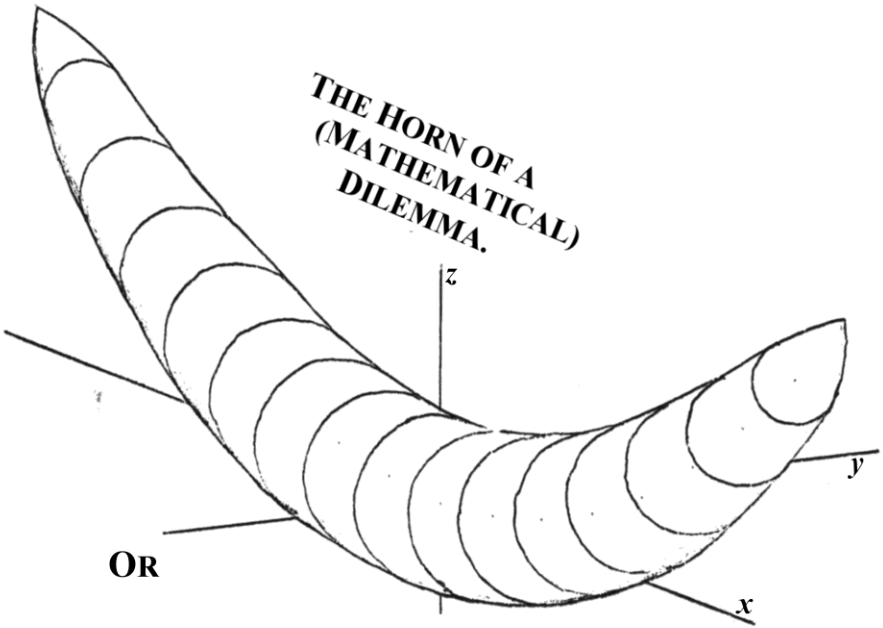
M500 57

THE HORN OF A (MATHEMATICAL) DILEMMA.



OR

$$64(z^2+y^2)+x^4-16zx^2+4x^2-256=0 \text{ ?}$$

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Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

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‘AMUSEMENTS IN MATHEMATICS’ TONY BROOKES

I recently purchased a book which I believe will be of interest to many M500 readers. It is *Amusements in Mathematics* by H E Dudeney, Dover Books, $2.75. Dudeney lived from 1847 to 1930 and was one of the greatest of English puzzelists. The book was originally published in 1917 and the Dover edition is a reproduction of the original edition complete with a number of charming Edwardian illustrations.

Dudeney’s book contains over four hundred mathematical puzzles on a wide range of topics (arithmetic, algebra, geometry, chess games, mazes, paradoxes, etc.) and with widely varying difficulty. Fortunately, to prevent the reader losing sleep over apparently unsolvable problems the book also contains solutions to each one. Unfortunately the method of solution is often not given so you are left wondering how he obtained the answer.

With such a large variety of puzzles and problems in the book I can guarantee that anyone with even a passing interest in mathematics will find plenty to interest them.

Problem 129 has a rather involved setting about determining the number of men in King Harold’s army at the battle of Hastings. Basically the problem involves finding an integer solution to the equation $km^2 + 1 = n^2$ for $k = 61$. The problem by way of illustration does give the solutions for $k = 60$ (when $m = 4$ and $n = 31$) and $k = 62$ (8,63). The number of men in Harold’s army is supposed to have been n^2. These relatively simple solutions are supposed to lead the reader to think that the solution for $k = 61$ is not so hard to find, but therein lies the trap. $k = 61$ has been chosen for the problem because it is the most difficult value of k for $1 \le k \le$ l00. The actual values of m and n for this k are very large. I won’t spoil anyone’s fun by giving the answer now; however to save a lot of wasted time I suggest you start with $n > 10^9$.

With the use of an HP29 calculator a friend and I have established solutions (where this is possible) for most values of k under 100. The only values of k for which there is obviously no solution are when k is a perfect square. Other values of k yield fairly simple solutions. Some examples are

k	m	n
p^2-1	1	p
p^2+1	$2p$	$2p^2+1$
$p^2\pm2$	p	$p^2\pm1$
$p(9p+2)$	3	$9p+1$
$p(4p+1)$	4	$8p+1$.

There are values of k for which the HP29 found solutions but we have not been able to find a simple relationship for k,m,n. It is here we would like M500 readers to help. Some typical (i.e. the worst!) examples are given below

k	m	n	k	m	n
29	1820	9801	58	2574	19603
46	3588	24335	67	5967	48842
53	9100	66429	86	1122	10405.

Finally we have values of *k* which defeated the computer. This is where we would really like some help; perhaps some reader knows a technique for any *k*, or has access to a bigger number-cruncher than we have. The values of *k* in question are:

61 - for which I have a solution, 73, 76, 85, 89, 94 and 97.

MATHEMATICS - ART OR SCIENCE? BRIAN WOODGATE

Perhaps the answer depends on the person rather than on the subject. i.e., as an individual progresses then it changes for him or her.

Firstly, for people at say A-level or HNC or for some doing degrees it is a technology. Mathematics is seen as a tool that is necessary to carry out some task. We are taught to use a tool and we become craftsmen in its use. We then use this skilled craft to pass exams or in our jobs or perhaps we teach the craft to others: i.e. the next generation of apprentices. Let us not despise the above despite its lack of ‘202’ purity; it *is* skilled mathematics.

Secondly, I suggest that for most people at university level it is a science. Laws (or theorems) are proposed, investigated and tested. If a flaw is found then it may lead to new laws or even to a new branch of the subject; e.g. non-Euclidean geometry. Eventually this pure science is used by the first group in industry; e.g., matrix theory.

