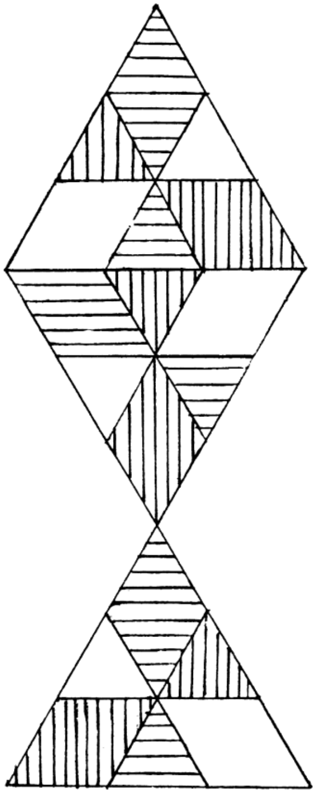

M500 58

M500 is a publication of The M500 Society for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for publication of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names, addresses, telephone numbers and details of past and present courses of voluntary members, by means of which private contacts may be made to share 0U and general mathematical interests or to form self-help groups by telephone or correspondence.

MATES is a special list of MOUTHS members who have explicitly volunteered for their MOUTHS data to be distributed to members in closed institutions such as prisons and special hospitals.

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Cover design is by Tony Brooks. This design is based on a self-replicating seven sided re-entrant polygon. Nine of these polygons fitted together form a larger polygon identical in shape to the original polygons. This process could clearly be repeated indefinitely. For obvious reasons this design is usually known as the fish.

Printer Graphics Ad Lib, 40*a Oxford Street, Southampton.*

ADVERTISMENTS

Colin Mills - A plea to all ex-M202 students—well all right then a small ad to go in the next M500. The last MOUTHS list will have told anyone interested (that's interesting, I wonder who the other person was?) that I'm doing M211 this year and I can now tell you for nothing that I probably will do M311 when it's available. I've been trying to get all four textbooks for M202 as it would be useful background reading for both courses, but Eddington Hook tell me that

M. Minsky: *Computation - Finite and Infinite Machines* is now out of print.

Does anyone in M500 have a spare copy of this priceless tome (well, it is because you can't get it new and it'll remain so unless someone lets me have one) or know someone who will part with their copy. I don't mind about the condition providing that the cover is still in one piece and there aren't any pages falling out (if you don't know what I mean you haven't read Marion's article in M500 55 3) .

Joyce Smith - New M332 Units and TMAs, exam papers etc, also Broadcast Notes. The Units are duplicate and unused due to repeat of course. £5 plus postage (or best offer).

Joyce Smith (C0109038) Bournemouth.

Pauline Eldridge-Lusted - For sale:

1. Texas TI-30 calculator; Texas RK-3 recharge unit; Texas lined carrying case. About £20 the lot.

2. Nering, E. D., *Linear Algebra and Matrix Theory*. £3.50.

3. Seymour Lipschutz, *Theory and Problems of Finite Mathematics*, Schaums Outline series; as new. £3.

Pauline Eldridge-Lusted.

To equations simultaneously Pellian
My approach is Machiavellian
Anything goes, rather than resort to such actions
As covering the walls with continued fractions.

J A Lindon (unpublished)

PHD? OR NOT PHD? BRIAN STEWART

I am in what will be my last year with the OU, and appalled at the thought of returning to a life filled with evenings in front of the box, and no more of those lovely brown envelopes dully thudding through the letter box (nobody else writes to me) I sent away for a copy of the Postgraduate Prospectus. I was slightly surprised at what I found.

Whereas the undergraduate student in the OU is obviously seriously disadvantaged by the need to study at a distance with little contact with other students and is therefore given a great deal of encouragement (the tutoring and counselling network, Open Forum, *Sesame*, self-help groups), the graduate who then wishes to do postgraduate study part-time, at a distance, away from his supervisor, and obviously seriously disadvantaged by the very solitude of it all - is given not any help, but an extra hurdle to overcome before he can start. He/she has to write a 5-600 word essay in his intended research topic (which immediately strikes me that you have to know a fair bit about it in advance).

In one sense this is fair enough: with so few places available there has to be some method of natural selection, and the people with the best ideas should end up with the chance to do the actual research. But someone like myself who has done quite well in the undergraduate courses and has the prospect of a good honours degree at the end of this year (unless M321 is as bad as everyone says it is) probably feels the idea of research is fairly pleasant; the natural next step, where the climax to years of study is to actually create a piece of mathematics. The vagueness of the research headings leaves me slightly bemused, and I know that once I start seriously studying this year I won’t have much time to spare or mental energy to investigate the topics and write the essay - the only way I can see to do it is to take next year off and work out what to do.

The headings may or may not mean much to people - they certainly mean little to me at present. I did M332 last year and it’s about Complex Analysis; so a research topic on Complex Analysis - well, we know what’s going on there. What we’re not quite so sure about is what areas have been developed and what kind of thing the research is going into. Also other cryptic titles like Graph Theory: clearly about the theory of graphs - but what kind of background is needed? If you are good at analysis, and enjoy it, is Graph Theory going to be your thing, or is your forte just irrelevant? What is it about? What are they trying to do?

