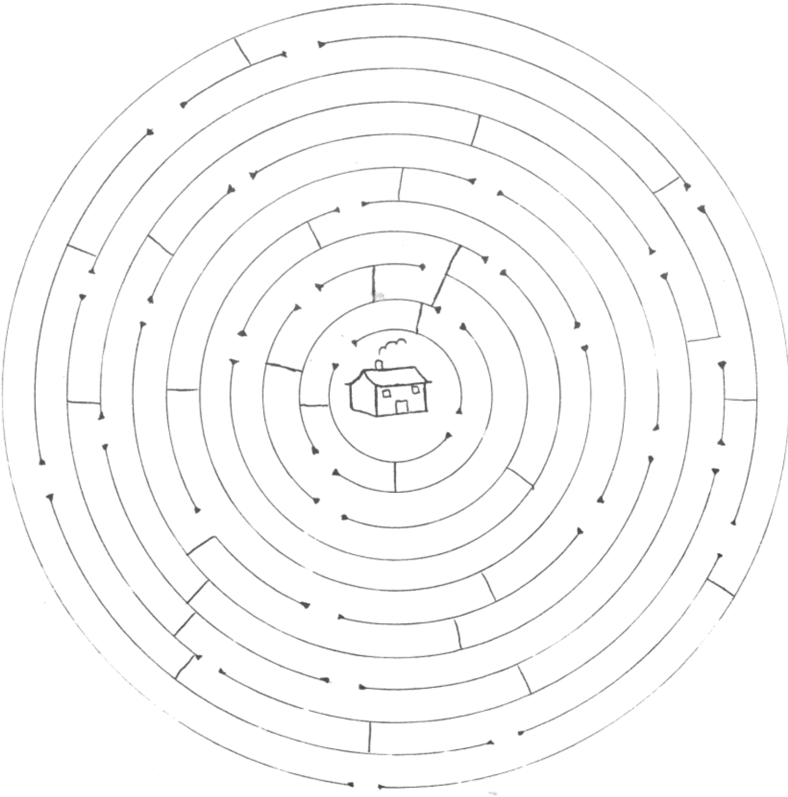

M500 56

M500 is a publication of The M500 Society for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for publication of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names, addresses, telephones and details of past and present courses of voluntary members, by means of which private contacts may be made to share OU and general mathematical interests or to form self-help groups by telephone or correspondence.

MATES is a special list of MOUTHS members who have explicitly volunteered for their MOUTHS data to be distributed to members in closed institutions such as prisons and special hospitals.

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The cover design is an original maze by Ann Wigmore.

The letter from Professor Halmos, on page 4 of this issue, refers to the cover of M500 55, published October 1978.

SOME POTENTIAL CONCEPTS OF INFINITY AND SPACE AND THEIR POSSIBLE INTERRELATIONSHIP

SID FINCH

For mathematics to be logical it needs to be an expression of some phenomenon.

Many paradoxes which now exist would be avoided by not using the term 'infinity' in mathematics. Infinity belongs to the realms of metaphysics and, as such, is not logical where the expression of phenomena is concerned.

Para mathematical models which have no known relationship to phenomena would only become mathematical were some relationship to be discovered. It is suggested that the difference between par mathematics and mathematics is similar to that between science and science fiction.

A point is the smallest perceivable mass, which may be called a quantum.

A straight line is the minimum number of points that can be maximised between two points. If there are, for example, n points then there are $(n–1)$ space intersections as a consequence.

A plane has $(xn).(yn)$ points, with $x(n–1).y(n–1)$ connected interspaces.

Number notation is only mathematics between intervals. It has both an upper and a lower limit beyond which 'mapping' cannot be defined. Just as, for example, to divide a number by zero cannot be defined.

It is suggested that mathematicians might calculate a finite number, defined by the expression: quantum/mass of our universe, of which all phenomena will constitute a subset.

If there is to be a 'breakthrough' of our present boundaries, this will increase accordingly within our new perception.

Whilst the effect of space upon matter is perceivable (e.g. a ray of light bends near a large mass), space per se is not perceivable, for it has no subdivision that may be perceived. However, space does have an overall influence. Thus, as space and matter come into contact then the local space 'intensifies' within the matter and its associated gravitational field.

The impact of the space, and its localised containment in the matter results in a loss of energy, and hence a wave motion within the space and the matter is generated.

The 'intensified' space in the matter, and its associated gravitational field would, it is suggested, exhibit an ultraviolet shift within its spectrum, whilst the infrared shift would occur due to its 'diminished' space.

Further, it may be possible to define (or describe) the limits of our Universe by the absence of space.

Additionally, it is tentatively postulated that matter and space may be akin to reason and emotion.

In conclusion, these largely intuitive concepts originate from the opinion of the writer that there has never existed a concept of infinity within the history of mathematics that has been either logical or rigorously defined.

A POLYNOMIAL GENERATING ALL THE PRIME NUMBERS CONCLUDED

ALAN SLOMSON

Let us begin by recalling briefly what we said last time. We observed that the set, Pr, of prime numbers is *recursive* because there is an algorithm for deciding whether an arbitrary natural number is prime. It follows that Pr is *recursively enumerable*, that is,

$$\Pr = (f(0), f(1), f(2), \ldots, f(n), \ldots\}$$

where *f* is a function whose values can be computed by means of an algorithm. (For example, we could define *f* by

$$f(n) = \begin{cases} n, & \text{if } n \text{ is prime} \\ 2, & \text{if } n \text{ is not prime} \end{cases}.$$

Since there is an algorithm for determining which alternative holds there is an algorithm for calculating the values of *f*.) The work of Davis, Putnam, Robinson and Matijasevič outlined in Part I now enables us to deduce that Pr is in fact a *Diophantine set*. This means that there is a polynomial P(k, x, y, z ,...) with one numerical parameter k and variables x, y, z ,... such that for each natural number k

$$k \text{ is a prime number} \Leftrightarrow \exists x \exists y \exists z \, (P(k, x, y, z, \ldots) = 0) \qquad (4)$$

where the quantifiers ∃x, ∃y, etc mean "there exists a positive integer x" etc.

