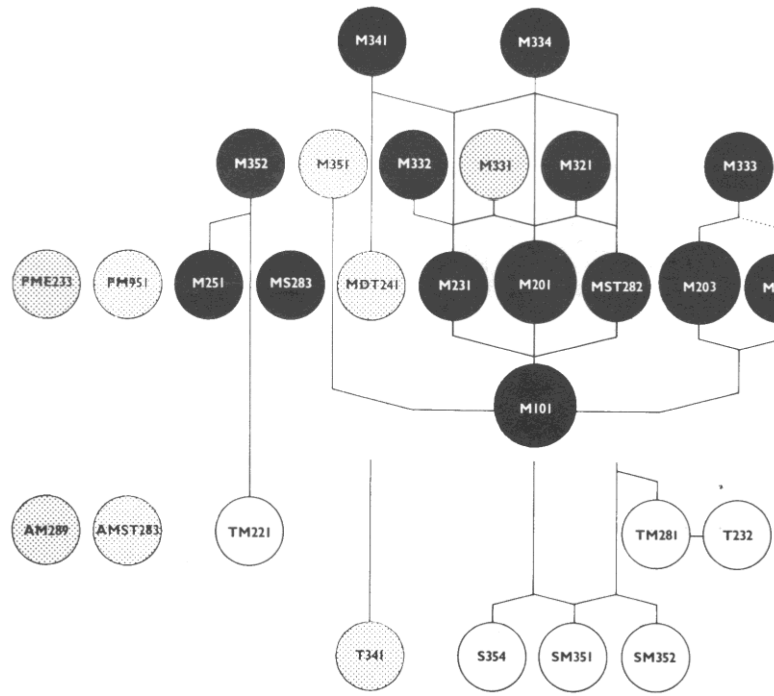
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Articles and solutions are not necessarily correct, but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

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The cover design is by Anthony Froshaug, and was originally designed for the M500 Special Issue 1979. Unfortunately, the Special Issue was abandoned just prior to printing. The cover design shows the relationships between OU mathematics courses and is intended as an aid to Conditional Registration, 1979.

****

MAP COLOURING PROBLEMS

ROBIN WILSON

Introduction

Until 1976, the four-colour conjecture was one of the foremost unsolved problems in mathematics. This famous conjecture - so simple to state yet so difficult to prove - is as follows:

> *Four-Colour Conjecture*: Suppose that we are given a map consisting of a number of countries, and suppose that we wish to colour these countries in such a way that any two neighbouring ones (that is, countries which share a common boundary line) are assigned different colours; then it is always possible to effect this colouring using only four colours.

At first sight it may seem difficult to believe that every map, however complicated, can be coloured in this way with only four colours, but a little experimentation is all that is needed to increase one's confidence in the conjecture. On the other hand, it is easy to see that "four" cannot be replaced by "three", since there are many maps which actually need four colours, such as the one in Figure 1.

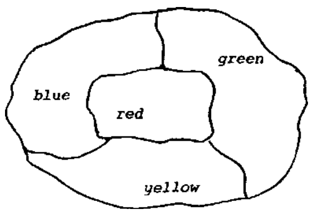


Figure 1

Our aim here is to outline the history of this celebrated conjecture from its first appearance in 1852 to its eventual proof in 1976 by two Illinois mathematicians, K Appel and W Haken. We shall see that the methods used in its proof include techniques which date back as far as a paper written in 1879, and we shall describe this early work in some detail.

We shall also mention an important related conjecture which extends the four-colour conjecture, and which also remained unproved for many years. This is the Heawood Conjecture, and concerns 'maps of genus k', that is, maps which can be drawn on the surfaceof a sphere with *k* 'handles' attached to it (or, equivalently, a doughnut with *k* holes in it), but not on the surface of a sphere with fewer handles. (These ideas will be discussed more fully later on.) Using this terminology, the Heawood Conjecture may be stated as follows:

> *Heawood Conjecture*: Suppose that we are given a map of genus *k*, and suppose that we wish to colour its countries in such a way that any two neighbouring contries are assigned different colours: then it is always possible to effect this colouring using $\left[\frac{1}{2}\left(7+\sqrt{1+48k}\right)\right]$ colours, but it is not possible, in general, to do this with fewer colours.