Finally, for the fortunate few, mathematics becomes an Art. To support this argument I can only suggest the reading of *A Mathematician’s Apology* by G H Hardy, one man who did reach the top. He made it clear that he considered his subject akin to the Arts, in fact almost a religion.

NOTHING IS VERY IMPORTANT BOB ESCOLME

More correctly the empty set Ø is very important. And the empty set is not nothing. It is something. It is a set, though there’s nothing in it.

Perhaps because there is nothing in it Ø tends to get overlooked on occasion, with dire results. Even the bright ones (I mean the OU professors, and I hope they can afford the odd martini) have been known to overlook it - as shall be recounted later.

For example, one can produce a fallacious proof that a subset of a ring is a ring if one forgets the fact that one of the properties required of a ring is that it be not empty. If you are required to prove that a given set or subset is a ring then, among other things, it must be known or proved that the set or subset contains at least one element.

As another example, consider that axiom for the real numbers $\mathbb{R}$ which states that every non-empty subset A of $\mathbb{R}$ which is bounded above has a least upper bound. The requirement that $A \neq \varnothing$ is important for one can easily show that $\varnothing$ is bounded above, yet does not have a least upper bound. In fact $\varnothing$ possesses a whole set of upper bounds: the whole of $\mathbb{R}$ itself is its set of upper bounds. And $\mathbb{R}$ is also the set of lower bounds of $\varnothing$.

Thus let b be any element of $\mathbb{R}$. Then for all $a \in \varnothing$, $a \leq b$. That last sentence is correct since it is incorrect if and only if we can show there exists at least one $a \in \varnothing$ such that $a > b$. But since $\varnothing$ is empty there is no such a. Thus $b \geq a$ for all $a \in \varnothing$, so b (which is any element of $\mathbb{R}$) is an upper bound of $\varnothing$. $\mathbb{R}$ of course has no least member, so $\varnothing$ does not have a least upper bound.

So, if you are required to show that a given subset A of $\mathbb{R}$ has a least upper bound then either it must be known or you must prove that A has at least one member (as well as proving that A is bounded above). Much the same goes for greatest lower bounds.

And now for the howler perpetrated by the martini boys. it concerns the M231 definition of the limit of a real one valued function near a point. This is the definition given in the course handbook.

> The function f (of one variable) approaches the limit l near a if for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $f(x)$ is defined and $|f(x) - l| < \varepsilon$.

We can omit the words '$f(x)$ is defined and' if we substitute 'for all $x \in$ the domain of f 'in place of 'for all x' where it appears in the definition. We then get

> The function $f{:}A \rightarrow \mathbb{R}$, $A \subset \mathbb{R}$, approaches the limit l near a ($a \in \mathbb{R}$) if and only if (this is a definition) for any $\varepsilon > 0$, $\exists \delta > 0$: $\forall x \in A$, $0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

The fallacy inherent in the definition becomes clear if we define a set X as

> $X = \{x \in A: 0 < |x - a| < \delta\}$

and then rewrite the definition as

> The function $f{:}A \rightarrow \mathbb{R}$, $A \subset \mathbb{R}$, approaches the limit l near $a \in \mathbb{R} \Leftrightarrow$ For every $\varepsilon > 0$, $\exists \delta > 0$: $\forall x \in A$, $x \in X \Rightarrow |f(x) - l| < \varepsilon$, where $X = \{x \in A: 0 < |x - a| < \delta\}$.

Now consider the following example. Let $A = \{0\}$ and define $f{:}A \rightarrow \mathbb{R}$ by $f(0) = 0$. Take $l =$ say $\sqrt{\pi}$, or e^3 or any other number $\in \mathbb{R}$ you care to think of. Given any $\varepsilon > 0$, choose δ in the range $0 < \delta < a$. Then $X = \{x \in \{0\}: 0 < |x - a| < \delta\} = \varnothing$. Following through the definition we get the following true statement: $\forall x \in \{0\}$, $x \in \varnothing \Rightarrow |f(x) - \sqrt{\pi}| < \varepsilon$. It is false $\Leftrightarrow$ we can exhibit an $x \in \{0\}$: $|f(x) - \sqrt{\pi}| \geq \varepsilon$. And since $\varnothing$ is empty there is no such x.

Thus the M231 definition enables us to define a simple function which tends to any limit near any point. The correct definition (as given in M100 (Unit 7 page 30)) must include the stipulation that our set $X \neq \varnothing$. Put it this way. If the above is baloney then I award myself 0 out of $\{0\}$. If it isn't the (M231) professors get 0 out of $\varnothing$.

LETTERS

From Eric Lamb Most OU students who took exams in 1978 will by now have been told that they have passed. In fact, if the pattern of results is like that published in *Sesame* in previous years some 90% of examined students will have been successful. The only reasonable conclusion that can be drawn from this is that the OU tries to ensure that as many satisfactory students as possible do pass. The aim is definitely not to stop students from receiving the fruits of their labours as seems to be the aim of some examining authorities whose success rate is about 30%.

Despite this I always suffer from exam nerves. This is not because of lack of experience (I've taken about fifty exams altogether), but is due to the nature of examinations. Every paper I have taken seems to have been designed to test small parts of the course in great detail. This introduces a large element of luck into any exam. By this I don't mean that OU exams are designed such that there is a high probability of failure for a candidate who is familiar with most of the course. What I do mean is that the quality of the result is not necessarily a reflection of the knowledge and ability of the student. To give an example, I gained a distinction on M231 simply because the right questions turned up in my exam. It certainly wouldn't have happened with any other M231 paper I've seen. In contrast to this, despite the fact that I spend an average ten hours a week using or programming a computer, the M251 exam gave me a very rough time.