From the polynomial P(k, x, y, z, ...) it is very easy to manufacture a polynomial whose positive values coincide with the prime numbers. Let Q (t, x, y, z, ...) be the polynomial given by

$$Q(t, x, y, z, \ldots) = t - (t + 1)\,(P(t, x, y, z, \ldots))^2 .$$

If k is a prime number then by (4) there are positive integers $x_0 < y_0/z_0$ such that $P(k,x_0,y_0,z_0, \ldots) = 0$, and hence

$$Q(k, x_0, y_0, z_0, \ldots) = k.$$

Thus every prime number is a positive value of the polynomial Q (t, x, y, z, ...). Conversely suppose $Q(t_0, x_0, y_0, z_0, \ldots) = k > 0$. From the definition of Q (t, x, y, z, ...) it follows that if $P(t_0, x_0, y_0, z_0, \ldots) \neq 0$ then $Q(t_0, x_0, y_0, z_0, \ldots)$ is negative. Hence we must have

$$P(t_0, x_0, y_0, z_0, \ldots) = 0 \qquad (5)$$

whence

$$Q(t_0, x_0, y_0, z_0, \ldots) = t_0 = k. \qquad (6)$$

From (5) and (6) it follows that

$$\exists x \exists y \exists z (P(k, x, y, z, \ldots) = 0)$$

and so by (4), k is a prime number. This completes the proof that the positive values of the polynomial Q (t, x, y, z,...) are all the prime numbers, and only the prime numbers.

There is a well known theorem that if a polynomial takes only prime numbers as its values, then the polynomial must be constant. So the polynomial Q(t, x, y, z,...) must also have negative values which are not primes.

Notice we can repeat this procedure for any recursively enumerable set. So that there are, for example, polynomials whose positive values coincide with all the perfect numbers, or all factorials (numbers of the form *n*!) and so on. The work of Davis, Putnam, Robinson and Matijasevič not only proves the existence of such polynomials but also provides a recipe for constructing them. In 1971 Matijasevič showed that there was a polynomial of degree 21 and with 21 unknowns which generates the prime numbers. Since then several mathematicians have tried to find simpler prime generating polynomials.

The simplicity of a polynomial can be measured in more than one way. The polynomial of lowest degree that it is known generates the primes has 42 unknowns and has degree 5. The smallest number of unknowns so far discovered for a prime generating polynomial is 12. The article from the *Times Higher Education Supplement* that Eddie Kent referred to was reporting the work of another four mathematicians James P Jones, Daihachiro Sato, Hideo Wada and Douglas Wiens, who have been working on this problem and who published their results in 1976. The polynomial Eddie quoted from the story in the *Times Higher Education Supplement* is the simplest prime generating polynomial measured by the number of symbols needed to express it. The index Eddie marked with an asterix should be 3. Notice also that a left hand bracket is missing in line 4 just before the sum "$n + l + v - y)^2$".

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1. Martin Davis, "Hilbert's Tenth Problem is Undecidable", *The American Mathematical Monthly* Vol. 80 1973, pp 233-269.
2. Martin Davis, Yuri Matijasevič and Julia Robinson, "Diophantine Equations: Positive Aspects of a Negative Solution" in *Mathematical Developments Arising From The Hilbert Problems* edited by Felix Browder, American Mathematical Society 1976 pp323-378.
3. James P Jones, Daihachiro Sato, Hideo Wada and Douglas Wiens, "Diophantine Representation of the Set of Prime Numbers," *The American Mathematical Monthly* Vol. 83 1976, pp449-464.

Reference 1 is a very good account of the details of the proof of the negative solution to Hilbert's Tenth Problem. Although the proof uses a lot of technical ingenuity very little knowledge of mathematics is needed to follow it, specifically basic facts about divisibility of positive integers, and about modular arithmetic. Reference 2 discusses some of the repercussions of the negative solution of the problem. The volume from which it comes also contains a reprint of Hilbert's 1900 lecture in which he listed his problems, and articles discussing the work done in solving them. The article on the Tenth Problem is certainly the easiest to understand.

$$(k+2)(1-(wz+h+j-q)^2-((gk+2g+k+1)(h+j)+h-z)^2-(2n+p+q+z-e)^2$$
$$-(16(k+1)^3(k+2(n+1)^2+1-f^2)^2-(e^3(e+2)(a+1)^2+1-o^2)^2-((a^3-1)y^2$$
$$+1-x^2)^2-(16r^2y^4a^2-1)+1-u^2)^2-(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1$$
$$-(x+cu)^2)^2-(n+l+v-y)^2-((a^2-1)l^2+1-m^2)^2-(ai+k+1-l-i)^2-(p+l$$
$$(a-n-1)+b(2an+2a-n2-2n-2)-m)2-(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)$$
$$-x)^2-(z+pl(a-p)+t(2ap-p^2-1)-pm)^2).$$

This is the polynomial in question – guaranteed !

LETTERS

From Paul R Halmos: Dear Jerry Humphries; My wife liked Roger Stangar's drawing, but we agree that it isn't enough evidence to hang a man on. In any event, you have my permission to use the picture; in return I ask that you send me a copy of the issue of M500 in which it appears.

Sincerely yours,
P R Halmos,
University of California, Santa Barbara,
3 April 1978.

