 0.01)$, which is $0.01/0.2 =$ <u>0.05</u>.

MICHAEL GREGORY says

$$\text{Prob } |x-y| < |xy| = 4\left(\int_0^{0.1} x\,dx - \int_0^{0.1} \frac{x}{x+1}\,dx\right) / 0.2 \times 0.2 = \underline{0.031}$$

<u>SOLUTION 48.3 SCRAMBLE</u> *Think of a* 4*-figure number, p. Rearrange the digits to make another number, q. Tell me any three of the digits in* $|p-q| = r$ (*including leading zero if necessary*). *What is the probability that I can tell you the* 4*th*?

Nobody answered this which is a pity because you can use real (ie OU) mathematics on it. A couple of people did express puzzlement so here is what it means:

We can apply the natural mapping $f: \mathbb{Z} \to \mathbb{Z}_9$ where $\mathbb{Z}_9$ is the set of residue classes modulo 9. *p* and *q* map to the same element of $\mathbb{Z}_9$, therefore $(p-q)$ is in the kernel of the mapping and so is divisible by 9. So, when I get the three digits, I add them and then subtract the sum from the next highest multiple of 9. This gives the 4th digit. The trick fails when the three digit sum is a multiple of 9; then I don't know if the 4th digit Is 0 or 9. If all the numbers 0,1, ..., 9 are equally likely to be the missing one I am certain to get it right eight, and 50% certain two times out of ten. Therefore the probability that I am right is 90%.

Note that this is theoretical. In practice the failure rate for this party trick is quite high, especially after the pubs close.

<u>SOLUTION 49.1 POETRY</u> *f*: JACK AND JILL → JAJAN ICDLK L

f: *x* → TVADH MHKTC HCWAC THJLE OUEM EEORU JOELL CWCEA RUIDP OLLHW STLTH AAAPE EGRIR EEND.

Find x and name the poet.

x is: The village all declared how much he knew; 'Twas certain he could write, and cipher too.

It's from *The Deserted Village* by Oliver Goldsmith.

The main point of the question is to get the words back - either you know the author or you don't. Marion says it is scarcely feasible to check every book of poetry line by line. However, if you did go searching you must have found something worth seeing even if it wasn't what you were looking for. There are worse ways to pass the time.

Correct answers came from LEON DUNMORE, MICHAEL MASTERS (who also found the author) SIDNEY SILVERSTONE and MARION STUBBS.

<u>SOLUTION 49.2 LOCK AND KEY</u> *A company has* 4 *directors. The safe is to have a certain number of locks, all different, and each director is to hold keys to some of the locks so that any* 3 *directors can open the safe but no* 2 *can. How many locks should the safe have and how many keys should each director have*?

The best solution I got for this said: put six locks on the safe and give each director three keys. This came from LEON DUNMORE, MICHAEL MASTERS, MARION STUBBS and MICHAEL MCAREE, who set it. This is Marion's solution and her reasoning:

DIRECTORS	A	B	C	D
	4	2	1	1
KEYS	5	3	3	2
	6	6	5	4

Triples can open it. ABC,ABD,ACD,BCD. No pair can open it- AB, AC, AD, BC, BD, CD. Each pair must lack at least one key. Any triple is the union of two pairs. Therefore any two pairs must lack different keys. Therefore there must be as many locks as pairs. The arrangement depends only on which key is not allocated to any particular pair. I said: AB not 1, AC not 2, AD not 3, BC not 4, BD not 5, CD not 6 to get my arrangement but the allocations could be different. In fact there are 6! ways of doing it.

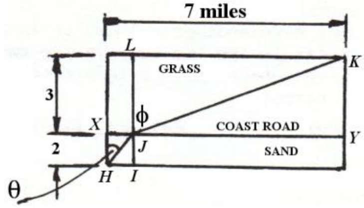
<u>SOLUTION 49.4 BILLIARDS</u> *Pot and Cannon agree to play* 5 *games of billiards. It is equally likely that Cannon will win three games or four games. What is the chance that he will win all five*?

Let *p* be the probability that Cannon wins any particular game. The condition is fulfilled if *p* is 1 or 0 since in either case the probability that Cannon gets 3/5 or 4/5 is zero. MICHAEL MASTERS and MARION STUBBS sent that answer. But there is another possible value for *p*. The binomial distribution gives:

$\text{Prob}(3/5) = \binom{5}{3}p^3(1-p)^2$ $\qquad$ $\text{Prob}(4/5) = \binom{5}{4}p^4(1-p)$

These are equal so $p = 2/3$. Then $\text{Prob}(5/5) = \binom{5}{5}p^5(1-p)^0 = (2/3)^5 = 0.1317$. SIDNEY SILVERSTONE found that one.