If, as I am sure is the case, the OU are setting their exams with a view to finding out what a student knows as opposed to trying to "trip him up", why do they persist with the old-fashioned tactics of restricting the examination questions to a few topics? Why not give such a wide choice of questions that the right one must turn up for any student for whom the right one exists?

A move in the right direction has been made by the maths faculty in that some examinations have compulsory short question sections. Why do these have to be compulsory? Why not set more questions and allow the student to select? Set some easy ones and some hard ones, and give marks accordingly.

Concerning the longer questions, it is customary to limit the number of these that the student is allowed to attempt. Furthermore, it is not uncommon to test more than one aspect of the course in one of these questions. This combination is a good idea if the aim of the exam is to create failures, but it is not such a good idea if the aim is to find out how much the student knows. If the number of questions to be attempted is not limited, then the student will do what he can. Again the marks can be suitably arranged to identify the good student.

If I could go into the exam room knowing that whatever I had revised would appear in some form on the paper, I'm sure that much of the tension would vanish. What do other people think?

From Frank Springall   I think my comments (M500 56) about Larry Niven make a lot more sense if you put the NOT back in and say that he has not won the Nobel Prize for physics.

Although it may have seemed something of a throwaway line the intention to form a

Science Fiction Society is serious. Even if there are no members of M500 interested we (I am not alone even if my partner in crime is only doing ‘A’ courses) would be grateful for any advice on what to do or pitfalls to avoid especially from anybody who helped found M500.

I cannot see why Sidney Silverstone thinks £33 for a Rockwell 63R is reasonable, I have seen them for £17. (I think it was in Curry’s but it was a year ago.) Also I do not think they are that good. I have a CB17 414SR which although it has only single parentheses and no x! it does have two memories, polar to rectangular conversion and vice-versa, a x/y button (very useful) and limited statistical functions. More important for me there is no function button, this saves me an enormous amount of time as I always forget to push the function button. This does lead to its one major fault: there are 48 keys which means they are rather small.

But even I, one of the illiterate working classes for whom the glorious OU was founded (this means I have not got piano-players fingers) can manage the calculator OK.

PS - To be honest I think the OU is the best thing to happen in education since it was made compulsory.

From Garnett Marriott From Halmos *Naive Set Theory* page 8, we know that “the empty set is a subset of every set”, in other words, “everything contains nothing.” Also from Halmos, page 6, we know that there cannot exist a set that contains all sets. In other words, “nothing contains everything”. Thus $A \supset B$ and $B \subset A \Rightarrow A = B$ forces the result

everything = nothing.

--mm-- who’d have thought it?

From Steve Murphy Just one short item, which I’m sure will have been covered by dozens of similar letters.

AD INFINITUM.

In M500 54 Brian Stewart wrote about the set of real numbers between 0 and 1 that contain no zeros in their decimal expansion. We could express this set as

$T = \{x\colon x \in \mathbb{R},\ x = \sum_1^\infty x_r\ 10^{-r}\ \}$

where x_r represents 1, 2, 3, ... , 8 or 9 and we exclude numbers such that $x_n = 9$ for $n >$ some integer k (to avoid writing 0.3 as 0.29999... and so on).

However although T is both uncountable and null it has as many members as all the reals in [0,1). We can see this by considering the set

$S = \{x\colon x \in \mathbb{R},\ x = \sum_1^\infty (x_r{}_{-1})9^{-r}\ \}$

with the same restrictions on the x_r as above.

Now we can obviously place the elements of T in a one–one correspondence with the elements of S, but S is really just an elaborate way of writing the reals in [0,1) in the scale of 9. So we’re back to Aleph-one after all.

From Tom Dale. I’m glad to see that my contribution based on *The World of Mathematics* has been of some use to you.

I have now managed to find out how to evaluate the fractions given at the bottom of 55 page 6 - in fact I am wondering how I was so dim that I did not realise that the first one is just the converse of the problem dealt with in Krysia Broda’s article in issue 47. For we can find x and y in the equation $x^2 – Ny^2 = (–1)^{n–1}$ - they are numerator and denominator of the convergent A_n/B_n. So in the continued fraction 1;3,2,3,2,... , $n = 1$, $A_n/B_n = 4/3$. Hence $16 – 9N = 1$, $N = 5/3$. So the continued fraction is equal to $\sqrt{{}^5/_3}$.

The second one depends on a theorem which says that in a periodic fraction of this type, where $P’/Q’$ and P/Q are the last two convergents of the first period, then $Qx^2 + (Q’–P)x – P’ = 0$ where x is the fraction. In the example given, $P’/Q’ = 3/2$ and $P/Q = 10/7$.

So $7x^2 – 8x – 3 = 0$ and $x = (4\pm\sqrt{37})/7$. The positive root is taken. (The negative root, as it happens, is equal in magnitude to the continued fraction which has the same quotients in inverse order.)

PS. - Notations can be troublesome. Krysia gives the first subscript of continued fractions as 0, whereas my book starts at 1. So nth convergents aren’t the same. Fascinating things though, continued fractions. Did you know that any series can be expressed as one?

From Bob Bertuello I hope that the following will help answer Tom Dale’s query at 55 6 on continued fractions.

To find the value of a recurring continuous fraction, it is important to note that a RCF is a subset of itself, and this allows the setting up of a quadratic equation whose solution is the required value.

Take the last case:

x = 1;2,3,l,2,3,l, ... ,

i.e., $x–1 = 1/(2+1/(3+1/(1 + ...)))$.

Since the right hand side recurs, the three dots can be replaced by $x–1$, i.e.

$$(x–1) = \frac{1}{2+\frac{1}{3+\frac{1}{1+(x-1)}}} = 1/(2 + x/(3x + 1)) = \frac{3x+1}{7x+2}$$

whence $7x^2 – 8x – 3 = 0 \therefore x = {}^1/_7\,(4 + \sqrt{37})$, negative value not applicable.