From Sidney Silverstone: Although I am not doing M351 this year by coincidence I bought myself a Rockwell 63R calculator last October. It's a marvellous instrument with a tremendous capability particularly because of the double-nested parentheses. An hour or two spent studying the instruction book and trying some of the various calculations there (particularly with your own data) will be amply rewarded. You will also find it fascinating to calculate various "infinite" sequences and show they converge, e.g.

$$\lim_{n\to\infty} \sqrt[n]{x} = 1.$$

(Use the Y^x and 1/x function keys.) Apart from the ability to calculate numbers such as

$$Y = \frac{\ln\left(\frac{x-z}{z-axb}\right)}{\ln\,(1+t)}$$

it has function keys for 1/x, ln x, e^x, log x, 10^x (antilog), Y^x, √x, sin x, cos x, tan x, arc cos x, arc sin x, arc tan x and x!. It has a fully addressable memory (i.e. you can perform +, –, ×, and ÷ within it) . It can produce the number π at the press of a button, operate in either degrees or radians and automatically convert one to the other. All that for a very reasonable price (£33, MRP £44.95 in my case).

From Henry Jones: One evening towards the end of last month (May) having earlier visited the bungalow which I had designed for my daughter Sheila and her husband Rowan, I began to wonder why the numbers 3, 4 and 5 happened to be so convenient for setting out right angled walls. This problem took just a minute or two to solve by means of arithmetic progressions. But then it occurred to me that since the squares of natural numbers can be equated to arithmetic progressions, then every power of a natural number might be expressed in the same way; and of course I found that this could be done. Thereupon I noticed that whereas the common difference of successive terms was the same, 2, for the squares of all natural numbers, such was not so for any particular power greater than 2. This raised the problem of how to sum the latter and, since it had been so easy to sum powers of 2, a transformation of arithmetic progressions into a 'common' common difference of terms seemed worth trying. The result suggests that Pierre de Fermat's 'marvellous' proof was not by his very difficult method of 'infinite descent' as is generally assumed, but by a simple 'transformation of progressions'.

From Peter Weir: Eddie - an old poem I dug up recently!

ON GOING TO BED WITH M332

Oh, come to bed, M332
Let daytime actions count for nought
Ourselves reveal for what we are
Let nothing stain the mortal bond
Let nature act as nature will.
Come! Let me of your treasures thrill
Your proofs intense and theorems deep
I wish to plunder, ere I sleep.
All through your units will I roam
Before the dawn has chance to steal
This night away: your rigour must
I have before the night is old
Your deepest proof must I unfold
Hide not from me your inner gold
No coyness must you have with me
No secret place I cannot reach
No SAQ I cannot breach:
This is the night, M332,
And twixt the sheets I'll be with you.
Oh, come to bed, M332.

John P W Donne

From Frank Springall: In reply to John Millar's comments about mandatory grants, I not only study for recreational purposes but also in the hope of getting a better job, with obviously more money; and I am not a teacher either.

In any case such a statement about the motivation of such a large number of people (the last figure I saw was over 50000) is meaningless.

Personally I have not got that much money that I can turn down any amount that the government is prepared to give me. It's my taxes after all.

Whether M500 associates (5.30 behind the gasworks) with the OUSA or not I do not really care as long as they do not take over or poke their noses in and try to change anything. To be honest I do not care about the OUSA - they leave me alone and I leave them alone.

I have just done M202 and still find the problems (in M500) difficult. Like Tom Dale I assumed that was where the notation comes from; a lot of it does but it's still no help to me.

What actually drove me to write to you after two years of non-communication with anybody in M500 was your editorial in M500 54. As a confirmed science-fiction nut (anybody want to form, a society I can be contacted via MOUTHS) I think your editorial takes the wrong view.

Larry Niven is not a mathematician, in fact he is rich enough not to have done anything but write. His work has won the Hugo and Nebula awards for science fiction, and the Nobel Prize for physics.

If you want to read books where the science and the maths are correct I would suggest *Lucifer's Hammer* by Larry Niven and Jerry Pournelle. Pournelle worked for NASA and is very well qualified with the science facts that come into the book. Also *Colony* by Ben Bova. (I have an American copy as it has not yet come out in this country but is definitely worth getting when it appears.)

In general science fiction is not written by or for mathematicians or to a lesser extent scientists. Like all fiction, science fiction is storytelling and to that extent it is no different from Enid Blyton or primitive man telling his favourite joke around the campfire. Its purpose is to entertain and no matter how accurate or sophisticated the mathematics, if it does not entertain it fails. Also no matter how poignant the message the author wants to get across.

Sorry this letter is so long but you pushed the wrong button. Whether you want to publish any of this letter is up to you but the section on science fiction is the part I feel strongly about. I hope you can read this (No - Ed) and make any corrections to the spelling etc that you feel are necessary. Maybe my inability with words (my cockney accent can make life confusing at times, half the people don't know who I am talking about the when I pronounce Lebesgue and many other names) is the reason I am attempting to become a mathematician.

PS. Why is it that apart from a cockney accent the way the way people speak is a regional accent whereas I speak bad English? Maybe this is not the case I often wonder, perhaps other members from different parts of the country are told they speak bad English.

PPS. The first phone call from somebody on the MOUTHS list came last Sunday (8th October) after being a member for two years. I was out.

From Tom Dale: Coincidence can be a funny thing. I have been reading *On Growth and Form* by Sir D'Arcy Thompson (a chapter is reproduced in *The World of Mathematics* edited by Newman) and in one chapter there is a lot of discussion on the shape of the cells in a honeycomb. (It is a subject which seems to have appealed to many mathematicians.) The cross-section is hexagonal, but the ends are different. To quote "Kepler had deduced from the space-filling symmetry of the honeycomb that its angles must be those of the rhombic dodecahedron" - and lo and behold issue 52 arrives with a rhombic dodecahedron on the cover! In this solid every plane cuts every other plane at 120° and Maclaurin (who apparently did a lot of work on bee cells) was one of those who determined the obtuse angle of the rhombus to be 109° 28' 16". Perhaps I should make it clear that a honeycomb is in two layers - it is where the layers meet that you find the rhombic shape. The outer ends are, I think, flat or domed. Apparently it was once thought that the rhombic dodecahedron was the solution to the problem of dividing space with a minimum partitional area but Lord Kelvin found "by means of an assemblage of fourteen sided figures, or tetrakaidecahedra, space is filled and homogeneously partitioned" even more efficiently. Fascinating stuff, eh?

