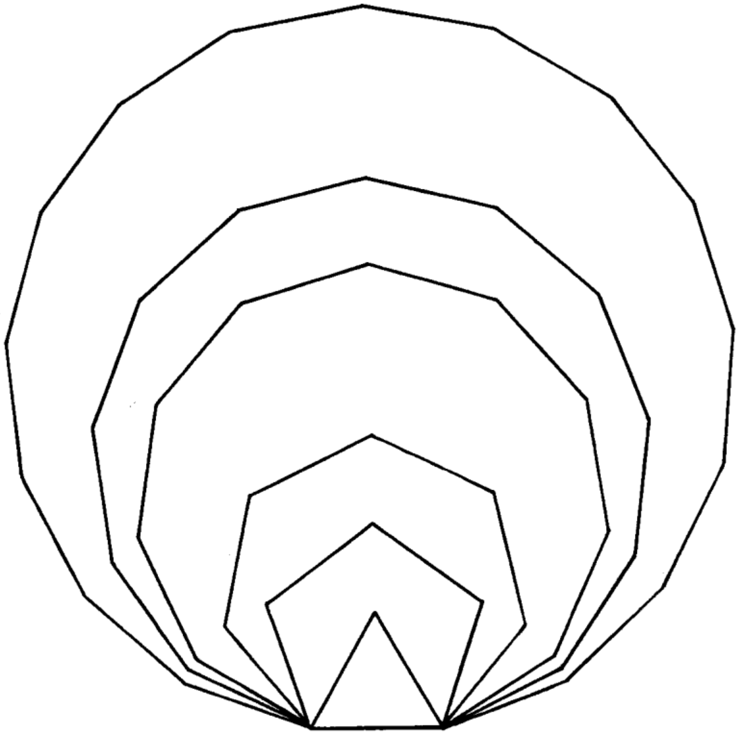

M500 54

M500 is a student-owned, student-operated magazine for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for public discussion of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names, addresses, telephones and details of past and present courses of voluntary members, by means of which private contacts may be made to share 0U and general mathematical interests or to form self-help groups by telephone or correspondence.

MATES is a special list of MOUTHS members who have explicitly volunteered for their MOUTHS details to be distributed to members in closed institutions such as prisons and special hospitals.

PLEASE NOTE NEW EDITORIAL ADDRESS:

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ANYTHING SENT TO ANY OFFICER OF THE SOCIETY WILL BE CONSIDERED FOR POSSIBLE PUBLICATION IN THE MAGAZINE UNLESS OTHERWISE SPECIFIED.

The cover design is by Alan Gurr and is called "Prime Polygons", constructed with ruler, compasses and cosec tables.

MAKING GOLD

BOB PHILLIPS

The mathematics in this short article is strictly limited to simple arithmetic. What is intriguing is the speculation that may arise from the nature of the rather arbitrary physical constraints here conjured. --- There is also an Ulterior Motive!

We consider the equations

1. 200 + 32 = 197 +35 80 + 16 = 79 + 17
2. 202 + 32 = 197 +37 80 + 16 = 79 + 17.

Nice!

As mathematicians (sic) this is as far as most of us like to get involved in common or garden arithmetic.

Now we consider a physical model - the notation should be familiar to anyone who has any knowledge of science.

1. $\genfrac{}{}{0pt}{}{200}{80}\mathrm{Hg} + \genfrac{}{}{0pt}{}{32}{16}\mathrm{S} \rightarrow \genfrac{}{}{0pt}{}{197}{79}\mathrm{Au} + \genfrac{}{}{0pt}{}{35}{17}\mathrm{Cl}\uparrow$
   mercury sulphur gold chlorine
2. $\genfrac{}{}{0pt}{}{202}{80}\mathrm{Hg} + \genfrac{}{}{0pt}{}{32}{16}\mathrm{S} \rightarrow \genfrac{}{}{0pt}{}{197}{79}\mathrm{Au} + \genfrac{}{}{0pt}{}{37}{17}\mathrm{Cl}\uparrow$
   mercury sulphur gold chlorine

Just to refresh our memories, numbers *x* written in the fashion $\genfrac{}{}{0pt}{}{x}{y}$ represent 'atomic weights' of the elements in question - for *y* mercury the isotopes are the two most common - that with atomic weight 200 representing 23.1% and that of 202 representing 29.8% of an average natural sample. For chlorine the figures are 35 = 75.5% and 37 = 2 4.5%. Numbers *x*' written $\genfrac{}{}{0pt}{}{y}{x}$ represent atomic numbers.

THE PROBLEM for the reader is to find similar numbers (answers please if you find any) $x, x', y, y', (z, z', \alpha, \alpha', \beta, \beta', \gamma, \gamma', \ldots$ optional) such that

3. $x + y\ (+\ \alpha + \beta + \ldots) = 197\ (+\ z + y + \ldots)$ $x' + y'\ (+\ \alpha' + \beta + \ldots) = 79\ (+\ z' + y' + \ldots)$.

THE DIFFICULTY is essentially that whereas for the lighter elements $x = 2x'$, for the heavier elements such as gold $x > 2x'$. ie there is an increasing excess of neutrons over protons in the nucleus.

THE SPECULATION - Since mercury and sulphur have been considered the traditional ingredients for making gold since Egyptian Times (with a lot of writers mentioning common salt - sodium chloride - which would dissociate at the temperature used, and presumably aid the extraction of chlorine) we are left to wonder why they should have selected these particular elements in the absence of the theoretical knowledge available to us today unless they could actually do the trick 3000 odd years ago.

THE RATIONALE is that transmutation of elements involving no spare bits and pieces (ie radiation) may take place at quite moderate energy levels ; and the production of minute quantities of gold in mercury vapour lamps was reported in the 1920s.