<u>SOLUTION 49.5 HORSE RIDING</u> *The lover must get from H to K on horseback in the shortest time, when progress is twice as fast on grass as on sand. What is the optimum length XJ*?



Several people sent quite a lot of work on this. THURSTON

HEATON, L S JOHNSON, SIDNEY SILVERSTONE and MARION STUBBS said one mile. Most of the points they mentioned come in this poem, by Heathcliff.

Set by 'The Minch'[1], seven miles of beach Two miles from sea to shore Where runs the straight coast road, then grass A full league wide, no more.

He takes a mark and rides to it Slowly through swirling sand, The time is short; his line is set By light and skilfull hand[2].

Now at the mark he changes course, Leaving the clogging dust, His sin is now a double sin[3] Compounding speed and lust.

She sees his way across the grass, The sand-stormed way he went And takes his old and dusty line To be a complement[4].

Similar pictures in his rnind[5] That mark no dusty ghost. No transcendental sign at all But just the first mile post[6]!

Notes: 1. The Minch: a sailing vessel; or the setter of this problem.

2. The line of shortest time *HJK* is the path that light would follow from *H* to *K* if *XY* represented the interface between two materials whose refractive indices were in the ratio 1:2.

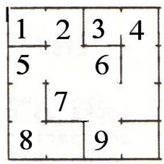
3. By Snell's Law, $\sin\phi = 2\sin\theta$.

4. This verse is a bit of a cheat. In order to get a 'nice' answer the setter must have introduced some relationship between ϕ and θ. One obvious one (true in this case in fact) is: $\phi + \theta = \pi/2$. His way across the grass is the complement of the old and dusty line.

5. If the above is true then the triangles *HIJ* and *JKL* are similar. It follows that $x/2 = 3/(7-x)$ (for $\phi \neq \theta$) and obviously the required solution is $x = 1$,

6. Thus the answer is not irrational - just hit the road one mile along!

PROBLEM 52.1 CROSSNUMBER MICHAEL GREGORY



This is Michael's own puzzle and he says, If you enjoy it keep your table of x^3-y^3 values for a future puzzle, which will be harder.

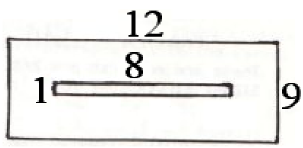
The solutions to all clues are differences between two cubes, (x^3-y^3 : $1 \le y < x \le 10$). None is used more than once.

Integers *a,b* are prime; these and integer *n* may have different values in different clues

ACROSS 1 A triangular number; 3 $a \times b$; 5 3 across + 8 across; 7 Sum of first *n* cubes; 8 $a \times b^2$; 9 $a \times b^3$,

DOWN 2 A Mersenne prime (2^n-1); 4 4 × a palindromic number; 5 a^2; 6 8 across + 1 across + 2 down + 5 down.

PROBLEM 52.2 CARPET



My study carpet measures 12 ft by 9 ft. Pacing back and forth in a symmetrical fashion, trying to think of TMA answers, I have worn a hole 8 ft by 1 ft exactly in the middle.

I have another room which is 10 ft square. How can I make two cuts in the carpet and rearrange the pieces to fit this second room. A cut is any continuous line lying on the carpet

(A continuous line is one which has no gaps. It is not necessarily smooth.)

PROBLEM 52.3 AGE

When my sister is four times as old as she was when I was twice as old as my brother, my brother will be two-thirds as old as I will be. My brother and I are teenagers; how old was my sister on her last birthday?

PROBLEM 52.4 STICKY SQUARES JOHN HULBERT

John says that this very sticky problem has been going round the office, and is a glorious timewaster.

Find three different integers such that the sum of any two of them is a perfect square and the modulus of the difference of any two of them is a perfect square.

PROBLEM 52.5 OLYMPIAD LXVIII/2 (IMO 1968)

ABCD is a parallelogram, AB = a, AD = 1, the angle DAB is α and the three angles of the triangle ABD are acute.

Prove that the four circles K_A, K_B, K_C, K_D, each of radius 1 and with centres A,B,C,D respectively, cover the parallelogram if and only if $a \leq \cos \alpha + \sqrt{3} \sin \alpha$.

(*Taken from* Mathematical Spectrum 2/1 *page* 4.)