GAUSS X

JEREMY GRAY

No sooner were Gauss's interests in astronomy consolidated and the most important of his theoretical discoveries published than he turned to another field of enquiry remarkable for the prodigious calculations it demands: geodesy. The chief problem in geodesy is the determination of the shape of the earth. Distance from the centre determines the effect of gravity upon objects and so, for example, the rate at which a pendulum beats - a matter of importance in navigation and the calculation of longitude. The shape of the earth must also be determined before accurate maps can be constructed. Together with this empirical problem is a more theoretical one: the derivation of maps from the curved surface of the earth to the flat page of an atlas. No map can be an exact scale copy; the choice of what properties are to be preserved is a matter calling for some ingenuity. (Quick proof, due to Lambert, that no map can be perfect. Assume the earth is a sphere. Then angle sum spherical triangle > π = angle sum plane triangle.) Gauss worked on both problems, going surveying himself (and inventing the heliotrope to improve his work!) doing the calculations (which suggested new methods in mathematical statistics) and publishing one of the greatest papers ever written on differential geometry. It is this paper, *Disquisitiones generates circa superficies curvas* (1828), that I intend to discuss because of its subsequent importance for non-Euclidean geometry, so let me briefly note two of Gauss's other observations on geodesy. He discussed whether the earth may be considered spherical (as Legendre had) or must be taken to be a spheroidal and found the hypotheses agreed with observations to three significant figures; and he found that

$$f: \mathbb{R}^2 \to \mathbb{R}^2$$

is angle-preserving ('conformal') if

$$\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0,$$

and formulated analogous conditions on maps between any surfaces.

In the eighteenth century mathematicians had largely studied surfaces by studying the curves that can be drawn upon them. Thus Euler had taken a surface, *S*, as z = f(x,y), where z is the height above the (x,y)-plane, taken P a point on *S*, erected the perpendicular n at P to *S*, and passed planes p through n. Each plane p cuts *S* in a curve C_p. Euler studied the curves that arise in this way; figure 1.

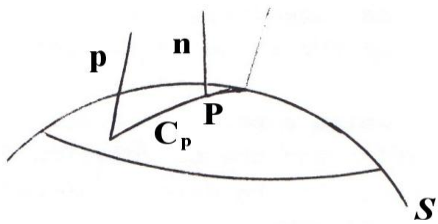


figure 1.

He found that if you measure the curvature of C_p by the reciprocal of the radius of the circle best approximating C_p at P (so sharp bends have high curvature and the straight line has zero curvature) then, in general

1. there are precisely two curves C_1 and C_2 which have maximum and minimum curvature respectively (say k_1 and k_2)
2. these curves meet at right angles at P.

(Proof, due to Meusnier: $z = f(x,y) = ax^2 + bxy + cy^2$ + higher terms, if the (x,y)-plane is the tangent to *S* at P. Indeed, $z = Ax^2 + Cy^2$ + higher terms w.r.t. suitable x,y axes but now *S* is, nearly, a bowl or a saddle obtained by rotating a circular arc about an axis in its plane. For these surfaces, which are parts of tori, the theorem is true.)

k_1 and k_2 are interesting quantities. Their sum $k_1 + k_2$, called the mean curvature, vanishes if the surface can be that of a soap bubble. Their product $k_1k_2 = K$ has a significance only discovered by Gauss. Recall n, the perpendicular to *S* at P. It points in a certain direction, coordinatized, say, by the celestial sphere. Gauss would therefore define a map

$$g : S \to S^2$$

by

P $\rightarrow$ celestial coordinates of n.

The map is highly degenerate on the plane (which is mapped to a point) or the cylinder (mapped to an arc) but interesting on all other surfaces. Draw a little triangle around P, say $\triangle$, and calculate

$$\lim_{\triangle \to P} \frac{\text{area } g(\triangle)}{\text{area } \triangle} = K_p$$

This turns out to be K, the quantity above, nowadays called the total or Gaussian curvature of *S* at P. It can vary from point to point on *S*, or be a constant. For a plane it always 0; for a sphere of radius R it is $1/R^2$.

The chief significance is this. Suppose you have two surfaces *M* and *N*; when can you find a map $f: M \to N$ such that $f(M) \subset N$ is an exact copy of *M* (for which the technical phrase is directly isometric)? The necessary condition is that the Gaussian curvatures are equal at corresponding points. This condition is not sufficient unless the curvatures are also constants. Gauss discovered this remarkable result after many examples had suggested it to him. It is remarkable because k_1 , k_2 and k_1+k_2 are not isometric invariants. Note that the condition . = 0 does not determine the global geometry of *S*, for it is satisfied by both the cylinder and the plane, but the local geometries are identical (roll the cylinder over the plane; i.e., print). Yet again we see that there is no isometry between the sphere and the plane.

Furthermore, K is intrinsic to the surface. To measure it, it is enough to draw a geodesic triangle $\triangle$ around P and calculate (angle sum) $-\pi$ as $\triangle \to P$. In this too K is unlike k_1 and k_2. Since K determines all the local geometry Gauss had shewed that each surface can have its own geometry independent of the surrounding $\mathbb{R}^3$. Sometimes the intrinsic geometry of *S* agrees with the geometry induced from $\mathbb{R}^3$ by defining as geodesies on *S* the curves of shortest length with respect to the idea of distance in $\mathbb{R}^3$ - the sphere is a case in point. But sometimes the surface can carry a new geometry.