THE ULTERIOR MOTIVE is to bring to the attention of the academic community the fact that the subject of 'low energy' as opposed to 'nuclear style energy' transmutation is already part of graduate curricula in France, the home of a M. Kervran whose work Biological Transmutations published by Crosby Lockwood '72 is available in English —

and — as they say on Tomorrow's World —

"We don't want the Frogs to have all the fun do we?"

HALF WAY THERE ...

SOME REFLECTIONS ON OU MATHEMATICS BY JOHN HAMPTON

It all started for me in 1975. I had spent the last seven years working in computer science in a new university. Before this I had been involved in industrial computing and applied statistics for many years. I had had a highly successful career, but it had been marred by persistent ill-health. After yet another period away from work it had been suggested that to aid recovery I should train as a mathematician. Sound advice? I had no idea. On the one hand I had no formal background in mathematics and had failed O-level abysmally. On the other, I knew a good deal about computing and had acquired an elementary professional qualification in statistics. Despite some misgivings the thought was appealing so I decided to give it a try. Since full time study was impossible the OU seemed to offer an ideal solution. I joined up for M100.

Three full credits mathematics courses later I have half my BA. I am thoroughly enjoying it and there is certainly no question of turning back. What of the last three years? What of the three to come?

M100 was a voyage of discovery with only two landmarks in sight. Seeking new horizons I did not dwell on these: instead I explored ones entirely new to me. An enthralling and deeply rewarding experience - how I hope that M101 will serve future students as well. Next I found M201 exceptionally well integrated, progressive in difficulty and offering quantities of gold in something for everyone taking it. An excellent course which, whilst it remains available, must be recommended to any student having a serious interest in mathematics Last year M202 proved a difficult and demanding challenge but it has changed mathematics for me into beauty beyond words. Before I simply enjoyed it. now I love it.

This year I am attempting M231 with M331. In planning my first five credits I have deliberately avoided computing and statistics so choices have been somewhat difficult. In 1980 I intend to round off my BA with S101 and in 1981 the long promised Galois Theory course.

The message for others in all this? Do not be discouraged by preconceived notions of difficulty. Continue your mathematical studies, discover incredible intellectual delights, and encourage your friends to join M500.

INTEGRATION - LEBESGUE ∨ RIEMANN JOHN READE

On reading M500 Special Issue 1977 again I felt that some answer should be given to questions raised by Allan Solomon in his article on M331 *Integration and Normed Spaces*. What is the difference between Lebesgue Integration and Riemann Integration? Why have two kinds of integration? Particularly when is it clear that one supercedes the other, thereby apparently making it redundant?

There is no doubt in my mind, though all may not agree, that Lebesgue Integration is harder than Riemann. It's more complicated, more sophisticated, more 'advanced' and requiring a higher degree of 'mathematical maturity' to understand it. What confuses the issue is that Riemann Integration is itself quite difficult, but it is a difficulty of a different order. Riemann is 'elementary but complicated' whilst Lebesgue is 'advanced and still complicated'.

Historically the Riemann integral was designed (about 1840) to put Newton and Leibniz's calculus on a rigorous footing. Riemann's definition makes precise the idea of the definite integral being the limit of a sum, in symbols

$$\int_a^b f(x)dx = \lim_{\delta x \to 0} \sum f(x)\delta x.$$

The connection with the indefinite integral of *f*, that is, the 'pre-derivative' of *f* defined as any function *F* whose derivative *F* ' is equal to *f*, is made by the Fundamental Theorem of Calculus which says that the most general indefinite integral of *f* is

$$F(x) = \int_a^x f(t)dt + \text{constant}$$

Also that the definite integral of *f* is given by

$$\int_a^b f(t)dt = F(b) - F(a)$$

where *F* is any indefinite integral of *f*. Riemann's definition is quite adequate to give a rigorous proof of the Fundamental Theorem of Calculus, under the single and natural assumption that *f* is continuous. This is all one needs to be able to cope with the problems that arise in elementary applications of Calculus.

The inadequacies of the Riemann Integral only became apparent in the present century, with the development of the great abstract theories of Analysis, mainly by Banach and Hilbert in the 1920s. The key ideas are 'norm' and 'inner product' (see M331, M201). Also the idea of 'completeness' which originally goes back to Cauchy (1825) who observed that any sequence of numbers (x_n) for which

$$|x_m| - |x_n| \to 0 \text{ as } m, n \to \infty$$

(such a sequence is now called a Cauchy sequence) always has a limit. Function spaces with this completeness property are called Banach spaces (if normed) or Hilbert spaces (if there is an inner product).

Now the trouble is that Riemann integrable functions do not form a Banach space when normed by

$$\| f \|_1 = \int_a^b |f(x)| dx$$

or a Hilbert space when normed by

$$\| f \|_2 = \left(\int_a^b (f(x))^2\, dx\right)^{½}.$$

However Lebesgue (in 1902) had constructed a more general theory of integration (originally motivated by his researches into Fourier Series and Integrals) which turned out to give enough integrable functions to make a Banach space for $\| \ \|_1$, and a Hilbert space for $\| \ \|_2$. It is because of the fundamental importance of Banach spaces and Hilbert spaces in modern analysis (now called Functional Analysis) that Lebesgue integration is invariably used today.

Lebesgue's original definition of the integral $\int_a^b f(x)dx$ was as the 'measure' of the plane set of points lying under the graph of *f*. This necessitated setting up the Theory of Measure of plane sets, which is a very sophisticated subject, and a source of great pain and frustration to all who study it, at least at their first few attempts. In 1918 an American mathematician, Daniell, came up with a new definition of the Lebesgue integral which avoided the prior construction of Measure Theory. Daniell's idea was to regard the integral as a linear functional, defined in the obvious way for 'step functions' and extended in a natural way to the more complicated classes of functions via the notion of convergence. Measure Theory is still needed, however, because when abstracted to a theory of general mass distributions it was found to be the ideal setting for Probability Theory (Kolmogorov, 1933).