What I’d like to see is more guidance. Not to be nursed into a project, but to be given the information so I can decide, on the basis of my passes and strong points, which areas I could be expected to be interested in, or if in fact I should just abandon the idea. I could sort out the problems of doing the research - such as being ninety miles from the nearest University (Aberdeen) - myself, and I’d feel prepared to write that initial essay.

I’ve seen honours graduates at a conventional university being shepherded into a research project. They have had the advantage of being in a research atmosphere for years. They know what's going on, which fields are being opened up and which merely being tidied up. They know the supervisors and they have easy access to previous material such as past theses so that they can see the standard asked for and can work out whether it is for them or not.

So how about being given a bit more idea of what these research topics involve? Then perhaps it will be clearer to us if, say, someone is particularly good at and fascinated by topology which topics, if any, would involve that as a major component, and so where he should look to for a research idea. If no idea comes, or if the idea isn't good enough, then OK. But the thinking would be educational in itself, and not just a futile stab in the murk.

Just give us more of an idea. Tell us the titles of the recent books in certain fields of research. Give us the names of the magazines that carry research publications. Let us see past theses. Then we can make up our minds if we want to fight for these pitifully few places, or if we'd be better retiring to the real world (and our wives).

CONTINUED FRACTIONS   LEWIS JOHNSON

The recent spate of interest in continued fractions (Tom Dale (55), Steve Ainley (56)) prompts me to pen the following note.

Most mathematicians are surprised to see continued fractions figuring prominently in the very prosaic syllabuses for production engineers; however, consider the following very basic problem. Helical milling is a process for cutting a helical groove on a cylindrical surface and is a fundamental operation for a wide range of production problems, e.g. making twist drills, gear-cutting and the like. In this operation the cutter rotates about a fixed vertical axis and the cylindrical workpiece moves horizontally whilst rotating at a suitable speed.

The horizontal travel is produced by a lead screw of fixed pitch (of the order of 3 mm) and this is positively geared to the workpiece through a gear train of ratio, say, n. Then

$$n = \pi d \tan \theta / p$$

where d is the workpiece diameter, and θ the required helix angle. Now since gears must necessarily have an integral number of teeth n is rational whereas $\pi d \tan \theta / p$ is clearly irrational. Herein lies the production engineer's dilemma. He must find a rational fraction which differs from a given irrational number by an amount not greater than the allowable design error (tolerance in engineering parlance).

This is where continued fractions come in. Suppose by way of example $d = 50$, $\theta = 20°$ and $p = 3$. Then $n = 19.057$ correct to one part in 2 E5. It is simpler to reciprocate and try to approximate the fraction 1000/19057. By a modification of Tom Dale's ladder technique the work proceeds as in the following table:

		PQ	p_r	q_r	diff	error
1000	19057	19	1	19	+1 6E–4	+1.6E–4
57	1000	17	17	324	–1.0E–5	–6E–6
31	57	1	18	343	+4.4E–6	+4E–6
26	31	1	35	667	–4.0E–7	–4E–7
5	26	5	193	3678	+1.4E–8	+1E–8
1	5	5	1000	19057	0	0

PQ signifies a partial quotient obtained by dividing each remainder into the previous divider. The successive convergents are p_r/q_r obtained as described by Steve Ainley (56 p.l2).* The column marked 'dliff' shows the difference between successive convergents. It can be shown that $p_r/q_r - p_{r+1}/q_{r+1} = 1/q_r \cdot q_{r+1}$ so can readily be estimated; since the signs of these differences alternate and it is clear that 1/19 is too large then the sign of each difference is known; by adding we could, if desired, estimate the error of each convergent. The coming of the pocket calculator has robbed this trick of much of its value.

The above example, although chosen at random, is a little exceptional in that all the errors are very small. Even the second convergent 17/324 is only –6E–6 in error and would be quite acceptable in the given circumstances. Paradoxically the engineer would, in this case, choose the convergent 18/343 as the solution to his problem. Gear wheels suitable for machine tools cannot be readily produced with tooth numbers less than 18 or greater than 150 (the latter would have a diameter of about 300 mm). Since 18/343 = 30/140 ×24/98, or better still, 24/40 × 20/56 × 20/35 × 24/56, gears are readily available to provide this ratio.

Ed - As I remember, the rule for finding p_r is multiply PQ_r by p_{r-1} and add p_{r-2}. Similarly for q_r. It would help if authors remembered that it is possible that some people have never seen a particular article they are referring to: a brief note could save a lot of frustration.

OBITUARY - DON MANSFIELD

The first annual report of the School Mathematics Project covers the year 1961-2. Eight schools (four grammar schools and four public schools) formed the initial core out of which the project developed. One of these schools was Holloway School, and its senior mathematics master was Don Mansfield. The report records that

> "... in September 1962, pupils in all the schools started according to the following schemes:
>
> 1. At 11+, in the four Grammar Schools, Mansfield and Thompson's book *Mathematics: a new approach* - Book 1 (Chatto and Windus, 1962) is being used. This is the first of three texts on material which has been tried out for some years at Holloway School."