The other cases are amenable to the same treatment.

Ed - *Other treatments have been received for the same problem. I might print them later if I forget.*

GAUSS XI JEREMY GRAY

In episode four I set some problems connected with Gauss’s work. Here I give the solutions to some of them. At the end of this episode you will have a proof of one of the greatest theorems in number theory: a prime number is a sum of two squares if and only if it is of the form 4n + 1. The theorem is due to Fermat, the proof here is ‘after Gauss’ and reveals a nice connection with quadratic residues. I hope you will enjoy looking at the surprising inner life of the integers, even if the details are too hard. It is, after all, the source and the ‘why’ of modern algebra.

Now for the number theory. We have

$$1 = 1^2 + 0^2$$

$$2 = 1^2 + 1^2$$

but

$$3 = 1^2 + 1^2 + 1^2$$

so requires three squares. Looking only at primes one finds $5 = 2^2 + 1^2$ but neither 7 nor 11 is a sum of two squares, whereas $13 = 3^2 + 2^2$, and generally primes which are a sum of two squares are

5, 13, 17, 29, 37, 41, ...

primes which are not are

7, 11, 19, 23, 31, 43, ...

All those in the first class are of the form $4n + 1$; all those of the second class are of the form $4n$ – l. We conjecture that this is always the case, and indeed it is. Before proving it observe that the next question helps reduce the question ‘is $n = a^2 + b^2$?’ for non-prime n to the case n

prime, so we prove N3 first.

(Find a reformulation of $(a^2 + b^2)\ (c^2 + d^2\ )$ which shows that if two numbers are each a sum of two squares so is their product.)

$$(a^2 + b^2)(c^2 + d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \ = a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2$$
$$= (ac + bd)^2 + (ad - be)^2$$

which is equal to $X^2 + Y^2$ where $X= (ac + bd)$ and $Y= (ad - be)$.

What about proving a prime of the form $4n + 1$ is always a sum of two squares? Here first of all is an elegant observation of Dirichlet, taken from his *New proofs of some results in number theory* of 1828. “With a and p as before [p a prime, a not divisible by p] let us consider the following $p - 1$ multiples of a.

$a, 2a, 3a, ..., (p - l)a.$

“It is easy to see that two of them cannot give the same remainder when divided by p; for if the remainder coming from ma and na were equal, $ma - na = (m - n)a$ would be divisible by p which is impossible since a is not divisible by p and $m - n$ is $< p$ and not zero. The remainders which we obtain by dividing the $p - 1$ multiples by p being all different and not zero, it is easy to see that these remainders must coincide with the numbers of the series 1, 2, 3, ..., $p - 1$ when one disregards the order between them. It follows from that that the product of the $p - 1$ multiples of a should give the same remainder as the product $1 \cdot 2 \cdot ... \cdot (p - 1)$.

“The difference of these products is then a multiple of p. But this difference is easily put in the form

$(a^{p-1} - 1) \cdot 1 \cdot 2 \cdot 3 \cdot ... \cdot (p - 1)$

and, $1 \cdot 2 \cdot 3 \cdot ... \cdot (p - 1)$ not being divisible by p, we conclude that $a^{p-1} -1$ is a multiple of p or, which is the same thing, a^{p-1} divided by p gives the remainder 1.”

Expressing this conclusion in the language of congruences

$a^{p-1} \equiv 1 \bmod p$ or $a^{p-1} -1 \equiv 0 \bmod p.$

Therefore p divides either $a^{(p-1)/2} - 1$ or $a^{(p-1)/2} + 1$, but which?

Here quadratic residues enter the picture. If $a \equiv b^2$ mod p then p divides $a^{(p-1)/2} - 1$ (because $a^{(p-1)/2} - l \equiv b^{p-1} - 1 \equiv 0$ mod p); but if not, not. So suppose $p = 4n + 1$, let us show it is a sum of two squares. -1 is a quadratic residue mod p (use $(p - 1)! = -1$ mod p and take $x = 1 \cdot 2 \cdot ... \cdot 2n = (-1)(-2) ... (-2n) \equiv (4n)(4n - l) ... (2n + l)$) so $-1 \equiv x^2$ mod p, i.e. p divides $x^2 + 1 = (x + 1)(x - 1)$. At this point we have entered the domain of Gaussian integers, discussed in questions N7-N12. Accepting for the moment that factorization into primes is possible, if p is prime (we shall soon see that it isn't) it must divide either $x + 1$ or $x - 1$. Neither $p^{-1}(x + i)$ nor $p^{-1}(x - i)$ are Gaussian integers, so p can't be a Gaussian prime. But then p must be factorizable as

$$p = (a + bi)(a - bi) = a^2 + b^2$$

and so p has been written as a sum of two squares.

It is elementary that any prime which is a sum of squares is either 2 or of the form $4n + 1$, since any square is congruent either to 0 or to 1 mod 4. So we may conclude: all primes of the form $4n + 1$ are a sum of squares, and conversely; no prime of the form $4n + 3$ can be a sum of squares.

This argument has taken care of some of the questions N7-N12, but we haven't yet shown unique factorization of Gaussian integers. To do that, first recall that factorization of integers is unique only up to multiplication by +1, which numbers do not alter the modulus: $|a| = |-1 \, . \, \mathrm{a}|$. Factorization of Gaussian integers is similarly unique only up to multiplication by $1, -1, i, -i$ which do not alter the norm $N(a + bi) = a^2 + b^2$. Very well, to prove factorization is possible it is enough to do it; Either you can't in a given case and the Gaussian integer you have picked is prime already, or you can, but the norms of the factors are less than the norms of the original number and so the factorization process stops in finitely many steps. To prove factorization is unique, suppose it isn't and for some Gaussian integer two different factorizations are possible. Now show that if a prime $a + bi$ divides $(c + di)(e + fi)$ it must divide one factor, and hence obtain a contradiction by successive cancelling of factors ($c + di$, $e + fi$ also prime). The proof exactly follows the unique factorization theorem for integers.