The main reason for retaining Riemann Integration is educational. It is adequate for elementary calculus, such as is encountered in Spivak. Lebesgue Integration is far more sophisticated, since it relies on either Measure Theory, or the idea of Convergence of Functions and Extension of Linear Functionals (M201 again), and is only needed when one comes to do Functional Analysis or Probability Theory.

It is said that some peasants have a method of multiplication which involves multiplication and division by 2 only. For instance 38 × 43 is calculated by dividing 38 progressively by 2, ignoring any remainder at the same time doubling 43:

38	19	9	4	2	1	0
	0	1	1	0	0	1 (remainder)
43	86	172	344	688	1376	
	86 +	172	+		1376	= <u>1634</u>.

The sum of the multiples of 43 corresponding to odd numbers in the first row gives the required result. Each odd number leads to a one in the remainder row and the remainders 100110 represent 38 in the binary scale so we have $38 \times 43 = (1 \times 2^5 + 0\times 2^4 + 0\times 2^3 + 1\times 2^2 + 1\times 2^1 + 0\times 2^0)\times 43 = (32 + 4 + 2) \times 43 = 1376 + 172 + 86 = 1634$.

Michael Gregory.

AD INFINITUM BRIAN STEWART

If the number of numbers in an infinite but countable set is Aleph-nought (writtten as $\aleph_0$) ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$), and the number of reals (infinite and uncountable) is) $\aleph_1$ where do the null sets of Lebesgue integration come in?

A null set is one which can be contained in a union of intervals the total length of which is arbitrarily small. Any countable set (such as $\mathbb{Q}$) is null - you cover them with intervals of lengths

$$\frac{\varepsilon}{2}, \frac{\varepsilon}{2^2}, \frac{\varepsilon}{2^3}, \ldots$$

the total length of which is ε which can be as small as you like.

The fun starts with *uncountable* null sets. One example is Cantor's ternary set (see M331), another is the set of numbers in [0, l] with no zero in their decimal expansion - as follows:

1. Eliminate the first tenth of the interval (representing numbers of the form $0.0a_1a_2a_3\ldots$) . This leaves 9/10.
2. Eliminate the first tenth of each of the nine intervals left (numbers of the form $0.b_10b_2b_3\ldots$) leaving 81/100.
3. Now repeat, eliminate numbers of the form $0.c_1c_20c_3c_4, \ldots$, &c.

At each stage you have 9^n intervals left each of length $l/10^n$, a total length of $(9/10)^n \to 0$.

So the set is null, but we can show (in a similar way to the method of proving that the reals are uncountable) that it is uncountable.

This means that it has more than $\aleph_0$ members, but surely it has fewer than $\aleph_1$ members, ie fewer than the set of *all* reals in [0, l] .

Another transfinite number then? Help!

PROOF ALAN SLOMSON

Bob Margolis (52 6) is quite right about the difficulty of explaining to students just what counts as an acceptable proof. This problem is particularly acute when it comes to setting examination questions. Suppose you want to ask for a proof of a moderately hard theorem in group theory. You don't want the students to quote an even harder theorem from which the theorem you are interested in follows in a one-line corollary. Nor do you want the students to start from the definition of a group and re-establish all the group theory needed for the problem.

This problem arises everywhere and not just in group theory. How do you tell students exactly what they can assume and what has to be proved? After pondering this question for many years I have come to the conclusion that the only acceptable form for an examination question of this kind is:

Give the following proof of the XYZ Theorem: "Suppose .... QED".

"THE WORLD OF MATHEMATICS" TOM DALE

In the editorial of issue 49 you refer (rather disparagingly, I thought!) to *The World o£ Mathematics* edited by J R Newman. Funnily enough, I have been perusing this book over the last year or two and have found it one of the most fascinating books I have come across. If I hadn't had to go into hospital for a spell I would have written before now in response to the suggestion by Norman Lees on page 3 of issue 48.

Actually 'book' should read 'books' as it is in four volumes; and I don't actually own a set but nobody else seems to borrow it from the local library! The book, first published in Great Britain in 1960 by Allen and Unwin, is subtitled ‘A small library of the literature of mathematics from A'h-mosé the Scribe to Albert Einstein, presented with commentaries and notes by James R Newman’.

There are 26 parts, each dealing with some aspect cf mathematics and varying from one to twenty two sections. Part one - General Survey - consists of the reproduction in full of a book by Philip E B Jourdain on *The Nature of Mathematics*. Part two - Historical and Biographical - starts with another book - *The Great Mathematicians* by H WTurnbull, Professor of Mathematics at St Andrews. This is followed by *The Rhind Papyrus* written by Newman himself; and then *Archimedes* by Plutarch, Vitruvius and Tzetzes. Other items in this part include part of *The Geometry* by René Descartes in the original French (with a translation!); *The Analyst* by Bishop Berkeley (1734) ; and articles on Kepler, Newton, Gauss, Cayley, Silvester and Ramanujan.

Part three is on Arithmetic and Counting (includes articles by Isaac Newton and Richard Dedekind), Part four is on Mathematics of Space and Motion (eg The Seven Bridges of Königsberg by Euler; Topology by Courant and Robbins; Geometrical Axioms by Helmholtz). Part five - Mathematics and the Physical World - includes writings by such famous names as Galileo, Daniel Bernouilli, Sir William Bragg, Heisenberg and Schrödinger. Intriguing articles on *The Soap Bubble*, by C Vernon Boys; *On Being the Right Size* and *Mathematics of Natural Selection* by G B S Haldane.

It would take too long to list all the parts but a few more examples will show the diversity of this anthology. For instance - *Statistics of Deadly Quarrels*; *Mathematics of a Lady Tasting Tea* by Sir Ronald A Fisher; *The Vice of Gambling and the Virtue of Insurance* by G B Shaw; *Mathematical Analysis of Logic* by George Boole; *Can a Machine Think* by A M Turing. Later parts include Mathematics - in Warfare; in Literature and Music; and finally Amusements, Puzzles, Fancies, eg *Assorted Paradoxes* by Augustus De Morgan and *Mathematics for Golfers* by Stephen Leacock.