Typical of Don Mansfield, more or less alone and unaided, he had "for some years" before SMP came into being, been working on a modernisation of the school mathematics curriculum. Not with any pretensions of effecting a national reform, but because he wanted to teach mathematics more interestingly and to a better result for the boys of Holloway School. And also typically, within a very short time, he had dropped out of the SMP; his freedom to act in the way he thought best for his students and colleagues was threatened.

Don taught at Holloway School for some seventeen years. This was followed by many years when he effectively worked "freelance" from his home in East London. He was in the Nuffield Mathematics Project at its inception and edited (after writing large sections) the Nuffield Teachers' Bulletin, which later became an independent journal for teachers. Television and radio programmes included the two year Mathematics Today series, which attracted a record audience. He examined for more than one Board, both GCE and CSE, where again he was there at its inception. Beginning with the school texts mentioned above, he also wrote a good deal.

In many of these activities he had colleagues, but when it came to hard work he always did the lion's share. Excuses were not part of his repertoire. He lived up to his promises, if at all humanly possible, and when he failed, often through no fault of his own, it troubled him greatly. Always too generous with his time, he would spend the "normal" working hours helping others, and then burn the midnight oil to finish his own work.

In 1970 he was persuaded to join the embryonic Open University as a staff tutor, and two or three years later he became a senior lecturer. As usual, having agreed to work for the OU he threw himself into his new activities whole heartedly. His "province" as staff tutor stretched over the vast area of eastern England from Bedford to Great Yarmouth. The extreme study centres had also to be visited, even if it meant coming back at one in the morning.

What has been written so far is a mere superficial listing of some of Don Mansfield's activities in thirty prolific and hectic years. Those who knew and worked with him took his immense energy and creativity for granted. Many worked with him for a long time before they realised that he had two major "handicaps"; he had never been to university and thus had had no formal mathematics education beyond school and, more seriously, from the beginning of his working life to the end, he was a critically ill man. In the late forties and early fifties he was operated on repeatedly for cancer of the kidneys, until it reached a stage where the doctors decided that further operations were impossible; if the disease spread any further there was nothing left to remove. Miraculously, it did not spread. After the last operation, he was left with a catheter sticking out of his back, and for the next twenty five years the catheter was changed once a month. And yet the only outward signs were hardly noticeable: the slightly odd shirts, the stick he used as added security and the stoop.

The incredible thing was that Don Mansfield not only found a way of living with his handicaps, but effectively turned them to his advantage. He learned new areas of mathematics as needed. At the Open University he worked on the tough pure mathematics courses, having to learn most of the material for himself first. The result was that he acquired his own insights and a method of exposition that was uniquely his own. His physical condition also meant that he preferred to work at home and avoid the social and semi-professional trappings of an academic profession. The result was a unique independence and integrity, and explains why so often he was involved in new departures.

In November 1976, Don Mansfield was awarded a BA by resolution of the Senate of the Open University. Typical of him, he was rather surprised that someone had bothered and mildly delighted. Modest and generous at all times, he was possessed of a fierce intellectual independence coupled with an almost total disregard for the value of his own work. When he died on Boxing Day, he left no enemies, but many students and colleagues heavily indebted to him and who were grateful that, against all the odds, he had lived and that they had had the privilege of knowing him.

Maxim Bruckheimer.

It is with regret that I record the deaths of two more people whose work might have struck wonder in the heart.

Last August Mr Edward Kleinschmidt, the man who invented the teletype machine, moved on at the age of 101. He was born in Bremen and moved to America when he was eight where he received no formal education but wound up with an honorary degree from Brooklyn Polytechnic. I believe he also made quite a lot of money.

And then, on the 9th of February, Professor Denis Gabor died. He’s the man who got the Nobel Prize for the hologram

I would love to write more about both these gentlemen but (a) I’ve had complaints about my obsession with death; (b) I no longer have *The Times* to help me; and (c) I’ve not yet managed to find a library in London. I believe they do exist, but not like back on the farm.

Eddie Kent

GAUSS XII

JEREMY GRAY

I begin with answers to the remaining problems set in Episode 4, and some comments concerning them.

To construct a regular pentagon is equivalent to solving $x^2 + x - 1 = 0$ or constructing a segment of length $x = \frac{-1+\sqrt{5}}{2}$. Pacioli's elegant method situates a square ACBD of side √2 in a circle of radius 1 centre 0. If E is the midpoint of AO, EO has length ½ and EC is $\frac{\sqrt{5}}{2}$ so OF is $\frac{-1+\sqrt{5}}{2}$ and is the base of an isosceles triangle of side 1 and angle 36°. Ten such triangles fill up a decagon exactly.

To construct a regular 15-gon construct a pentagon (by taking alternate vertices of the decagon). At each vertex construct an equilateral triangle inscribed in the circle and with one vertex the chosen vertex of the pentagon; the $5 + 5 \times 2 = 15$ vertices form a regular 15-gon.

This method enables one to construct 5·17-gons, for example, but not 3^n-gons ($n > 1$), 5^n-gons or any other p^n-gons except 2^n-gons.