Care is needed in recognising a Gaussian prime. N10 asked you to show 5 is not a Gaussian prime; $5 = (2 + i)(2 - i)$. As an example that –1 is a quadratic residue mod 5 one has

$$3^2 \equiv -1 \bmod 5$$

or

$$5 \mid (3 + i)\,(3 - i)\,.$$

But 5 does not divide $3 + i$, nor need it as $3 + i = (2 - i)(1 + i)$. But we can easily recognise which of the usual primes is a Gaussian prime. If p, a prime, is a Gaussian prime we must fail to write

$$p = (a + bi)\,(a - bi) = a^2 + b^2\ ,$$

so only those usual primes are Gaussian primes which are of the form $4n + 3$.

Other interesting rings are made up as

$$a + b\sqrt{-n}.$$

If one has $a + b\sqrt{-3}$ in mind, so

$$(a + b\sqrt{-3})(a - b\sqrt{-3}) = a^2 + 3b,$$

it turns out that factorization is still unique, and the usual numbers which can be written in this way are either squares, or primes of the form $6n + 1$ and products of such. Quite different behaviour occurs in the ring of ‘integers’ of the form

$$a + b\sqrt{-5}.$$

(N13). Here

$$21 = 3 \times 7 = (1 + 2\sqrt{-5})\,(1 - 2\sqrt{-5})$$

are two distinct factorizations into primes. (They must be primes because their norms are so small by comparison with numbers of the form

$$a^2 + 5b^2$$

and they can’t factorize as integers.)

PERFECTION EDDIE KENT

Some time ago I reprinted a problem from an American magazine in M500. It asked for a proof that the only ‘almost perfect’ numbers (those that miss being perfect by one only) are the powers of two. No one took up this challenge, or at least admitted to it, and I thought that everyone else agreed with me that it was a pretty silly thing to look for. In fact even the concept of the perfect number (one which is the sum of its proper divisors; 6, for instance, which is 1 + 2 + 3; or 28 = 1 + 2 + 4 + 7 + 14) seemed perverse. In counting the divisors of a square number do you count the square root once or twice? Once obviously, but that means you lose one factor compared with other numbers. Also there is only the most tenuous connection with the Euler phi-function (which counts the number of positive integers less than x and relatively prime to x) and that, though useless, has at least been saved by the groupies.

I then discovered that Euclid and Euler (in that order) showed that each and every even perfect number has the form $2^{n-1}(2^n - 1)$ where $2^n - l$ is prime. You will notice of course that $2^n - l$ is the sum of the divisors of $2n$.

It has now been pointed out to me that if you replace all the even numbers in Pascal’s Triangle with dots you get, amongst other things, a series of upside down triangles straight down the middle, almost each of which contains a perfect number of dots. The exceptions correspond to the occasions when $2^n - l$ is not prime; but all the even perfect numbers are there. Why?

Pascal’s Triangle is conventionally numbered from the top with the first row labelled row 0, the second row 1 and so on. (This makes sense, the nth row then gives the nth power expansion in the binomial theorem; also all the numbers in the pth row, p prime, are divisible by p. It is not so convenient for us as the number of numbers in row n becomes $n + 1$, but never mind.)

Because of the way the triangle is constructed, each number the sum of the two immediately above, it is obvious that even numbers will occur in equilaterally triangular chunks. It is also obvious (can be shown but how boring) that the rows which are entirely even (except at the ends) are those which are labelled with powers of two.

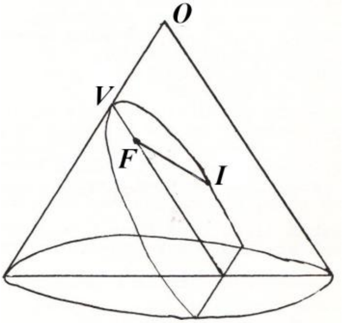
Since there are $n + 1$ numbers in row n, this becomes $2^q + 1$ for some q. Knock off the two ones at the ends and you get $2^q - 1$, and if this is prime you have a Mersenne prime. So it only remains to show that the formula above for perfect numbers is equivalent to a triangular number in Mersenne primes. But that is easy:

$$\begin{aligned} 2^{n-1}(2^n - 1) = \tfrac{1}{2}(2^n(2^n - 1)) &= \tfrac{1}{2}(2^n(2^n - 2 + 1)) \\ &= \tfrac{1}{2}\,(2^{2n} - 2^{n+1} + 2^n\,(+\,1 - 1)) \\ &= \frac{(2^n - 1)^2 + 2^n - 1}{2} \end{aligned}$$

which is triangular in $2^n - l$.

Two last questions and a thought. Are there any even almost perfect numbers? Is there an odd perfect number? If you put $n = 0$ in the formula you get 0×0 which equals 0. In other words, nothing’s perfect only if nothing is prime. I find that quite cheering.

PROBLEMS COMPILED AND EDITED BY JEREMY HUMPHRIES



Bob Escolme wrote to me again about problems 54.4 and 54.5, Cone I and II. He says that his solution published in 56 is unsatisfactory and he has reworked it for an arbitrary cone. . He gets the result *OV* = *VF* + *FI* (see figure) where *O* is the vertex of the cone, *V* the vertex of the parabola and *I* is either of the two points which become points of inflexion when the cone is flattened.