ON CERTAIN POLYHEDRA LEWIS JOHNSON

Since the subject has been raised intermittently it might be of interest to consider the set of quasi-regular solids known, I believe, as Federow's zonahedra.

Take a sheaf of concurrent rays in 3-space, no three coplanar. By constructing parallelograms on each pair of rays and displacing as necessary a solid is formed whose faces are $F = n(n-1)$ parallelograms with $2F$ edges and $F + 2$ vertices.

If the rays are equal in length the faces are rhombs.

A few examples: Three orthogonal rays (all at right angles to each other) generate a cube.

The four solid diagonals of a cube form a rhombic dodecahedron, (see cover. To produce the drawing I had to make the wretched thing first and then my wife says the sketch doesn't look like the real thing - however it’s the best I can do.) and so on.

The cube has nine planes of symmetry; three parallel to pairs of faces and six through pairs of diagonally opposite edges. A most interesting solid is generated by the nine rays perpendicular to these planes. This is a symmetrical solid whose faces consist of 12 squares, 8 hexagons and six octagons, which is in agreement with $n(n-1) = 72$ rhombic faces since a hexagon consists of three rhombs and an octagon six.

Incidentally this particular solid is very easy to construct and I can supply a recipe.

EDITORIAL

There have been so many references recently to the new large number mentioned by Martin Gardner in his November 1977 *Scientific American* column that I thought I'd better look at it.

It is, as has been reported, to do with graph theory. A graph is a collection of points with lines joining them, or not, as the case may be. If every point is joined to every other you have K_n, the complete graph for that number, n, of points. If there are six points then there is no way of drawing the connecting lines using two colours without completing a mono-coloured triangle This is well known and is the basis for the game of Sim. Six is the smallest number of points for this to happen. Symbolically, $R(3,3) = 6$; If K_6 is two-coloured it forces a monochromatic triangle.

Joining all the corners of a cube gives K_8 with a Euclidean structure. It is possible to two-colour this so that no monochromatic K_4 results lying on a plane. The problem can be generalised to the hypercube with 2^n corners. If $n = 4$ or 5 a two-colouring can be produced that has no planar one-colour K_4. So what is the smallest n that forces a planar monochromatic complete graph of four points?

The answer is not known; the hypothesis is six. However Ronald L Graham of Bell Laboratories has proved that the number cannot exceed $3\uparrow\uparrow\ldots\uparrow 3$. That is the big number everyone is talking about.

The arrows represent powers, in the form $a\uparrow a = a^a$, $a\uparrow\uparrow a = a\uparrow(a\uparrow a) = a^{a^a}$, and so on. Unfortunately I can't put arrows in place of the dots in the last paragraph as there isn't enough room in the universe.

Two words on calculating it. First, $3\uparrow\uparrow 3 = 3^{3^3} = 7625597484987$. This is then the number of exponents In $3\uparrow\uparrow\uparrow 3$.

Secondly, write the number $3\uparrow\uparrow\uparrow\uparrow 3$ and call it Row 1. Row 2 is $3\uparrow\ldots\uparrow 3$ where there are $3\uparrow\uparrow\uparrow\uparrow 3$ arrows between the threes. The number of arrows between the threes in Row 3 is the number represented by Row 2. And so on. Graham's Number is in Row 64.

*   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *

There has been some voting recently and although there is no room in this issue to give full details I will give an extract here and promise you the full and enthralling results soon.

First: the Constitution. Amend rule 2 to allow for the publication of the directory MATES, circulated to M500 members in closed institutions by special request of those named on it. Number voting, 257. For: 231; against: 3; abstain: 23.

Next, an Opinion Poll to determine whether renewing members were in favour of the M500 SOCIETY affiliating with OUSA (the Open University Students Association.) This is in no way binding as no firm offer has been made, but the results were interesting: for affiliation, 93 (36%), against, 62 (24%), don't care, 89 (35%), abstain, 13 (5%). Not a very clear result. There will be a proper vote on all the issues involved in this matter soon.

The last thing is those who voted, as it were, with their feet by not rejoining M500. Several of these, under pressure, gave their reasons. I will not give away any names but $2\frac{1}{3}$ people found it too expensive, $3\frac{5}{6}$ too difficult and $11\frac{5}{6}$ are no longer with the mathematics faculty. (The reason for the fractions is obvious and should indicate that at least one person found M500 too difficult, too expensive and had finished with mathematics. That one enjoyed the weekend though.) Still, 21% found M500 too difficult. Please, everybody, write and tell me that it isn't so.

Eddie Kent.