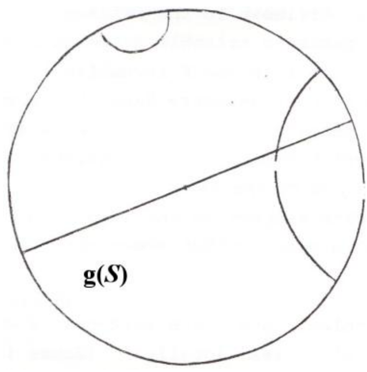
Consider this problem: are there surfaces of constant *negative* curvature $K = -1/R^2$? Yes, locally. (Gauss 1823-27 in an unpublished note.) Yes, globally (Minding 1839, Codazzi 1857, Beltrami 1868) but they cannot be mapped isometrically into $\mathbb{R}^3$ (Hilbert 1900). A conformal map can be given, however, due to Beltrami and Poincare (1880). $g(S) \subset \mathbb{R}^3$ is the interior of the unit disc, geodesics in $g(S)$ are arcs of circles perpendicular to the boundary of the disc (see diagram below).

And what is the geometry of this map which, you will recall is *intrinsic* to $g(S)$ and S (but *not* the geometry induced from $\mathbb{R}^3$)? The answer is *non-Euclidean* geometry. Now it is a curious fact that the work of Lobachevskii and Bolyai had been cast in the following form:

If non-Euclidean geometry exists then it has such and such a trigonometry.

But no conclusive proof of its existence was given. It was not until Riemann in 1854 marked their work with Gauss that a conclusive argument for non-Euclidean geometry was given, thereby shattering over two thousand years of confidence in the truth of Euclid and the *a priori* validity of mathematics.

On non-Euclidean geometry see M101 and M203. For more on the history see AM289 (Unit 8). For differential geometry take M334.



WHAT NEXT? MICHAEL GREGORY

If you are wondering what to do in your spare time after completing your degree here are some ideas and references. Any other ideas?

	What to do	*Outlet*	*Comments*
a	Advanced study perhaps survey and assessment of various authors' work.	OU BPhil or MPhil degrees by dissertation other universities.	Reference (1). See university prospectuses available in larger libraries.
b	Original research under supervisor appointed by university.	PhD by thesis.	As above. And (2). It can be a long haul if done part time.
c	Solving problems in journals (3).	The journal, or private satisfaction.	It can be individual work or collaboration between a group.
d	Problems suggested by books on topics of interest.	Letter or short paper to a relevant journal.	Access to a library specializing in mathematics is useful.
e	Development of an idea in depth or breadth.	Publish as a book.	Sadly, few books make authors rich. For publishers' addresses etc see (5)
f	Compilation of puzzles and invention of games.	Submit to a magazine. Offer to game makers or market yourself.	(6) especially useful.

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"For example" is not proof. - Jewish proverb.

CONTINUED FRACTIONS: HOW TO DO (SOME) THINGS WITH THEM STEVE AINLEY

This note is written in response to Tom Dale's claim of ignorance in M500 55 6.

Notation: $a + \frac{1}{b+\frac{1}{c+\cdots}}$ is written $(a; b, c, \ldots)$. $(a; \dot{b}, c, \dot{d})$ means $(a; b, c, d, b, c, d, \ldots)$

1. to convert one way

$$\tfrac{1}{7}(4 + \sqrt{37}) = \frac{7+\sqrt{37}-3}{7} = 1 + \frac{\sqrt{37}-3}{7}.$$

$$\frac{7}{\sqrt{37}-3} = \frac{7(\sqrt{37}+3)}{37-9=28} = \frac{\sqrt{37}+3}{4} = \frac{8+\sqrt{37}-5}{4} = 2 + \frac{\sqrt{37}-5}{4}.$$

$$\frac{4}{\sqrt{37}-5} = \frac{4(\sqrt{37}+5)}{37-25=12} = \frac{\sqrt{37}+5}{3} = \frac{9+\sqrt{37}-4}{3} = 3 + \frac{\sqrt{37}-4}{3}.$$

$$\frac{3}{\sqrt{37}-4} = \frac{3(\sqrt{37}+4}{37-16=21} = \frac{\sqrt{37}+4}{7} = \frac{7+\sqrt{37}-3}{7} = 1 + \frac{\sqrt{37}-3}{7},$$

which is where we came in. So $^1/_7\,(4 + \sqrt{37}) = (1; \dot{2}, 3, \dot{1})$.

2. to convert the other way

$$x = (\dot{1}; 2, 3, \dot{1}) = 1 + \frac{1}{2+\frac{1}{3+\frac{1}{x}}} = 1 + \frac{1}{2+\frac{x}{3x+1}} = 1 + \frac{3x+1}{7x+2} = \frac{10x+3}{7x+2}$$

so that

$$7x^2 + 2x = 10x + 3; \qquad 7x - 8x - 3 = 0; \qquad x = {}^1/_7\,(4 + \sqrt{37}).$$

(Obviously, since x is positive.)

3. to use a cf to find a 'best' rational approximation

(that is, one such there is none closer without a larger denominator). Here, and generally with cf's, actually) I find it easiest to use simply 'pure' cf's, i.e. those with no integral part; they are represented as $(a,b,c,\ldots)$ with no semicolon. (You can get a pure cf from an impure one either by taking the reciprocal or by subtracting the integral part. The reciprocal of $(a; b , c ,$ ...) is
$(a ,b , c,\ldots)$.)