I also got some stuff on continued fractions from BOB BERTUELLO and HOWARD PARSONS which I have passed to Eddie. *Continued Fractions* by A Ya Kinchin is a nice little book published by the University of Chicago Press in 1964 - I don't know if you can still get it. Main topics are representation of numbers as continued fractions, convergents as best approximations, order of approximation, and introduction to measure theory of continued fractions including a treatment of various averaging operations. A review says Khinchin achieves a clarity of exposition which it would be difficult to surpass.

The record for an unsupported circle of seated people - each sits on the knees of the person behind - is now 3394. They did it for one minute at Balment Park, British Columbia, last May.

Two mathematical curiosities heard recently on the radio:

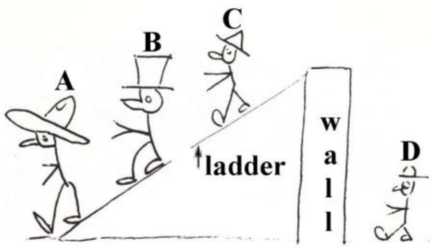
...40 ambulances, each carrying between two and three people... ...the ninety-five dollar question....

There is an article on Hilbert's tenth problem by Martin Davis and Reuben Hersh in *Scientific American* November 1973 (ref Alan Slomson M500 55 and 56). And anyone who watched the BBC Christmas lectures by Christopher Zeeman and wants to know more can find an article on catastrophe theory in *Scientific American* April 1976. (The November 73 also has Martin Gardner on programmed worms - a must.)

Libby Drake's hat problem had no number. Let's call it 55.0

SOLUTION 55.0 HATS *Four men, Archibald, Bartholomew, Clarence and Dorothy are arranged as shown. Two hats are white, two are black. No one can see his own hat. B can see A, C can see B and A, D can't see anyone. When a man knows the colour of his hat he says so. What can happen*?

ROSEMARY BAILEY, HOWARD PARSONS, STUART POTTER and TOM DALE answered this. Rosemary said she liked the problem but why wasn't D called Douglas? She thinks two things can happen. In the first case Archibald and Bartholomew have the same colour hat, say white. Then Clarence shouts ‘black’ and this is followed by the others simultaneously: Duncan says ‘black’ - Arthur and Benjamin say ‘white’. In the second case Alastair and Brian have hats of a different colour, say black and white respectively. Then Christopher and David say nothing. Faced with this silence Barnabus says ‘black’ whereupon Alfred says ‘white’. Cuthbert and Derek continue to say nothing.



What Rosemary really wants to know is how they got up onto the ladder without Albert and Barry seeing Cedric’s hat. Please send your answers to this and to why Dennis is called Daphne as soon as possible.

(My practical friends say that since no-one can see Diane he should take off his hat and look at it.)

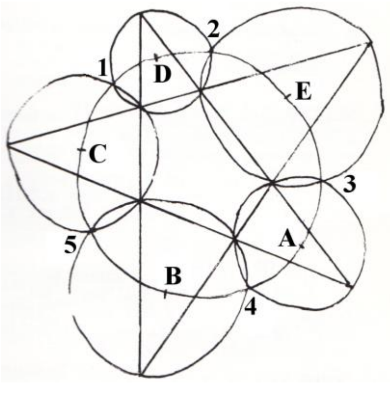
SOLUTION 55.1 CIRCLES

Richard Ahrens has sent an explanation of this. The drawing is complicated. I suggest you draw your own and add the bits as you come to them.

1 2 3 4 5 are any five points in that order on the circle. ABCDE are the mid-points of the arcs 34, 45, 51, 12, 23.

1E4C2A5D3B1 is a ten-pointed star. ACEBDA is a five pointed star

5D and C2 meet at X
D3 “ IE “ Y
IB “ C4 “ P
B3 “ 5A “ Q



FACT 1 X lies on the circle, centre C, passing through 1 and 5.

proof $\angle CX5 = \angle C25 + \angle D52$ (ext ang of $\triangle X25$) = $\angle C51 + \angle 15D$ (equal arcs subtend equal angles) =$\angle C5D$.

therefore $\triangle CX5$ is isosceles; therefore CX = C5. therefore X lies on a circle centre C radius C5. Similarly X lies on a circle centre D radius D5. This means that X is the second point of intersection of the circles centred at C and D which meet at 5.

Similarly, Y, P and Q are the second points of intersection for the circles centred at D & E, C & B and B & A respectively.
Thus XY and PQ are two lines of the star in the problem. We must show that these lines meet on the circle with centre C.

FACT 2 XY is parallel to CE.

proof $\angle$1YX = $\angle$12X (1 X Y 2 lie on circle, centre D)
$\angle$12X = $\angle$12C = $\angle$1EC (1 2 E C lie on original circle)
$\therefore$XY || CE ($\angle$1YX & $\angle$1EC are corresponding angles)
Similarly PQ is parallel to CA. Hence the angle between XY and PQ equals $\angle$ECA.
Now $\angle$ECA = ½$\angle$2C4 since arc EA is half arc 24.
$\therefore$The angle subtended at C by arc XP is twice the angle between XY and PQ.
$\therefore$XY and PQ meet on the circle with centre C. ($\angle$ at circum = ½$\angle$ at centre

Extensions, which Richard suggests are:
1. *What happens with four circles instead of five*?
2. *Sometimes this still works if the points* 1 2 3 4 5 *are not arranged in order on the circle. You can also try using the alternate centres to draw the secondary circles*.
3. *What is the general theorem of which the problem* 53.1 *was a special case*? (*I don't know*; *R.A.*)

SOLUTION 55.1 BLACK AND WHITE *Remove two black and two white squares from a* $(2n)^2$ *chessboard. If at least one of each is an interior square show that the remaining board can be tiled with* 2×1 *tiles.*

Nobody has sent me a proof yet. ANGUS MAC DONALD has solved it by induction and says that his solution is very long and pedantic. STEVE AINLEY says "Hm... " (about the problem, I mean).