The area of a spherical triangle is a classical argument; I was sent a marvellously coloured ball by Mr Wilkins together with impeccable reasoning to the desired result. Let the triangle be ABC with angles α at A, β at B and γ at C; extend the sides to great circles. The double line at A which includes ABC has area (proportional to) 4α - if we assume the radius of the sphere is 1 the area of the double line actually is $4\alpha \left(= \frac{2\alpha}{2\pi} \cdot 4\pi = \frac{\alpha}{\pi} \cdot \text{area of sphere}\right)$. The area of the three double lines is then $4(\alpha + \beta + \gamma)$. But this area equals the area of the triangle counted twice in each line and therefore six times altogether, together with the rest of the sphere counted exactly once. So, letting the area of the triangle be δ, we have

$$4(\alpha + \beta + \gamma) = 6\delta + 4\pi - 2\delta$$

or

$$\delta = (\alpha + \beta + \gamma) - \pi.$$

This formula has a remarkable analogue in non-Euclidean geometry, as we shall see. I had hoped to set some very hard problems in number theory. Now I can!

$\widehat{N}$1. Prove that there are infinitely many prime numbers. This result goes back to the Greeks, and a proof can be found in Euclid (Book IX, prop 20). Assume that there are only finitely many primes: $p_1, p_2, \ldots, p_n$, and consider $p_1 \cdots p_n + 1$. We know there are infinitely many numbers - how many of them are prime? As you go out along the number line do the primes thin out, or are they distributed evenly? The first result which suggested there were a lot of primes was Euler's. Recall that the sum of the reciprocals diverges, written $\sum_{p=1}^{\infty} = \infty$ if you feel permissive. Euler showed that $\sum_{p \text{ prime}} 1/p$ also diverges, so there are more prime numbers than squared numbers, because $\sum_{n=1}^{\infty} 1/n^2$ is finite.

Problems then:

$\widehat{N}$2. Show $\sum_1^{\infty} 1/n^2$ is finite.

$\widehat{N}$3. Show $\sum 1/p = \infty$, p prime.

$\widehat{N}$4. In fact, $\sum 1/n^2 = \pi^2/6$. Can you prove it?

Euler's 'proof' of the result $\sum 1/n^2 = \pi^2/6$ is so delightful that I shall give a hint of it here.

A polynomial $p(x)$ with roots $\varepsilon_1, \ldots, \varepsilon_n$ can be written

$$P(x) = C_0(1 - x/\varepsilon_1)(1 - x/\varepsilon_2) \ldots (1 - x/\varepsilon_n).$$

Thinking of $\sin \pi x$ as a polynomial (!) gives

$$\sin \pi x = C_0 x(1 - x/1)(1 + x/1)(1 - x/2)(1 + x/2) \ldots .$$

C_0 can be evaluated by considering $\lim_{x \to 0} \frac{\sin \pi x}{x} = \pi = C_0$, so

$$\sin \pi x = \pi x(1 - x^2/1)(1 - x^2/4) \ldots .$$

But, by Taylor's Theorem,

$$\sin x = \frac{\pi x}{1!} - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \ldots$$

so, equating coefficients of x^3 in the two expressions

$$-\frac{\pi^3}{3!} = \pi\left(-\frac{1}{1} - \frac{1}{4} - \frac{1}{9} - \ \ldots\right)$$

i.e. $-\frac{\pi^2}{6} = -\sum 1/n^2$; i.e.

$$\sum 1/n^2 = \pi^2/6.$$

This breathtaking argument is not, of course, a proof. Euler knew that. But calculate both sides to thirty places of decimals (Euler did) and see if you feel like denying the validity of the conclusion. Twenty years later Euler came up with a proof. (His first argument, of 1743, is in *Opera* (1) 14; 138-155; his second I haven't found yet.)

This leaves me with $\sqrt{-163}$ and $e^{\pi\sqrt{163}}$. I shall defer this to a later episode, because this is a non-elementary jewel of number theory that deserves a detailed discussion. I must finish therefore with N4 and N5.

N4. *Show by examples that* $Ax^2 + 1 = y^2$ *has any solutions at all.*

Some examples were given of solutions for small A. The smallest solutions for A = 2, 3, 5, 6 are x = 2, 1, 4, 2 respectively. For A = 92 (Brahmagupta's problem), one has x = 120. But for A = 109 or 149 which Fermat proposed to his English contemporaries the least value of x is 15 140 422 455 100 and 2 113 760 020 respectively. That he picked these examples out is one reason we have for supposing he could solve the general case, since his method has not come down to us. The English, Wallis and Brouncker, solved the method by an iteration procedure which is impressive enough, but could not prove its validity (a much harder task). Their method follows on page 10.

Stage 1. Take $x_1 = 0$; $y_1 = 1$.