Take $(1; \dot{2}, 3, \dot{1})$ as example. Use its reciprocal $(\dot{1},2,\dot{3})$ then write

n	F_n	0 1	1 0	approximation	error
1	1	1	1	$^1/_1 = 1.0$	0.30575
2	2	3	2	$^2/_3 = 0.\dot{6}$	02759
3	3	10	7	$^7/_{10} = 0.7$	00575
4	1	13	9	$^9/_{13} = 0.69231$	00195
5	2	36	25	$^{25}/_{26} = 0.69\dot{4}$	00019
…					

The successive terms of the cf go in the F_n column. The

0 1
1 0

is standard. For the numbers below them, the rule is that the number on line n is F_n times the number on line n–1 plus the number on line n–2. e.g. 121 = 3×36+13. The approximations are successive 'convergents' - fractions obtained by ignoring terms of the cf after the nth. They are alternately high and low.

PROBLEMS COMPILED AND EDITED BY JEREMY HUMPHRIES

It is, as Eddie has probably said, my fault that this issue is delayed. I'm afraid I neglected the important work and wasted my time swotting the wrong things for MST282. Still - I think I've scraped through and I'm glad to see the back of it. Didn't much like it. On to the solutions.

SOLUTION 52.4 STICKY SQUARES *Find* 3 *different integers A, B, C such that the sum of any two of them is a perfect square and the modulus of the difference of any two of them is a perfect square.*

A few people have sent 'near misses' for this but only STEVE AINLEY sent a complete solution. JOHN HULBERT sent some work which went part of the way. He wrote:

Let integers be x,y,z with $x > y > z$. $x + y = A^2$; $x + z = B^2$; $y + z = C^2$; $x - y = D^2$; $x - z = E^2$; $y - z = F^2$. Then

$$A^2 = B^2 + F^2 = C^2 + E^2 \qquad (1)$$

$$B^2 = C^2 + D \qquad (2)$$

We note that if one solution can be found, that will generate any number of others. Let us consider only irreducible solutions.

Any prime of the form $4n + 1$ is uniquely expressible as the sum of two squares. If A is the product of two such primes then put

$$A = (a^2 + b^2)(c_2 + d_2) = (ac + bd)^2 + (ad - bc)^2 = (ac - bd)^2 + (ad + be)^2$$
$$= \mathrm{L}^2 + \mathrm{M}^2 = \mathrm{N}^2 + \mathrm{P}^2.$$

We see that A can be put equal to the sum of two squares in (at least) two different ways, since

$$A^2 = (\mathrm{L}^2 + \mathrm{M}^2)(\mathrm{N}^2 + \mathrm{P}^2) = (\mathrm{LN} + \mathrm{MP})^2 + (\mathrm{LP} - \mathrm{MN})^2 = (\mathrm{LN} - \mathrm{MP})^2 + (\mathrm{LP} + \mathrm{MN})^2.$$

This doesn't solve the problem - we still have to satisfy equation (2). I have found five solutions so far, for which the values of A are

925	1105	1394	2146	2210
5×5×37	5×13×17	2×17×41	2×29×37	2 × 5 ×13×l7.

From these you should be able to find the corresponding x,y,z.

Steve wrote: Let integers be A, B, C

1. $$\left.\begin{aligned} A &= m^4 + n^4 \\ B &= 2m^2 n^2 \\ C &= -2m^2 n^2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A + B &= (m^2 + n^2)^2 \\ A + C &= (m^2 - n^2)^2 \\ B + C &= (0)^2 \end{aligned} \qquad \begin{aligned} A - B &= (m^2 - n^2)^2 \\ A - C &= (m^2 + n^2)^2 \\ B - C &= 2mn^2 \end{aligned}$$

e.g., with m = l, n = 2,

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$$\left.\begin{array}{l} A = 17 \\ B = 8 \\ C = -8 \end{array}\right\} \Rightarrow \begin{array}{l} 25,\ 9 \\ 9,\ 25 \\ 0,\ 16 \end{array}$$

2. a) If $A > B > C > 0$, $A + B = x^2$, $A + C = y^2$, $B + C = z^2$, $A – B = l^2$, $A - C = m^2$, $B – C = N^2$

b) $A + B = x^2 = m^2 + z^2$
$A + B = x^2 \quad = n^2 + y^2$
$A – C = \quad m^2 \quad = n^2 \quad + l^2$

The relationship of x, m and n is crucial. Any Pythagorean triple (p,q,r) where $p^2 + q^2 = r^2$ has $p = s^2 + t^2$, $q = s^2 – t^2$ and $r = 2st$. Hence we need

$x = p^1p^2p^3$ $\qquad$ $z = p^1r^2p^3$
$m = p^1q^2p^3$ $\qquad$ $y = r^1p^2p^3$
$n = q^1p^2p^3 = p^1q^2q^3$ $\qquad$ $l = p^1q^2r^3$

c) So we need $\frac{p_1}{q_1} = \frac{p_2}{q_2} \cdot \frac{p_3}{q_3}$.

d) We luckily soon find

$$\frac{1105}{264} = \frac{33^2 + 4^2}{2 \times 33 \times 4} = \frac{65 \times 17}{33 \times 8} = \frac{7^2 + 4^2}{7^2 - 4^2} \times \frac{4^2 + 1^2}{2 \times 4 \times 1}$$

Thus

$$\left.\begin{array}{lll} p_1 = 1105 & p_2 = 65 & p_3 = 17 \\ q_1 = 264 & q_2 = 33 & q_3 = 8 \\ r_1 = 1073 & r_2 = 56 & r_3 = 15 \end{array}\right\} \Rightarrow \begin{array}{ll} x = 1105 \times 65 \times 17 & z = 1105 \times 56 \times 17 \\ m = 1105 \times 33 \times 17 & y = 1073 \times 65 \times 17 \\ n = 264 \times 65 \times 17 & l = 1105 \times 33 \times 15. \end{array}$$

1105 will cancel from these, and we get

A = 733 025 $\qquad$ E = 488 000 $\qquad$ C = 418 304 .

e) Is this the smallest?