Further information concerning the history of these two problems, together with much of the original source material, will be found in [3].

THE EARLY HISTORY OF THE FOUR-COLOUR CONJECTURE

The earliest written reference to the four-colour conjecture occurs in a letter, dated 23rd October 1852, sent by Augustus De Morgan, Professor of Mathematics at University College, London, to Sir William Rowan Hamilton, in Dublin. In this letter, De Morgan described how one of his students had asked him whether every map could be coloured with just four colours; this student was identified some years later as Frederick Guthrie, who claimed that the problem was due to his brother Francis. Francis Guthrie had conjectured this result while colouring a map of England, and although he went on to a mathematical career he never published anything on the four-colour problem.

De Morgan quickly became intrigued with the problem, and communicated it to other mathematicians, so that it soon became part of mathematical folk-lore. In 1860 he mentioned it in an unsigned book review in the *Athenaeum*, and it is likely that this is the first printed reference to the problem. De Morgan's book review was read in the United States of America by the logician and philosopher C S Pierce, who soon obtained a proof of the four-colour conjecture (incorrect) and presented it at a seminar at Harvard University.

Another person who became interested in the four-colour conjecture was the distinguished mathematician Arthur Cayley, Sandlerian Professor of Mathematics at the University of Cambridge. At a meeting of the London Mathematical Society on 13th June 1878 Cayley asked whether or not the conjecture had been proved, and soon afterwards sent a short paper on the problem to the Royal Geographical Society. In this paper he described the problem in general terms, and showed that in trying to prove the four-colour conjecture one can make the simplifying assumption that exactly three countries meet at each point of intersection. To see this, suppose there are more than three countries meeting at a given point, as shown in Figure 2. If we stick a small circular patch over this point, and repeat this procedure for all other such points, we get a

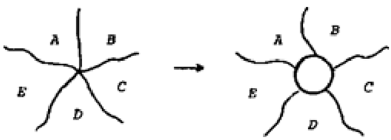


Figure 2

map with exactly three countries meeting at each point. It is now easy to see that if we colour the countries of the new map with only four colours, then we can immediately obtain a four-colouring of the original map, simply by shrinking each of the patches to zero. In view of this result we may assume that from now on all maps under consideration have exactly three countries meeting at each point.

KEMPE'S FALLACIOUS 'PROOF'

In 1879, A B Kempe, a London barrister who had studied mathematics under Cayley at Cambridge, produced what is now regarded as one of the most famous fallacious proofs in the whole of mathematics! Kempe's paper [5] purporting to prove the four-colour conjecture appeared in the newly founded *American Journal of Mathematics,* and the error in his proof remained undetected for more than ten years. In fact Kempe's argument, although incorrect, contains two important ideas which were to become the foundation for almost all subsequent attempts on the problem, including the successful one in 1976. In Order to understand the recent proof of Appel and Haken it will be necessary to describe these ideas in detail.

The first and less important of Kempe's ideas concerns the reformulation of the four-colour conjecture in terms of colourings of the vertices (or nodes) of a planar graph. If we choose a point (vertex) in each country of the map (for example, the capital of the country), and if we join two of these capitals by a line (edge) whenever they lie in neighbouring countries, we get a diagram of points and non-intersecting lines called a planar graph (see Figure 3).