Stage $n + n + 1$. Calculate $K_n = y_n^2 - Ax_n^2$. Multiply both sides by $r_n^2 - A = s_n$ to get

$$(y_n r + x_n A)^2 - A(y_n + x_n r_n) = k_n s_n,$$

where r_n and s_n are to be determined as follows. r_n is positive, and as large as possible provided $r_n^2 < A$ ($s_n < 0$) and K_n divides $y_n + x_n r_n$. Then K_n divides $y_n r_n + x_n A$. Set

$$y_{n+1} = \frac{y_n r_n + x_n A}{|K_n|}; \qquad x_{n+1} = \frac{y_n + x_n r_n}{|K_n|}$$

and repeat until some $K_n = 1$ at which point you are finished; *mirabile dictu*, this process always works. For example when $A = 13$, so $r_n = 0, 1, 2, 3$ only the results are as in the table below and $649^2 - 13{\cdot}180^2 = 1$.

n	1	2	3	4	5	6	7	8	9	10	11
x_n	0	1	1	2	3	5	33	38	71	109	180
y_n	1	3	4	7	11	18	119	137	256	393	649
$K_n = y_n^2 - 13x_n^2$	1	–4	3	–3	4	–1	4	–3	3	–4	1
r_r	3	1	2	1	3	3	1	2	1	3	

The Wallis-Brouncker iteration procedure used to solve $13x^2 + 1 = y^2$

PELL'S EQUATION STEVE MURPHY

M500 56 seemed to be full of continued fractions and then in issue 57 Tony Brooks investigates the equation $km^2 + 1 = n^2$ whose solutions are clearly related to the continued fraction for $\sqrt{k}$. Solutions always exist provided k is not a perfect square. The equation is usually called Pell's equation.

The first systematic method for solving Pell's equation was given by William Brouckner in 1657. Wallis was said to have expounded the method which according to John Aubrey was rather an ingrained habit:

> Tis certain that John Wallis is a person of reall worth ... yet he is so extremely greedy of glorie, that he steals feathers from others to adorne his own cap; e.g.- he lies at watch, at Sir Christopher Wren's discourse, Mr Robert Hooke's, Dr William Holder, &c; putts downe their notions in his Note booke, and then prints it, without owning the Authors.

The association of the equation with John Pell is said to be a confusion of names (by Euler, no less) . Incidentally Pell seems to have been a friend of Aubrey's so he says nothing nasty about him though 'he dyed of a broken heart'.

If we expand $\sqrt{k}$ as a continued fraction we get an expression of the form

$$\sqrt{k} = (q_0;\, q_1,\, q_2,\, \dots ,\, q_r,\, 2q_0)$$

(for notation see *Continued fractions* by Steve Ainley M500 56 12) . Now work out A and B where

$$\frac{A}{B} = q_0 + \frac{1}{q_1 +}\ \frac{1}{q_2 +} \cdots \frac{1}{q_r}.$$

Then $A - kB^2 = (-1)^{r-1}$. If r happens to be odd then A and B give the lowest solutions of the equation $n^2 - km^2 = 1$. If r is even then they are solutions of the equation $n^2 - km^2 = -1$. But we find n and m in this case such that

$$(n + m\sqrt{k}) = (A + B\sqrt{k})^2.$$

For example, for $k = 29$ we have the continued fraction $\sqrt{29} = (5;\, 2,1,1,2,10)$.

$$\text{Thus } \frac{A}{B} = 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}} = \frac{70}{13}.$$

Now $(70)^2 - 29(13)^2 = -1$ and we get the lowest solution of $n^2 - 29m^2 = 1$ from

$$(n + m\sqrt{29}) = (70 + 13\sqrt{29})^2 = 9801 + 1820\sqrt{29}.$$

References: *The Higher Arithmetic* by H Davenport, Hutchinson 1952 (and lots of standard books).
Brief Lives, Aubrey, Penguin edition page 136.
Continued fractions by S Ainley, M500 56 12.

PROBLEMS COMPILED AND EDITED BY JEREMY HUMPHRIES

Partly in deference to Eddie, mainly because I can't think of anything, this preamble is not prolix.

SOLUTION 55.1 BLACK AND WHITE *Remove two black and two white squares from a* $(2n)^2$ *chessboard. Show that if at least one black and one white square removed is interior the remainder can be tiled with* 2×1 *tiles.*

Eddie has written to say he is surprised nobody answered this and he submits his own solution:

A $(2n)^2$ board can always be tiled. Tiling remains possible after some squares are removed provided that 1) parity is preserved; 2) there remain no isolated squares.

Here 1) is ensured since $2 \times 2 = 4$ and $(2n)^2$ are both even squares.

2) is ensured by stipulating that at least one of the two missing black squares is interior and ditto for white. The only white square that can be isolated by the removal of two blacks is a corner square, when both removed blacks are edge squares. And ditto again.

There is an algorithm for tiling what remains. Let S_1 be a square which joins only one other untiled square. Look for an S_1. If there isn't one tile a $(2x)^2$ square of the board, which x you may as well make as large as you can, and look for an S_1 again. Proceed until an S_1 appears. Begin tiling at $S_1$and continue until there is more than one 'next move'. Stop. You have completed an unambiguous path U. Complete every U. Then start tiling $(2x)^2$ squares again (x any number of course) until more Us appear. Complete Us and so on. At some point in this process the board will be tiled.

SOLUTION 56.1 HIDE AND SEEK *Hider and Seeker simultaneously choose points on a closed disc of unit radius. H escapes if his point is more than* 1 *unit distant from S's point. Show that if both play optimally H will be caught with probability* $^1/_7$.

Nobody solved this completely though SIDNEY SILVERSTONE was well on the way to it with the right ideas. In the question as set the probability was printed as 1:7 which may have been confusing.

The closed unit disc can be covered by seven closed discs of radius k, centred at A,B,...,F,G where G is the centre of the unit disc and A,B, ... ,F are uniformly distributed at a distance $\sqrt{3}/2$ from G.