CHRIS PILE sent a good ‘near miss’ which was A = –40, B = 104 and C = 185. The only sum/difference which is not a square is A+C which is 12^2 +1. (And another which was A = –10080, B = 10369, C = 10656.)

SOLUTION 54.1 CROSSNUMBER

3	3	5	2	6
3	7	6	3	1
1	2	0	8	9
1	8	9	6	3
9	1	8	6	6

Solutions to MICHAEL GREGORY’S puzzle were sent by STEVE AINLEY, IAN GLASS, N F LEIGH, ANGUS MACD0NALD, SIDNEY SILVERSTONE, FRANK SPRINGALL and ANNE WILLIAMS. Anne says it was great fun and must have been more difficult to devise than to solve.

SOLUTION 54.2 MAGIC DOMINOES *Make magic squares with eight dominoes from the standard pack. The example, given is shown on the right*.

3	6	6	3
4	4	5	5
5	4	5	4
6	4	2	6

STEVE AINLEY says that for a start we can make more by replacing (2,3,4,5,6) by (1,2,3,4,5), (0,1,2,3,4), (6,5,4,3,2), (5,4,3,2,1), (4,3,2,1,0).

SOLUTION 54.3 NUTS *The walnut tree was loaded with many nuts, but now someone has stripped it. Parthenopea had from me the fifth part of the nuts, to Philinna fell the eighth part, Aganippe had the fourth and Orithyia rejoices in the seventh, while Eurynome plucked the sixth part of the nuts, and Muss got nine times nine from me. The remaining seven you will find still attached to the branches. How many nuts were there on the tree*?

I don't know what the answer is to this. Did the old Greek book from which DON HARPER took it give an answer? STEVE AINLEY suggests, not very confidently, that there were $762\,^{6}/_{97}$ nuts, and says that if Muss took 90 instead of 81 there would be 840. IAN GLASS also says 840 if Muss took 90. FRANK SPRINGALL got 7 170 240 and says a lower number may do, but he did not check.

ANNE WILLIAMS interprets the question in a way which gives a neat answer. I think this is probably what the question wanted. She says:

There were 223 nuts. 7 were left, 216 were picked.

Philinna took $^1/_8$ of these = 27	leaving	189
Orithyia took $^1/_7$ of these = 27	leaving	162
Eurynome took $^1/_6$ of these = 27	leaving	135
Parthenopea took $^1/_5$ of these = 27	leaving	108
Aganippe took $^1/_4$ of these = 27	leaving	81
Muss took 9×9.		

Note that the order in which the fractions are taken is immaterial. Muss's share is still $9 \times 9 = \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times$ total picked whence total picked = 216. Total on tree = 223.

Late comments received from ANGUS MACDONALD: "Are you trying to make monkeys out of us?" and SIDNEY SILVERSTONE: "This is driving me nuts - I can't crack it."

(*Ed - There is a solution given*: 1680; *but this is clearly wrong for the quoted problem.* 1680, *together with Frank's* 7170240 *and Anne's correct* 216 are *all examples of numbers from which you can take successively the fractions* $^1/_8, ^1/_7, \ldots, ^1/_2$ *and end up with zero. However,* 216 *is the only one which passes* 81 *on the way*.)

SOLUTIONS 54.4 AND 54.5 CONE I & II

A quadrant of a circle can be rolled up to form a cone of semi-vertical angle arcsin ¼. *Imagine a plane parallel to the 'join' line, which cuts the cone. The cut will be a parabola. What curve does the cut give when the cone is flattened again to a quarter circle?*

The curve has two points of inflection. Give a geometric description of these points.

BOB ESCOLME has sent detailed work on this, which I summarise here

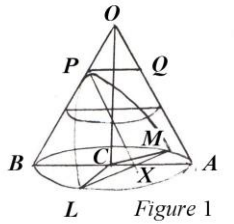


Figure 1

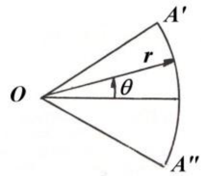


Figure 2

Figure 1 is the assembled cone with cut *LPM*. Figure 2 is the flattened cone, cut along *OA* to make *OA*' and *OA*''. The coordinate system is shown by r,θ.

Let *OP* = *l*/2. Then *XA* = 2.*l*/2 arcsin ½ = *l*/4. We want the locus of the point *M*. *OM* = *r* = *OA*. Let $B\hat{O}M = \theta$.. Then $M\hat{C}B = 4\theta$.

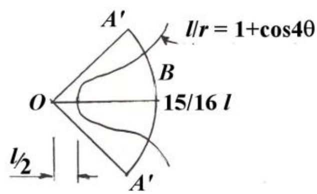
Also $\cos(M\hat{C}B) = -\cos(M\hat{C}A) = -CX/CM = -(CA - XA)/CA = -(1 - (l/4)/(r/4))$.

Thus $l/r = 1 + \cos 4\theta$.

The curve is contained within the lines $\theta = \pm\pi/4$ and is symmetric about $\theta = 0$. It has no asymptotes and two points of inflection. Set $u = 1/r$, then $u = \frac{1}{l}(1 + \cos 4\theta)$.

A necessary condition for inflection is $u = \frac{d^2 u}{d\theta^2} = 0$

(*Differential Calculus*, J Edwards, 1948, e.g.)
whence $1 - \cos 4\theta + 16 \cos 4\theta = 0$; $\cos 4\theta = 1/15$.
Which gives two values for θ6 in $(-\pi/4, \pi/4)$.



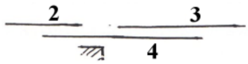
Let x be the focus of the parabola in Figure 1. Then
$MX = 2XP = r/4 \sin 4\theta$ XP = $r - l/2$.

So $\sin 4\theta = 8 - 4l/r$. Also $\cos 4\theta = l/r - 1$ which gives $r = 17l/32$.