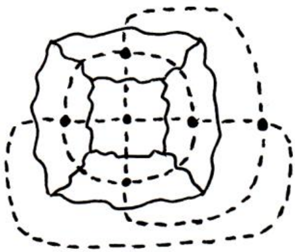


Figure 3

If we now assign to each vertex of the planar graph the same colour as the country containing it it is easy to see that the four-colour conjecture for maps is equivalent to the statement that the vetices of every planar graph can be coloured with only four colours in such a way that any two vertices which are joined by an edge are assigned different colours. The latter formulation of the problem was the one used by Appel and Haken in their successful proof. Since the formulations are interchangeable, we shall feel free to change from one version to the other whenever convenient.

The second, and more important, idea introduced by Kempe is the use of what is now called a 'Kempe-chain argument'. This involves looking at a part of the map which involves only two colours and then interchanging these colours. To illustrate this idea we shall prove a weaker result - the 'five-colour theorem' - which states that the countries of any map can be coloured with only five colours. Kempe's error was that in adapting this method of proof to the four-colour conjecture he performed two interchanges of colour simultaneously. It turns out that this is not permissible. (The following section will not be needed in what follows and may be omitted if preferred.)

THE FIVE-COLOUR THEOREM

Theorem: Suppose that we are given a map consisting of a number of countries, and suppose that we wish to colour these countries In such a way that any two neighbouring ones are assigned different colours, then it is always possible to effect this colouring using only five colours.

Proof. We shall suppose that we have a map which cannot be coloured with five colours, and we shall suppose, for simplicity, that this map contains the smallest possible number of countries subject to this restriction. Our aim is to show that these assumptions lead to a contradiction. Using a result known as Euler's Polyhedral Formula it is easy to prove that this map must contain a country *F* which is bounded by at most five other countries. If we now shrink this country down to a point then the remaining map has one fewer countries, and can therefore be coloured with five colours, by our original assumption. Replacing our original country *F*, we can now see that *F* must be surrounded by exactly five other countries, all differently coloured, since otherwise the countries around *F* would use up less than five colours, leaving a colour free to colour *F*; this is impossible since the map cannot be coloured with five colours. So the situation is now as in Figure 4.

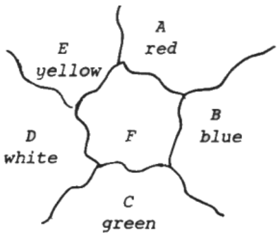


Figure 4

We now look at that part of the map which is coloured red and green. In this red-green part, the red and green countries *A* and *C* next to *F* may be joined by a chain of red and green countries, or they may not. We shall look at each of these possibilities in turn.

1. If there is no red-green chain of countries joining the countries *A* and *C*(as in Figure 5) then we may take the red-green part containing *A* and interchange its colours. This interchange of colours will not affect *C*, but will recolour *A* green, thereby enabling us to colour *F* red. But this means that the map has been coloured with five colours, contradicting our original assumption that this is impossible.

2. If there is a red-green chain of countries joining the countries *A* and *C* (as in Figure 6) then there cannot be a blue-yellow chain of countries joining the countries *B* and *E*. We may therefore take the blue-yellow part containing *B* and interchange its colours. This interchange of colours will not affect *E*, but will recolour *B* yellow, thereby enabling us to colour *F* blue. But this means that the map has been coloured with five colours contradicting our original assumption. This contradiction completes the proof.