Let S play at random on one of the seven points A,B,... ,G. Then the probability, P, that S catches S is > $^1/_7$ regardless of H's strategy.

Now let H play on G with probability $^1/_7$ and on an arbitrary point on the boundary of the unit disc with probability $^6/_7$. We must show that S catches H with probability ≤ $^1/_7$ regardless of

S's strategy.

Let X be the disc of half unit radius centred on G. H's strategy ensures that the probability that S catches H is positive if and only if S and H either both play in X, or both play outside X.

If both play in X the probability that S catches H is $p = 1$.

If S and H both play outside X, let q be the probability of capture. Then $q \le {}^1/_6$, since an arc of the unit circle contained in a disc of radius ½ has length $\le 2\pi/6$ (since its chord is ≤ 1). Let r be the probability that S plays in X. Then probability of capture is

$$pr({}^1/_7) + q(1-r)\,({}^6/_7) = \frac{r+6q(1-r)}{7} = P.$$

But $6q \le 1$. Therefore $P \le {}^1/_7$. $\therefore\ P = {}^1/_7$.

(*Ed - Jeremy sent a diagram to go with this solution but it was in pencil and much too large so I couldn't reproduce it. Still I'm sure you can draw your own if you feel it necessary.*)

<u>SOLUTION 56.2 TRIANGLES</u> *Solve* $\frac{m(m+1)}{n(n+1)} = 2$ *for integer m, n.*

Eddie named this problem because, he says, "I noticed that $m(m+1)/n(n+1) = 2$ is equivalent to

$$\frac{m(m+1)}{2} = \frac{2n(n+1)}{2}$$

which means we are looking for pairs of triangular numbers (t, 2t). A glance at Pascal's triangle will show that 6 is twice 3 and 210 is twice 105; and (3,6) → (2,3); (105,210) → (14,20), etc, by the map

$$x \to -\tfrac{1}{2} + \sqrt{\frac{1+8x}{4}}."$$

STEVE MURPHY says that from m=3, n = 2 we can get succeeding values by applying the equations

$$m_{x+1} = 3m_x + 4n_x + 3, \qquad n_{x+1} = 2m_x + 3n_x + 2.$$

JOHN HADLEY and ANGUS MACDONALD used triangular numbers, like Eddie.

1	**1**
2	**3**
5	**7**
12	**17**
29	**41**
.	.
.	.
.	.
a	***b***
a+b	***2a+b***

HOWARD PARSONS found values of (n, n+1) from the square root ladder generated by (1, 1). Cross multiply successive rungs of the ladder, e.g.

$$12 \times 7 = 84 = n \qquad 5 \times 17 = 85 = n + 1$$

whence $m = 119$.

STEVE AINLEY used a similar method. SIDNEY SILVERSTONE pointed out that any pair (m,n) generates three more if we allow negative solutions. e.g. (3, 2) gives

(–4, 2), (3, –3) and (–4, –3).

LESLIE GLICKMAN, a recent recruit, found that if we let $m(m + 1)$ equal $(x – 1)x$ and $n(n + 1)$ equal $(y – 1)y$ so that $y(y – 1)/x(x – 1) = ½$ we have Fred Mostello's (the MDT241 fellow) sock drawer problem (50 *Challenging Problems in Probability* F Mostello, Addison Wesley, 1965):

A drawer contains y red socks and z black socks. When two socks are drawn at random without replacement the probability that both are red is ½. What is the smallest possible number of socks in the drawer? The probability is $\frac{y}{y+z} \times \frac{y-1}{y+z-1} = \frac{1}{2}$. Let $z = x – y$ and we are back to our problem, except that we find all possible numbers of socks (and some impossible ones from Sidney).

Solutions also came from TONY FORBES, THURSTON HEATON, FRANK SPRINGALL, BRIAN STEWART and JOHN HULBERT.

<u>SOLUTION 56.5 REAL PROBLEM</u> *Part* 1. *Is there an integer x such that* $6x^2 – 105$ *is a square*? *Part* 2. If $5x^2 \pm 4$ *is a square to what family of numbers, F, does it belong*? *Show that* $5x^2 \pm 4$ *is square* $\Leftrightarrow x \in F$.

Part 1. No there isn't. I got several demonstrations, some of which were rather long. STEVE AINLEY's was compact.

$6x^2 – 105 = A^2 \Rightarrow 3(x^2 – 35) = A^2 \Rightarrow 3| A$, say $A = 3B \Rightarrow 3(x^2 – 35) = 9B^2 \Rightarrow x^2 – 35 = 3B^2$.

Now $x^2 = 0$ or $1 \pmod 4) \Rightarrow x^2 – 35 = 1$ or $2 \pmod 4$; but $3B^2$ is 0 or 3 (mod 4). Contradiction.

RON WHEELER said at the end of his solution that he looked forward to seeing it done in two lines, so here is TONY FORBES:

$6x^2 – 105$ is divisible by 3 but not by 9 therefore cannot be a square.

The other people who sent answers were JIM EZECHIEL, JOHN HADLEY, THURSTON HEATON, ANGUS MADONALD, A K MOSLEY, STEVE MURPHY, HOWARD PARSONS and JOHN HULBERT, who set this one, as well as the last one. John must be congratulated on these problems which produced a larger than usual response.