Therefore if *l*/2 is the distance from the apex of the cone to the curve which is cut, the points of inflection of the curve are at $\frac{17}{32} l \pm 21.544°$.

SOLUTION 49.3 PLANKS *Obtain maximum overhang with* 3 *uniform planks of length and weight* 2, 3 *and* 4.

SIDNEY SILVERSTONE reminds me that I did not give a solution for this. I think it is



which gives an overhang of $\frac{59}{18}$.

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PROBLEM 56.1 HIDE AND SEEK EDDIE

Two players, Hider and Seeker, simultaneously choose points in a closed disc of unit radius, Hider escapes if his point is more than half a unit from Seeker's point. Show that if play is optional in the sense of game theory, Hider will be caught with probability 1:7.

PROBLEM 56.2 TRIANGLES JOHN HULBERT Solve $\frac{m(m+1)}{n(n+1)} = 2$ for integer *m*, *n*..

PROBLEM 56.3 REAL PROBLEM JH

John writes, "Can anyone help ne with a real problem? Is there an integer x such that $6x^2 - 105$ is a square. I can't find one nor prove there isn't one.

I know several x's exist such that $5x^2 \pm 4$ is a square. To what family of numbers do these x's belong? Prove that $5x^2 + 4$ is square $\Leftrightarrow$ x belongs to this family."

PROBLEM 56.4 SQUARE DIVDING STEVE AINLEY

a. Divide a unit square by two straight lines into parts so that by taking one or more contiguous parts you can get area of $\frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \ldots, \frac{k-1}{k}, 1,$ for k as large as possible.

b. Same with three straight lines, or four, or ... .

PROBLEM 56.5 WIT'S END GOING ROUND AT WORK

"Graveside", Wit's End, Haunts. Dear Friend, Some time ago I bought this old house, but found it haunted by two ghostly noises - a ribald Singing and a sardonic Laughter. As a result it is hardly habitable. There is hope, however, for by actual testing I have found that their behaviour is subject to certain laws, obscure but infallible, and they can be affected by my playing the organ or burning incense.

In each minute, each noise is either sounding or silent - they show no degrees. What each will do during the ensuing minute depends, in the following exact way, on what has been happening during the preceding minute:

The Singing, in the succeeding minute, will go on as it was during the preceding minute (sounding or silent) unless there was organ playing with no Laughter, in which case it will change to the opposite (sounding to silent, or vice versa).

As for the Laughter, if there was incense burning, then it will sound or not according as the Singing was sounding or not (so that the Laughter copies the Singing a minute later) . If however there was no incense burning, the Laughter will do the opposite of what the Singing did.

At this minute of writing, the Laughter and Singing are both sounding. Please tell me what manipulations of incense and organ I should make to get the house quiet, and to keep it so.

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EDITORIAL

I wonder what proportion of our readers take *The Times*. One morning recently two more people sent me cuttings, and they were both from that eminent newspaper. Keep them rolling in, please.

Jim Ezechiel's was from *The Times* of 17 November and is headed "Prime Number Record Broken". It seems that two eighteen year old students, Laura Nickel and Curt Noll, using a computer at California State University have shown that 2^{21701} is prime. It is also stated that the previous largest known prime was 2^{19937} discovered by Dr Bryant Tuckerman. This information is probably enough to help those interested deduce what the two primes in question really are.

The other cutting, sent by John Millar is from *The Times* of 18 November. The article is called "The mathematicians of Spitalfields" and is written by that champion of the English language, Philip Howard. It is about some research done by Professor John Cassels, Sadleirian professor of pure mathematics at Cambridge who "last night discovered a golden age of numbers, when working men studied mathematics for pleasure". He was giving the presidential address to the annual meeting of the London Mathematical Society.

It seems that there was a Spitalfields Mathematical Society which flourished from 1717 until 1846 when it was absorbed in the Royal Astronomical Society. It had originally 8^2 and later 7^2 members and its Rule Book called for fines to be imposed on members for, amongst other things, not keeping silence between the hours of eight and nine (when all were to employ themselves on mathematical exercises), cursing, or introducing into a lecture on mathematics controversial points of divinity or politics (in which case the fine was two shillings and sixpence).

An exhibition of the archives has been opened at the Royal Astronomical Society.

Here is a third one; seen by me. It was in *The Times* of 15 November, headed "Three times winner". Mr Cyril Thornley, a taxi driver, aged 61, of Preston, Lancashire, beat odds calculated by Liverpool Polytechnic mathematicians at 64 000 000 000 000:1 by winning his third £1000 lottery prize in five months yesterday.

(Would anyone care to work out his chance of winning a fourth?)

In the past I have given cut-off dates for material submitted to M500 relating to particular events. After the debacle of the last issue where several functions were advertised after they had taken place this information is in need of revision.

For instance I had a notice from Jim Skinner addressed to all past and present students of M202. If they don't know what I am talking about there is no point me telling them; but I enjoyed the party. Jim is to be congratulated on his effort and I hope someone will write in and do it. Parsifal College was bristling with celebrities; even Professor Oliver Penrose was there, and I haven't seen him since OU-1.

By the way, re-reading my interjection to solution 54.3 I see it reads somewhat ambiguously. In the first place the set of numbers from which one can successively take the given fractions ($^1/_8$, $^1/_7$, ..., $^1/_2$) is precisely the multiples of eight. Secondly there are other such (than 216) which pass 81, for instance $81 \times 8 = 648$. But I was just trying to understand where the unsuccessful attempts came from.

Jeremy Humphries, in his preamble, unjustly accuses me of complaining about his tardiness.

Eddie Kent

However, since he raised the point, I would like to complain about his prolixity. I had allowed him three pages this issue and he ran to five, so some items which were to appear have been held over, but I'm not saying which.