SOLUTION 55.2 COINS A *tosses two coins hidden from* B. *Is there at least one tail*? B *asks*. A *says* "*Yes*" (a). A *drops one coin* (b). *When they find the coin it is a tail* (c). "*That's OK*" A *says*, "*it was a tail to start with*" (d). *At points* (a) (b) (c) *and* (d) *what is the probability that both coins are tails*?

Great fun with this. BOB BERTUELLO, LIBBY DRAKE and STUART POTTER got it right, and several more got it nearly right.

(a) $^1/_3$. Possibilities are (HT) (TH) (TT) .

(b) $^1/_3$. Possibilities, with dropped coin italicised are

H*T* T*H* T*T*   *H*T *T*H *T*T

H*H* T*T* T*H*   *T*T *H*H *H*T.

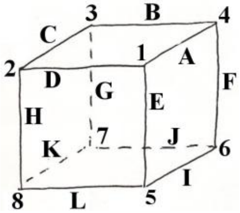
(c) $^2/_3$. Possibilities are H*T* T*T* T*T*; *T*T *T*H *T*T.

(d) $^1/_2$. Possibilities are H*T* *T*H T*T* *T*T.

SOLUTION 55.3 CUBE *I have a wire model consisting of the edges of a cube. How many different structures can I make by removing three edges*?

STUART POTTER says there are thirteen structures. Eight of these can be split into two sets of four so that each set is a reflection of the other. The remaining five have reflective symmetry. ROSEMARY BAILEY gave a theoretical solution as follows: Nine if reflections count as the same; thirteen if not. The method is based on G, the symmetry group of the cube. (See also DAVID ASCHE, M500 49.)

Symmetries of the cube



Rotations

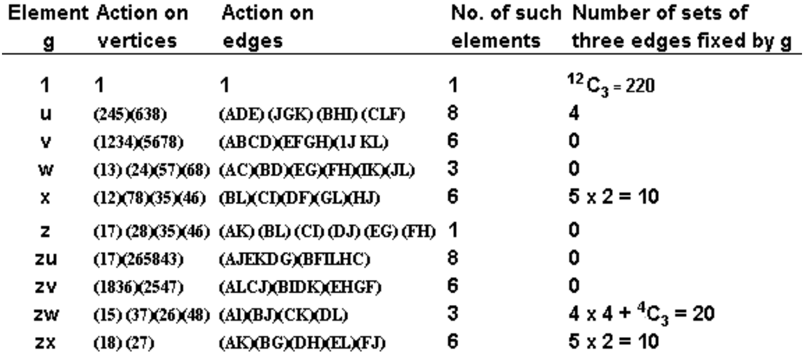
1 = identity

u = through 120° about a main diagonal.

v = through ±90°
w = through 180°
about an axis through the mid points of opposite faces

x = through 180° about an axis through the mid points of opposite edges

If reflections are allowed we also have
z = inversion in the centre of the cube.
zu zv zw = reflection in a plane parallel to a face
zx = reflection in a plane through opposite edges.



Element g	Action on vertices	Action on edges	No. of such elements	Number of sets of three edges fixed by g
1	1	1	1	${}^{12}C_3 = 220$
u	(245)(638)	(ADE) (JGK) (BHI) (CLF)	8	4
v	(1234)(5678)	(ABCD)(EFGH)(1J KL)	6	0
w	(13) (24)(57)(68)	(AC)(BD)(EG)(FH)(IK)(JL)	3	0
x	(12)(78)(35)(46)	(BL)(CI)(DF)(GL)(HJ)	6	5 x 2 = 10
z	(17) (28)(35)(46)	(AK) (BL) (CI) (DJ) (EG) (FH)	1	0
zu	(17)(265843)	(AJEKDG)(BFILHC)	8	0
zv	(1836)(2547)	(ALCJ)(BIDK)(EHGF)	6	0
zw	(15) (37)(26)(48)	(AI)(BJ)(CK)(DL)	3	$4 \times 4 + {}^{4}C_3 = 20$
zx	(18) (27)	(AK)(BG)(DH)(EL)(FJ)	6	5 x 2 = 10

The number of essentially different structures is

$$n = \frac{1}{o(G)} \times \sum_{g\in \mathrm{G}} (\text{No. of really different structures fixed by } g.)$$

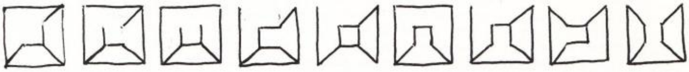
If reflections are not allowed $G \cong S_4$ and the number of fixed structures of each g is given in the top half of the table

$$n = \frac{1}{24}(1 \times 223 + 8 \times 4 + 6 \times 10) = 13.$$

If reflections are allowed on $G \cong S_4 \times Z_2$ and we can use the whole table

$$n = \frac{1}{48}(1 \times 223 + 8 \times 4 + 6 \times 10 + 3 \times 20 + 6 \times 10) = 9.$$

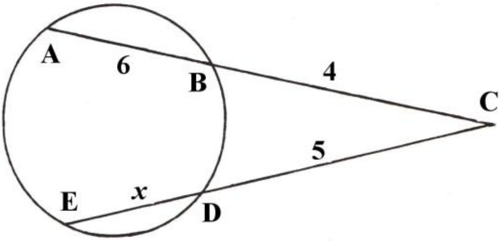
STEVE AINLEY actually drew the nine structures. They are



SOLUTION 55.4 PYRAMIDS *Take a regular tetrahedron and join all the midpoints of the edges by straight lines. How many tetrahedra are there now*?