Most of those who answered Part 1also spotted that in Part 2 the family F is the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... STEVE AINLEY says that the y form another interesting sequence: 2, 1, 3, 4, 7, 11, 18, ..., devised, he thinks, by Lucas.

This is TONY FORBES's proof for the iff part:

Put $w = \frac{1+\sqrt{5}}{2}$, $\overline{w} = \frac{1-\sqrt{5}}{2}$; $g_n = w_n + \overline{w}_n$, $f_n = \frac{w^n - \overline{w}^n}{\sqrt{5}}$, $n = 0, 1, 2, ...$, so that g defines the sequence 2,1, 3, 4, 7, ... and f defines the Fibonacci sequence. Note that $w\overline{w} = –1$ and it follows that $(y, x) = (\pm g_n, \pm f_n)$ is a solution of $y^2 – 5x^2 = \pm 4$, $y, z \in \mathbb{Z}$. (∗)

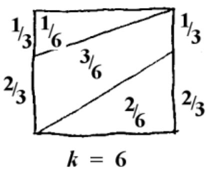
To show that there are no other solutions put $\langle\alpha, \beta\rangle = \frac{\alpha + \beta\sqrt{5}}{2}$ and note that $\langle\alpha, \beta\rangle\langle\alpha, -\beta\rangle = \pm 1$ $\Rightarrow \langle\alpha, \beta\rangle$ satisfies (∗). Also $\langle g_n, f_n\rangle = w^n$. Now suppose (a, b) is a solution of (∗) such that $w_n < \langle a, b\rangle < w^{n+1}$. Multiplying by $w^{-1} = \langle -g_n, f_n\rangle$ gives

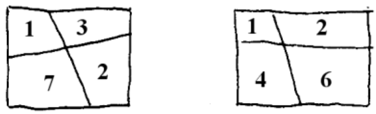
$$1 < \langle a, b\rangle\langle -g_n, f_n\rangle < w$$

and $\langle a, b\rangle\langle -g_n, f_n\rangle = \langle c, d\rangle$ for some $c, d \in \mathbb{Z}$. Furthermore $\langle c, d\rangle\langle c, -d\rangle = \pm 1$ and therefore (c,d) satisfies (∗), which is impossible.

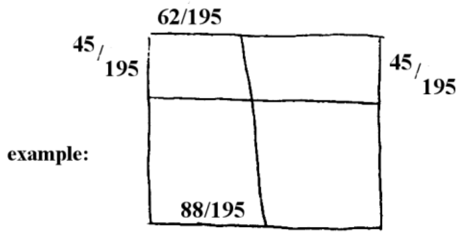
SOLUTION 56.4 SQUARE DIVIDING a) *Divide a unit square into parts by two straight lines so that by taking one or more contiguous parts you can get areas of* 1/*k*, 2/*k*, 3/*k*, ..., 1 *for largest possible k.* b) *Same with more lines.*

STEVE AINLEY set this problem and sent these suggestions for part a.





Area divisions; give $k = 13$



ANGUS MACDONALD showed that if we divide the square orthogonally we can get $k = 2^m - 1$ where m is the number of rectangles produced by n lines.

n	1	2	3	4	5	6
m	2	4	6	9	12	16 ···

(Which continues, I think, $m_n = (n + 1) \pmod 2 + 2m_{n-1} - m_{n-2}$) e.g. four lines can give $k = 511$. (For diagram see next page.) But, as Angus says, these are not the largest number of divisions we can get.

The series for divisions of the plane by n lines goes

n	0	1	2	3	4	5	6	...
d	1	2	4	7	11	16	22	...

which suggests $(2^d - 1)$ =

k	1	3	15	127	2047	... .

However, I'm not sure that you could divide the square in the correct proportions for these values. I know you can't divide the circle into seven equal areas with three lines. Would anyone care to comment on any of this?

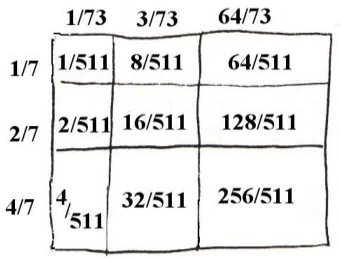


Diagram for the unit square divided orthogonally by four lines giving $k = 511$.

SOLUTION 56.5 WIT'S END *You'll have to look at* M500 56 *for this question which is much too long to repeat here.*

Great fun - everybody who sent in an answer got it right - I think. There isn't a unique solution.

We can set up a table with Singing, Laughter, Organ and Incense.

(O,I) \ (S,L)	00	01	10	11
00	01	01	10	10
01	00	00	11	11
10	11	01	00	10
11	10	00	01	11

Initially (S,L) is in (1,1) and we can see that one way to (0,0) is:

Stop the incense burning for one minute; next stop the incense and play the organ, then start burning the incense again, keep it burning and don't play the organ.