All told there are some 2500 pages - surely enough to keep anyone going between the exams and the start of the new session!

One item in Turnbull's article on *The Great Mathematicians* referred to above relates to Krysia Broda's contribution in M500 47 1: The Pell Equation. I shall have more to say on this subject next month.

GAUSS VIII JEREMY GRAY

The study of Gauss's diary is a demanding task - the cryptic Latin doesn't help, but the chief difficulty is the combination of range and profundity which makes so much of it dramatically unfamiliar to any one reader. Since writing episode six I have learned some things myself I wish I'd known at the time. Entries 14 and 31, for example, relate to the following charming result. Consider the integer lattice points on a Cartesian grid with integer coordinates; (m,n) is said to be visible from the origin iff the line joining (m,n) to (0,0) passes through no integer lattice point in between. So (2,3) is visible from the origin, but (91,104) isn't because (7,8) gets in the way. We can ask how many points in the square with corners $(\pm N,\pm N)$ are visible from the origin as a proportion of the total $(2N+1)^2$ and, better still, what is the limit of this ratio as $N \to \infty$. The answer is $6 : \pi^2$. A simple proof is given in Apostol: *Introduction to Analytic Number Theory* - I find that rather surprising and attractive, and I hope you do.

However, it is my purpose in this episode to explain what was going on with all those integrals and elliptic transcendents. Recall from the calculus that doing integration is difficult (!) and it helps to know the answer in advance. For instance

$$F(n) \int_0^\theta \frac{dx}{\sqrt{(1-x^2)}}$$

is difficult unless you know the substitution $x = \sin y$, $dx = \cos y dx$, which reduces $F(\theta)$ to $\int_0^{x=\theta} dy = y = \Big|_0^{x=\theta}$ or, writing $y = \arcsin x$, $F(\theta) = \arcsin\theta$. This works because we know all about sine; but no trick works for

$$G(\theta) = \int_0^\theta \frac{dx}{\sqrt{(1-x^4)}},$$

so, what to do? First answer, proposed around the turn of the century by Legendre, following much work by Fagnano, Euler and others is to find other useful expressions for $G(\theta)$, relating, say, $G(\theta)$, $G(\phi)$ and $G(\theta+\phi)$, or expressing $G(\theta)$ as the solution to a differential equation. Second answer, proposed by Gauss, is to realise that $G(\theta)$ is the wrong function to study. It corresponds to arc sin, which is unattractive, whereas the inverse function to arc sin is sin, and is very attractive. So if $G(\theta) = z$ the function to study is $\theta = G^{-1}(z)$. It was Gauss's genius both to recognise this essential simplification of the problem and to carry it out triumphantly.

Of course Gauss wasn't the first to study these integrals, which had been introduced into mathematics by Johann Bernoulli in 1698 (see AM289 4). It was known for instance that a formula analogous to sin 2*x* = 2 sin *x* cos *x* = 2 sin *x* (l–sin^2x)½ existed, as did a formula analogous to sin (*x*+*y*) = sin*x*cos*y* + sin*y*cos*x* = sin *x* (l–sin^2y)½ + sin *y* (l–sin2*x*)½. The point of replacing cos by (l–sin^2)½ is two-fold: only sines appear in the formula and only in algebraic combinations. No new functions and no transcendental expressions (other than sin itself) are involved; and the same was true of the arc length of the leminiscate.

If

$$s(u) = \int_0^u \frac{dt}{\sqrt{(1-t^4)}}$$

then $s(u) + s(v) = s(r)$, where Euler had showed

$$r = \frac{u(1-v^4)^{1/2} + v(1-u^4)^{1/2}}{1+u^2v^2}\ .$$

(cf sin $x = u$, sin $y = v$ and sin(x+y) = u(l–v^2)½ + v(l–u^2)½.)

Gauss's idea was to invert $s = s(u)$, thereby obtaining a function of *s* which he called the sinus leminiscaticus and denoted sl, was made to work by a bold extension of the variable off the real line into the complex plane. This was entirely new; the study of complex functions was not to begin, elsewhere until Cauchy (around 1821). The method of study of such functions is to look for points where they become zero or infinite. Away from such points they have locally valid power series expressions of the form $y = a_0 + a_1x + a_2x^2 + \ldots$. Near a 'zero of order *n*' the series begins

$$y = b_nx^n + b_{n+1}x^{n+1} + \ldots ,$$

near an *n*th order pole (= infinity) one has

$$y = c_{-n}x^{-n} + c_{-n+1}x^{-n+1} + \ldots .$$

The integral of a function of a complex variable taken around a closed curve in the complex plane depends only on the poles enclosed by the curve, and indeed only on the term c^{-1} multiplying x^{-1}. This remarkable result means that for the purposes of integration one curve can be deformed into another provided in so doing it is not pushed across a pole. Now, all these results concerning complex functions were speedily discovered by Gauss, who didn't bother to publish them (he did give a topological proof of the fundamental theorem of algebra in which functions were considered in the complex plane, in 1799, and described his views on complex functions in a letter to Bessel of 1811). It was these techniques which enabled Gauss to find the infinite series expressions for sl.