This is quite difficult to see. Only STUART POTTER got the right answer which is 25. He says: When the four regular tetrahedra are removed from the vertices an octahedron remains. This is dissected into eight tetrahedra. Pairs of these eight can be put together to make twelve more. That's 24 and the original one makes 25.

SOLUTION 55.5 CHORDS *Find x.*



x is 3. There are many ways to show it. For instance ACE and DCB are similar triangles (Ext angle of cyclic quad equals interior opposite.)

Solutions came from STEVE AINLEY, TOM DALE, BOB ESCOLME, PAUL GARCIA, ANGUS MACDONALD, SIDNEY SILVERSTONE and my friend LIBBY DRAKE.

PROBLEM 57.1 SEQUENCES 57 RICHARD AHRENS

Richard has sent this problem from the 1978 International Mathematical Olympiad.

a = f(l), f(2), f(3), ... , and b = g(l), g(2), g(3), ... are two sequences of positive integers. Every positive integer is in either a or b, but not both. a and b are increasing.
$\forall n$, $g(n) = f(f(n)) +1$.

Find f(240) without finding all previous terms.

PROBLEM 57.2 FUNCTIONS 57 BOB ESCOLME

Bob has sent this from the 1977 Olympiad.

Given a function f: ℕ →ℕ where ℕ is the set of positive integers, prove that if f(n+1 >

$f(f(n))\ \forall n\in\mathbb{N}$ then $f(n) = n\ \forall n\in\mathbb{N}$.

PROBLEM 57.3 GARMENTS RICHARD AHRENS

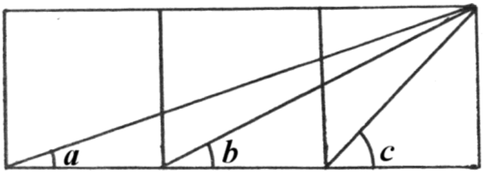
a) A and B work in the garment industry. A can make nine garments while B is making five. However, the quality of their work is such that three of B's garments earn the same profit as seven of A's garments.

Find numbers to represent the relative values of A and B to their employer.

b) Why do we add (or average) marks when determining an overall grade for a student? Would it be more sensible to multiply?.

Richard says that this problem owes a lot to Lewis Carroll - *A Tangled Tale.*

PROBLEM 57.4 ANGLES 57



The square things are squares. Show that $a + b = c$.

There are many proofs of this. Charles W Trigg collected fifty five. See how many you can find.

Since I expect the response to overwhelm me I shall not set a fifth problem this time. (Well - you got six in the Libby's Hats issue, didn't you?)

Brian Woodgate once asked for a book on prime number theory. I think I have mentioned before

Elementary Number Theory by Underwood Dudley, Freeman, 1969.

Section 21 is called "Formulas for Primes", section 22 "bounds for $\pi(x)$". I think the book is good. It contains a bibliography of books Dudley thinks are good.

One book included in the bibliography is L E Dickson's *History of the Theory of Numbers*. He says that when you make a new discovery in the theory of numbers look in Dickson to find out who has already used it before 1918.

(*Ed - He won't have out prime generating polynomial though. For something like that try* Concepts of Modern Mathematics *by I Stewart, Penguin* 1975.)

An interesting problem came up in one of those computer magazines recently (*Datalink*, 30 October). I suppose it's old hat to those involved in the business but I found it puzzling. First, how do you code a date, unambiguously, so as to be able to compare it with another. This is not difficult, of course. Two methods are mentioned in the article. One views the date as constructed from three fields: year, month and day. Multiply the first field by the maximum range of the second, then add the second, less one; and so on. To keep the numbers from growing too large DEC start from the year 1964 so the formula becomes

$$(\text{year} - 1964) \times 12 + (\text{month} - 1) \times 31 + \text{day} - 1.$$

The second method, used by ICL, merely counts the number of days elapsed since 31.12.1899.

The problem is, given a number, to produce the corresponding date as economically as possible. Obviously with the first method it is easy: you just reverse the formula (go on; do it:) Equally obviously the way to do it in the second case is to buy a set of calendars and count. However it should be possible to produce a formula. (Don't forget the leap years - there was not one at 1900 but there will be at 2000.)

Here are a couple more quotes from *The Times*. (Do you remember *The Times'*?) November 7: "St John the Evangelist has rewritten his Gospel after a course in OU-speak at the Open University: At the initial moment in time was a verbalization situation, and the verbalization situation was in the environmental totality, and the verbalization situation was the environmental totality. The same was in an ongoing linguistic doghouse situation."

I copied that for two reasons: first it isn't often the OU gets a mention of any kind; and secondly if anyone is being attacked it certainly is not the maths faculty. I can't think of any reason for putting in the next, except that it might raise a laugh though it deserves the opposite reaction. It was picked up by Philip Howard and printed on the 28th November.

"Least attractive event of today, unless you know NHS-speak: a lecture at the London School of Hygiene and Tropical Medicine by Professor M Aitkin. The synopsis goes: 'Exponential and Weibull models may be fitted to complex survival data in GLIM using the EM algorithm to give maximum likelihood estimates of the parameters when some lifelines are censored'."

I've just done a simple calculation. If M500 ran to twenty pages, and if it came out ten times a year, and if, furthermore, the SOCIETY had four hundred members, then all I would need to fill the magazine with no problem at all would be half a page from each member a year. Surely that couldn't possibly hurt.

Finally, since Jeremy has only given you four problems this month, here is one from me. It comes from *The Changeling* which I saw the other night:

A fool before a knave, a fool behind a knave, and between every two fools, a knave. How many fools and knaves?

Quick! Got it? Too late.

Eddie Kent.