TONY FORBES sent an algebraic solution. Let S,L,I,O be functions from $\mathbb{Z}^+$ into the Boolean algebra $(\{\emptyset,1\},\wedge, \vee, \sim)$. Then, given that

$$S_1 = L_1 = 1$$
$$S_{t+1} = O_t \wedge \sim L_t \wedge \sim S_t \vee (O_t \vee L_t) \wedge S_t$$
$$L_{t+1} = I_t \wedge S_t \vee I_t \wedge \sim S_t$$

we have to determine O and I such that S and L eventually become zero. Clearly $O_1 = \emptyset$, $O_2 = 1$, $O_t = \emptyset$ for $t > 3$ and $I_t = \emptyset$ for $t = 1$ and 2, $I_t = 1$ for $t \geq 3$ will do the trick. STEVE AINLEY sent a poem which went:

1) Absence of incense will stop the laughter

2) Organ (no incense) then stops the chaunts
3) With incense (no organ) for ever after
You'll have beautiful silence at Wit's End, Haunts.

Other solutions similar to one or more of these came from ROY ADAMS, JOHN HULBERT, ANGUS MACDONALD, HOWARD PARSONS, SIDNEY SILVERSTONE, MARION STUBBS, ANNE WILLIAMS and BRIAN WOODGATE. The one I liked best was sent by fifteen year old MARY CHAMPION, daughter of CECILIA. She wrote detailed instructions and underlined the important bits in red, which looked very fine.

PROBLEM 58.1 LIMERICK

Write me a limerick which has some connection, however tenuous, with mathematics.

PROBLEM 58.2 582 NUMBERS TONY FORBES

Show that $x^2 - 7$ cannot be a cube. ($x \in \mathbb{Z}$).

PROBLEM 58.3 583 NUMBERS TONY FORBES

Show that $n^4 + 4$ is composite for $n = 2, 3, 4, \ldots$.

PROBLEM 58.4 PACKING EDDIE KENT (No solution known)

Show that 41 bricks of dimensions $1 \times 2 \times 4$ can be packed in a $7 \times 7 \times 7$ box. Is there a packing of 42 such bricks?

PROBLEM 58.5 POINTS

Three points are chosen at random on the circumference of a circle. What is the probability that they are the vertices of an acute angled triangle?

I don't know the answer - I only just thought of the question. Does 'random' need to be defined?

What is remarkable about 'abstemious' and 'facetious', apart from the fact that neither applies to your good self? Can you find words with the reverse property?

Ed - No! I haven't the least idea. But can you tell me what order these words are in?

Able	*Riot*	*Unasked*
Rink	*Tour*	*Male*
Ivory	*Dullard*	*Heinous*
After	*Sight*	*Tort*
Bunk	*Bare*	*Vinous.*

And when you've done that write me a sentence with the word 'has' three times in succession; no tricky punctuation, please.

EDITORIAL

In an idle moment I looked at Ann Wigmore's maze on the cover of M500 56, wondering if there is an algorithm to solve all mazes. It is obviously not 'go in and keep your right hand against the wall' because if you try that with this one you merely fall out of the hole in the bottom. (Keeping your left hand on the wall merely reverses this path; again you fall out.)

One method that does work with this particular maze is to walk to the wall opposite the entrance, put your left hand on the wall, and walk. If you keep going when you get to the house you end up at your starting point.

If you trace this route out with a pencil you are forced to ask certain questions, all concerned with what we mean by 'a solution'. In the first place this route is obviously not the shortest. This probably doesn't matter too much - if you are stuck in Hampton Court maze you will no doubt be pleased to get out at all even if it does take a few minutes more than it might. But on the other hand it is not the longest route and this seems to me a far more serious fault. There are some paths which you do not enter at all and even of those you enter there are some where you only touch one wall. This suggests that the possibility exists that there is no algorithm to solve all mazes, let alone find shortest routes.

Steve Murphy's article on Pell in this issue reminds me that Steve Ainley sent in a note a while ago in which he mentioned the coincidence of David Asche's Equivalence relations on a set (M500 53); Martin Gardner's article in *Scientific American* (May or June 78) and the publication of Steve Ainley's own *Mathematical Puzzles* (which I can't find from the information given here - Dillons say they don't know it) which contains a puzzle about Pell's numbers. He says 'one of their charms is the following:

1	1	2	5	15	52
2	3	7	20		
5	10	27			
15	37				
52					

etc. You can construct them by putting the last term at the top of the next column of "differences".'

Some snippets of information, culled from various sources. Oliver Penrose has been elected a member of the SRC mathematics committee. He also has a publication submitted: *An H theorem for K-systems*. Max Bramer has: *Representing pattern-knowledge for chess endgames: an optimal algorithm for King and Pawn against King* in 'Advances in Computer Chess' 2 (ed M R B Clarke). Stanley Collings has been appointed Academic Advisor to Plymouth Polytechnic. Alex Graham has written a book, to be published by John Wiley and Sons: *Matrix Theory and Applications for Engineering and Mathematics*. Maxim Bruckheimer (who wrote the obituary for Don Mansfield) is now Head of the Science Teaching Department at the Weizman Institute. Professor Frank George, head of Cybernetics at Brunel (nothing to do with us but why be parochial? This is from *Datalink* 27 November) has written a program to beat the pools. It won for his wife £1000 plus 25 first dividends for other punters during four weeks. Who says mathematics isn't useful?

Ellie Kent.