As was mentioned in episode 2, Gauss decided to do mathematics when his method of studying the division of the circle into n equal parts led to the first successful construction of the regular 17-gon. He was led to this via an algebraic study of the relationship of sin nx to sin x, and it was natural for him to study the analogous question: what is the relationship of sl nx to sl x? When n = 2 the duplication formula quoted above (due to Fagnago 1718) is of degree 4. For the division of x into n equal parts the formula is of degree n^2, and we may now witness the remarkable perception of Gauss. The n solutions (mod 2π) of the equation relating sin nx and sin x can be displayed either on a circle as angles $x + 2k\pi/n$, $k = 0, \dots, n-1$ or along the real line, repeating every 2π units. To display the n^2 solutions for sl what better than n^2 points in a lattice, precisely a parallelogram lattice? The basic tile carries $n \times n$ points in a regular array, and is itself repeated regularly across the complex plane. In this way sl is obtained as a function from $\mathbb{C} \to \mathbb{C}$ which is not singly but *doubly* periodic. Corresponding to the identification of the points 0 and 2π which takes you from the real line to the circle is an identification of opposite sides of the tile which takes you from the complex plane to—the torus. A periodic function $\mathbb{R}$ goes over to a function on the circle; a doubly periodic function on $\mathbb{C}$ goes over to a function on the torus. eg $\sin(x + 2\pi) = \sin x$; $\mathrm{sl}(x + 2\varpi) = \mathrm{sl}\, x$; $\mathrm{sl}(x + 2\varpi \mathrm{l}) = \mathrm{sl}(x)$.

If this account has done anything it has, I hope, illuminated the extract (51) on the lemniscatic integral. What about the "whole new field" of extract 92, and the "height of universality" of number 105? The integrand $(1-t^4)^{-½}$ is a rather special quartic term with roots ± 1, $\pm i$. The general quartic $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ might present a rather different analysis. Gauss discovered that it did not. The inverse function to the integral $\int_0^x dx/(\text{quartic})^{½}$ was again doubly periodic, the shape of the lattice depending on the so-called cross-ratio of the roots of the quartic. This quantity, $(a_1-a_4)(a_2-a_3)/(a_1-a_3)(a_2-a_4)$ if the roots are a_1, a_2, a_3, a_4 is invariant under the map $z \mapsto \frac{az+b}{cz+d}$, or $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z = \frac{z_1}{z_2}$ and a, b, c, d are any complex numbers. Now let w_1 and w_2 be any two independent periods of a doubly periodic function, so w_1 and w_2 define two sides of a parallelogram with a vertex at the origin. This lattice, we say in an earlier episode, is determined by the ratio w_1/w_2 (which cannot be purely real) and is not changed by maps

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where $ad - bc = 1$; a, b, c, d integers. The occurence of lattices in number theory and complex function theory strikes deep into both subjects, and is one of the reasons for finding Gauss's mathematics so deeply unifying. It has remarkable consequences, as I intend to discuss in a later episode.

The height of universality Gauss referred to was his discovery of the complete generality of the treatment of quartics and doubly periodic functions, or as they came to be called, elliptic functions. In the late 1820s the mathematical world was fascinated by a tremendous competition between two mathematicians to (re)discover elliptic functions. Jacobi, the first German Jew to benefit from a relaxation of the anti-semitic laws of the time, and Abel (he of abelian groups) crowded the pages of Crelle's new journal with their new discoveries: Abel's were perhaps the more general. Both men said they had been stimulated by a part in §335 of the *Disquisitiones Arithmeticæ* in which the leminiscate was mentioned. The aged but still active Legendre praised both men for so transcending his own investigations; Gauss merely remarked that Abel had got about one third of the way (letter to Bessel 1828) , and incidentally never read his copy of Abel's memoir on the unsolvability of the quintic.

But I must end with a consoling misunderstanding. The quantity $z = w_1/w_2$ determines the elliptic function (we may insist $\mathrm{Im}(z) > 0$) so the space moduli, as the parameters z are called, is $H^+ = \{z \in \mathbb{C}: \mathrm{Im}(z) > 0\}$. The map $z \longmapsto (az + b)/(cz + d)$ preserves H^+ when a,b,c,d are real, so it is natural to study function f such that $f(z) = f(Az)$ where A is a 2×2 matrix of real integers and determinant one. Gauss did this and left in his notes a diagram of the canonical region of H^+ upon which these functions f takes each value once and only once: it is bounded by the circle centre 0 and radius ½ and the two verticals $\mathrm{Re}(z) = \pm½$. Schering, in the first edition of Gauss's *Werke* (1863), failed to understand the master and reproduced it only as a doodle - the second edition corrected the error and thereby reached the profundity of Gauss two generations before!

LETTERS

From Jim Ezechiel (M101): You have asked for comments on the proposal that M500 should affiliate to OUSA. For what they are worth here are my views (for which I claim no copyright).

My opinion of OUSA and its usefulness is strongly coloured by the fact that it is predominently dependent on the grant from the OU. This gives a specious appearence of being able to represent the views of all the students of the OU. In fact this is less true than it is of similar bodies. Since there is no financial commitment to the association it is impossible to judge how many students are positive members of it but to judge by comments at the tutor-counselling sessions I attend and the pleas recently issued by the local branch for more active participation I judge that the percentage must be pretty low.

There may be a need for a students association and I would be very willing to join one that made OUSA's charitable activities its main concern. On the other hand, so long as it depends for its existence on official funds I shall be of the opinion that those funds could be much better spent (eg by a great improvement in the tutor-counselling arrangements for second and third level students). And while this situation continues I shall discourage any action by M500, or others, which might add to the already high air of self-importance with which OUSA appears to regard itself.

My recommendation is, therefore, that we should decline the invitation to affiliate.

______

From Tom Dale: I was extremely annoyed to find from M500 50 that M352 has been delayed yet again - it seems to have been 'coming' for years. And yet they find time to replace M231, which seems to get more praise than any other course and has a text book admired by nearly all who took the course.

By the way, editorial 52: Probably those who find M500 difficult are those, like me, who haven't done M202! (At least I assume that's where all the strange notation comes from).

I enclose a copy made from an old maths book I have. Perhaps some of these curves would make cover designs? Why, I wonder, the 'Cissoid of Diodes' and the 'Witch of Agnesi'?

______

From Michael Masters: In one of the recent editions of M500 mention was made of including advertisements and tnerefore wonder if you can include the following

CARSHALTON BRANCH OUSA <u>FORTHCOMING EVENTS</u> ...Wine and cheese party Wednesday 13th September Passed the exams party Friday 24th November ... .

______

From Sidney Silverstone: In response to some queries about this year' s Mathematics Weekend; it is possible to book for tuition only. The cost is £14.80 per head which also includes tea/coffee and biscuits. Address queries to me ... .

From John Millar: You asked for views on association with OUSA: my general views were given in a letter printed in August *Sesame*. I will add a few reasons.

Sidney Silverstone, in his letter, refers to expenses three times, once by OU, once by OUSA and once by the University. He could replace each one by OU student. Perhaps without his jaunts we could have had M352 next year.

He uses the word 'democratic'. OUSA campaigned for an opt out option but did even better. They have got conscription as confirmed by the editor's comment on my letter.

O'Hare's comment is asinine. If there are only fourteen of us 'dissidents' what objection does he have to opt in. I don't think he would pass MDT241, I doubt if he can add.

OUSA woos NUS, I personally prefer to use computers rather than smash them.

Mandatory grants are wrong. OU students study for recreational purposes, that's me and I suspect most M500 members, or to get more money, that's teachers. In either case we are not overcharged. The student studying M334 so that he may better relieve the troubles of suffering humanity I will introduce to a gentleman from Missouri.

I wonder how much of that 120 000 is spent on OUSA officials and their trips.

Lets keep away from it. Anyone wishing to be involved with OUSA can do so through their normal facilities and they can keep their 95 pieces of silver or 190 pieces of cupro-nickel.

Sidney Silverstone mentions, casually, that OUSA might request our names and registration numbers. Well at no time do you have permission to do that regarding me. There might be a few more with the same ideas so the constitutional changes will become complicated.

From Ken May: I observe that in the M500 53 editorial you are showing an interest in Black Holes. As shattering as your information is what is even more shattering is that way out in the cosmos the physical laws as we understand them may be completely dethroned. However, on reading your article I thought members of M500 may wish to know that The Institute Of Mathematics And Its Applications is holding a symposium on black holes at Chelsea College London on 5th January 1979. The first lecture will begin at 10am with registration from 930am and the symposium will finish with tea at 415pm. Fees including VAT are

Symposium fee including coffee, lunch and tea	£10.50
Bona fide research students recommended by their Professor	5.00
Late registration fee for payments received after 1st December 1978	1.00

The purpose of the symposium is to introduce and explain black holes and will cover the subject in both its observational and theoretical contexts. Ne previous knowledge of black holes is necessary and the meeting is designed for students, graduate mathematicians and physicists. The symposium will be opened by Professor Sir Hermann Bondi KCB FRS FIMA (Department of Energy) who will take the Chair for the morning session. Speakers and sessions include:

What are Black Holes	Dr D W Sciama, Oxford.
Observational Aspects	Dr M Rees, I of A, Cambridge.
Theoretical Aspects	Dr B F Schutz, UC Cardiff.
Quantum Aspects	Dr P C W Davies, King's, London.

For members who require a background knowledge of the Cosmos the following book, which was advertised in Volume 13 (December 1977) of the IMA Bulletin, may be of interest:

Mathematical Cosmology (*An Introduction*) by Peter T Landsberg and David A Evans; OUP; £6.95.

This book gives readers a quantitative understanding of some important aspects of general relativistic cosmology, starting only with school mathematics and Newtonian ideas in physics. It is concerned mainly with various models that have been proposed to explain the large-scale structure of the universe: the Friedmann models, steady-state models, and model universes involving pressure. Other topics include Special Relativity, red shift magnitude relations, radio source counts, Olbers' paradox, horizons, and cosmological coincidences.

From Jeremy Humphries: ∃ a few items in 53 which may confusing to anyone misguided enough to read my section.

SOLUTION 48.3 line 7 should say 'equally likely'.
SOLUTION 51.2 The printed question is the correct one - it should be the incorrect one.
PROBLEM 53.1 line 9 1+1 should be 1–1.
Page 53 17 $g\colon \mathbb{R} \to \mathbb{R}^+$ should be $g\colon \mathbb{R} \to \mathbb{R}_0^+$.

DEFINITIONS NICK FRASER

Below is a supplementary list to the Mathematical Handbook. Please file it and have it ready for usage.

Augmented Matrix	Additional section for Scalextric.
Base Vectors	Rather common vectors.
Commutative Binary Operation	Position 91 *Kama Sutraa*.
Conjugate	What married people do.
Contrapositive	Laxative.
Inner Product	Baby in the womb.
Homomorphism	What consenting adults do.
Mean Proportional Parts	Well I'm alright!
Permutation	Pools coupon.
Simple Proposition	Will you?
Standard Deviation	See Homomorphism.

P R 0 B L EMS - JEREMY HUMPHRIES

I am very short of problems.

Having seen the disparaging editorial remark about James Newman (M500 49) the BBC felt moved to feature him in *Enquire Within* on a recent Friday evening (R4, 5 40pm).

A listener wrote to ask "Is a GOGOL the largest number anyone can think of?"

Well, of course to be precise we must say that it isn't. A Gogol is a Russian novelist (1809-1852) best known in England for his masterpiece *Dead Souls* (1837; English translation 1887), a picaresque romance satirizing the provincial Russian society of the day. What Sir is thinking of is a GOOGOL as the BBC man was quick to point out. He went on "The answer is no since somebody has thought of a larger one. A googol is 10100 which is large but is not infinite." (Yes, he said that.) "The best place to find out about the googol, and the googolplex which is very much larger but still not infinite" (curses - will none of these numbers be infinite?) "is in *Mathematics and the Imagination* by Kasner and Newman."

There followed quite a good account of what K and N say. Really the BBC are making an effort these days - World chess champ and all - and I take back some of the things I've said about them.

SOLUTION 48,3 SCRAMBLE *Think of a four figure number = p, rearrange the digits = q. Tell me any three digits of* $|p-q|$. *What is the probability that I can tell you the missing digit*?

I think that I now have the correct answer. We know that if the missing digit is 0 or 9 I cannot tell what it is. As STEVE AINLEY said all digits are not equiprobable so the answer may not be 0.9.

When I beqan to have another look at this I decided to consider numbers from 0000 to 9999 rather than 1000 to 9999 and to allow the null scramble ABCD $\rightarrow$ ABCD. That was for completeness; it did not alter the general pattern of the results but did of course affect actual values. The frequencies of occurrence of the digits 0-9 in $|p-q|$ for p = 0000 to 9999 are

0 -	279408	probability	0.29105
1 -	79632		.08295
2 -	76896		.08010
3 -	74160		.07725
4 -	71424		.07440
5 -	68688		.07155
6 -	65952		.06870
7 -	63216		.06585
8 -	60480		.06300
9 -	120144		.12515
Total	$960000 = 4 \times 41 \times 10^4$,		1.00000

Note the constant increment for numbers 1-8 inclusive. This happens with 2-figure and 3-figure scramble also. Therefore all scrambles have this property. (Proof by aesthetics.)

I haven't looked at 5-scramble - even curiosity has time limits. 4-scramble took 200 hours on the HP67. 5-scramble on the HP9821, which is ten times faster, would take 1250 hours and all the real computers I can get at send back invoices.

Now for the strategy. If the missing digit is 0-8 I know it. If it's 0 or 9 it's probably 0, therefore I choose 0. Therefore I get it wrong only if it's a 9. Probability of a 9 is 0.12515. Therefore probability that I get it right is 1 – 0.12515 = 0.87485.

In practice nobody chooses p < 1000 and nobody uses the null scramble. the frequncies then become: 0 - 217016; 1 - 72246; constant increment to 8 of –2688; 9 - 10820. Total = 108280. Probability of 9 = 0.13077 ... . Probability of success = 1 – 0.13077... = 0.86922... .

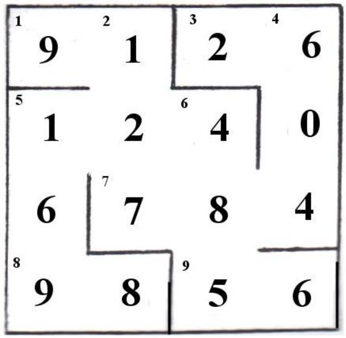
Is there any mathematics in all this?

SOLUTION 52.1 CROSSNUMBER

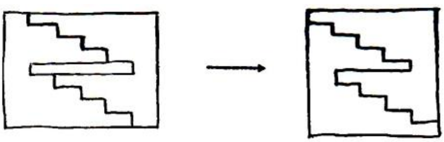
CAROLINE CABLE and ANGUS MACDONALD solved MICHAEL GREGORY'S easy crossnumber. His difficult one is PROBLEM 54.1.

(*For those who can't be bothered to look back here are the clues again*:

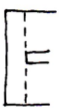
ACROSS 1. *A triangular number*;3. *a*×*b*; 5. 3 *across* + 8 *across*; 7. *Sum of first n cubes*; 8. $a\times b^2$; 9. $a \times b^3$. *DOWN* 2. A *Mersenne prime* (2^n–l); 4. 4 × *a palindromic number*; 5. a^2; 6. 8 *across* + 1 *across* + 2 *down* + 5 *down*.)



SOLUTION 52.2 CARPET *A carpet measures* 12’ *by* 9’ *and has a hole* 8’ *by* 1’ *symmetrically lengthways in the* middle. *Make two cuts in the carpet so that the resulting pieces can be rearranged to fit a* 10’ *square room*.



Nobody sent a valid solution to this though STEVE AINLEY said he had seen it in a book by Dudeney. Nevertheless I did get works of great ingenuity requiring, I'm sure, far more mental effort than that needed to produce the correct answer. The common misconception was that a cut can sever an edge of the carpet without crossing that edge

eg  and  . In each case the dotted line is two cuts, not one.

(*Stop Press: Sidney Silverstone sent a valid solution to this and to Crossnumber*.)

SOLUTION 52.3 AGE *When my siater is four times as old as she was when I was twice as old as my brother my brother will be two thirds as old as I will be. My brother and I are teenagers; how old was my sister on her last birthday*

My sister was eight years old according to STEVE AINLEY, CAROLINE CABLE, ANGUS MACDONALD and MICHAEL MASTERS. And SIDNEY SILVERSTONE

SOLUTION 52.4 STICKY SQUARES *Find three different integers such that the sum of any two of them is a perfect square and the modulus of the difference of any two of them is a perfect square.*

I have nothing complete yet on this problem so I'll hold it for a future issue.

SOLUTION 52.5 OLYMPIAD LXVII#2 ABCD *is a parallelogram*, AB = *a*, AD = 1, *the angle* DAB *is* α *and the three angles of the triangle* ABD *are acute. Prove that the four circles* K_A, K_B, K_C, K_D, *each of radius* 1 *and with centres* A,B,C,D *respectively, cover the parallelogram iff* $a \le \cos\alpha + \sqrt{3}\sin\alpha$.

This is a problem which Eddie set and no one has sent an answer so I'll have a go at it.

The figure will be symmetric about BD so we need only look at the triangle ABD.

First draw AD length 1. Construct perpendiculars (to one side) at A and D. In △ABD the angles A and D are acute; therefore B lies between these perpendiculars. Draw the circles K_A and $K_D$to intersect at P. Now it is necessary that K_B cover P. Draw the arc radius 1 centre P to cut the perpendiculars at Q and R and continue it to D. Note that APR is a straight line and ∠RAD = 60°. AP = PR = 1.

We see that arc QR and line RD are bounds for B since for any α=DAB, AB cannot cross these lines.

Now not only is it necessary that K_B cover P, it is also sufficient. That is, if K_B covers P it also covers any other point of ABD which is not covered by K_A or K_D. Simple to show - do a drawing to convince yourself.

Therefore if $\alpha > 60 = \alpha_1$, maximum $a = \alpha_1$ is given when $B = B_1$ is on QR. Then length $AB_1 = a_1 = 2\cos\alpha_1 - 60) = 2(\cos\alpha_1 \cos 60 + \sin\alpha_1 \sin 60) = \cos\alpha_1 + \sqrt{3}\sin\alpha_1$.

If $\alpha_2 < 60$ then B_2 is on RD for maximum a_2. We see that $AB_2 = a_2 < AX$ where X is on the arc RD, and. $AX = 2\cos(60 - \alpha_2)$. Hence

$$\alpha_2 < 2(\cos 60 \cos\alpha_2 + \sin 60 \sin\alpha_2) = \cos\alpha_2 + \sqrt{3}\sin\alpha_2.$$

Q as they say ED. Scrappy and not rigorous I suppose, but easy to clean up.

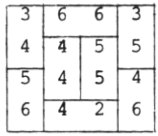
PROBLEM 54.1 CROSSNUMBER MICHAEL GREGORY

The solutions to all clues are differences between two cubes, ($x^3 - y^3$; $1 \le y < x \le 15$). Some of these differences are also realizations of the function $f(n) = A + Bn + Cn(n-1)$ where *A*, *B* and *C* are to be found. Integers *a*,*b*,*c* are prime and may have different values in different clues. All solutions and intermediate numbers are positive integers.

ACROSS. 1. $\sum_{n=1}^{5} F(n)$. 3. $f(0)+f(1) = a\times b$. 5. $f(2)$. 7. $f(13)$. 9. $a\times b^4$. 11. 3 × 13-across. 13. $a \times b^2$. 14. $\sum_{n=1}^{6} n^2$ 15. $f(10) + f(11)$.

DOWN. 1. $f(f(0)+2)$. 2. $f(l)+f(2) = a \times b^3$. 4. $f(3)$. 6. $x^3 - y^3$ for two sets (x,y). 8. $f(f(0)) + f(f(0) - 1)$. 10. $\sum_{n=3}^{8} f(n) = ab^2c^3$. 11. $f(l)$. 12. $2\times f(0)^2$.

PROBLEM 54.2 MAGIC DOMINOES JOHN HULBERT Take eight dominoes from the standard pack and form them into a magic square. Who can find the most? How many are there?



an example

PROBLEM 54.3 NUTS *Taken from A Greek Anthology Book XIV written about AD 400; and sent in by DON HARPER.*

The walnut tree was loaded with many nuts, but now someone has stripped it. But what does he say? Parthenopea had from me the fifth part of the nuts, to Philinna fell the eighth part, Aganippe had the fourth and Orithyia rejoices in the seventh, while Eurynome plucked the sixth part of the nuts, and Muss qot nine times nine from me. The remaining seven you will find still attached to the branches. How many nuts were there on the tree?

PROBLEM 54.4 CONE#I RICHARD AHRENS

Consider a circular cone with a semi vertical angle arcsin¼. ie if cut down the side and flattened out it will be a quadrant of a circle. Imagine the cone sliced by a plane parallel to this line along which we are cutting. The plane will intersect the cone in a parabola. What curve will we get when we flatten the cone into a quarter circle?

PROBLEM 54.5 CONE#II RA

The curve in 54.4 has two points of inflexion. Give a geometric description of the points which become points of inflexion when the cone is flattened.

EDITORIAL

A while ago we had people writing in M500 on the use of mathematics in fiction - particularly, of course, science-fiction. Many years ago I was hooked into this genre, but I grew away from it as it regressed into that ghost oriented sub species which required not even the modicum of hard thought needed for the ‘science’ of its parent.

I was interested, therefore, when someone gave me *Ringworld* by Lar Niven, Gollancz 1972, which promised on the blurb: "... the return to classical hard-science fiction of the kind popular in the Golden Age ..." - Frederick Pohl; and "... dizzying mathematics ..." - The *Observer*

For the science I cannot speak having seen none so far and not being sure I would recognise any if I saw it. But let me quote some of the mathematics :

> *CHAPTER* 5
>
> *There are singularities in the mathematics of hyperspace. One such singularity surrounds every sufficiently large mass in the Einsteinian universe. Outside of these singularities, ships can travel faster than light. Inside, they disappear if they try it.*

I assume here that he is talking of the sphere of convergence about the singularity, which is a point; but I wonder how big the units are. Imagine a converging space-ship.

> *CHAPTER* 6
>
> *... The ring is clearly rotating about its primary at 770 miles per second, a velocity hiqh enough to compensate for the pull of gravity from the primary, and to provide an additional acceleration of 9.94 metres per second.*

Oh well: Back to *Star Wars*.

* * * * * * * * * * *

There might be some subscribers who have not yet studied the inside front cover. They will not know the new editorial address. I now live at

20 Statham Grove, Stoke Newington, iondon N16. 01 249 6836

but my friendly postman has promised not to lose anything sent to the old address.

I hope nothing has gone astray in the move. If you feel that something of yours might have disappeared write and tell me. Ready for 55 is a review of *Problems and Proofs* by P R Baxandall and others (see M201 Weekend). Well it's not exactly a review, just a note by Graham Flegg which he has given me permission to use. I was going to do a review myself but I seem to have (temporarily) mislaid the book.

Also there will be Alan Slomson shooting down my Breakthru article; more from John Hulbert on Rational Sequencies; a book review by Tony Brooks; Infinity by G Marriott; and something on ladders for which I hope to find a signature soon.

Next week I'm going to get organised!

Eddie Kent.