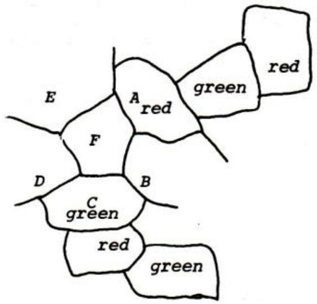


Figure 5

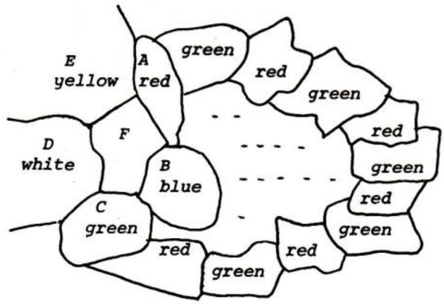


Figure 6

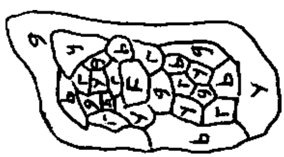
FROM KEMPE TO HEAWOOD

Kempe’s ‘proof’ of the four-colour conjecture was greeted with great enthusiasm, and Kempe was shortly afterwards elected a Fellow of the Royal Society. Among those interested in Kempe'’s work was P G Tait, Professor of Natural Philosophy at the University of Edinburgh, who reformulated the four-colour problem in terms of a colouring of the boundary edges of the map, rather than the countries. He then used this reformulation to obtain a very short (and incorrect) proof of the four-colour theorem.

Throughout the 1880s the four-colour theorem was regarded as an established fact, and it became very well known. Among those who mentioned it was Lewis Carroll, who turned it into a game in which one player draws a map for a second player to colour. The Headmaster of Clifton College in Bristol set the four-colour problem as a 'Challenge Problem' to the whole school, saying that “no solution may exceed one page, thirty lines of manuscript, and one page of diagrams”. This challenge problem appeared in the columns of the ***Journal of Education*** in 1887, and two years later in the same journal there appeared a’'solution’ by the Bishop of London, Frederick Temple (later Archbishop of Canterbury), who obtained it while “allowing his mind to wander” during a rather boring meeteing.

In view of all this it came as rather a surprise when in 1890 P J Heawood published an important paper **[**4**]** refuting Kempe's, well established proof. Heawood pointed out that Kempe’s error was in trying to carry out two interchanges of colour at the same time and gave an example, Figure 7, to explain why this is not permissible. However he did manage to salvage enough from Kempe’s paper to give a correct proof of the five-colour theorem similar to the one given above.

Figure 7



Heawood also considered the corresponding problem for maps drawn on various surfaces other than the plane. It is easy to show that any map drawn on the surface of a sphere gives rise to a corresponding map drawn in the plane, and conversely; to see this, we simply place the sphere on top of the plane, and project up or down using "stereographic projection" (see Figure 8). It follows that the four-colour conjecture and the five-colour theorem may be regarded equally as

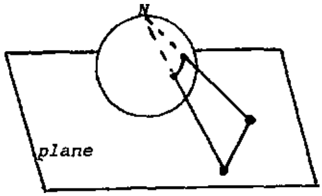


Figure 8

results concerning maps drawn in the plane or maps drawn on the surface of a sphere. In view of this it is natural to ask whether there are analogous results for surfaces other than the sphere, such as the torus (doughnut with one hole), and so on. Heawood was able to prove that every map drawn on the surface of a torus can be coloured with at most seven colours, and he gave an example of a map on the torus which actually needs all seven colours. So for maps drawn on the torus there is a 'seven-colour theorem'.

Unfortunately, not even Heawood was infallible! He believed that his method of proof for the torus generalized immediately to all surfaces of genus k (doughnuts with k holes), and he consequently claimed to have proved the analogue of the four-colour conjecture for all surfaces of genus $k > 1$. (This analogue was stated under the name of the 'Heawood conjecture'.) But Heawood's proof does not generalize in the way he imagined. His proof that every map drawn on such a surface can be coloured with

$$\left[\frac{1}{2}\left(7+\sqrt{1+48k}\right)\right]$$

colours is correct, but he failed to prove that there are maps which need this number of colours. This gap in Heawood's proof was not to be filled for almost eighty years.

FROM 1890 *TO* 1976

Progress on the two conjectures came slowly. In 1891, L Heffter proved the Heawood Conjecture for surfaces of genus $k \leq 6$, and in 1910 H Tietze obtained analogous results for graphs drawn on the projective plane or Mobius strip. Various writers between 1900 and 1930 such as Wernicke, Birkhoff and Franklin investigated the properties of maps on the plane which *fail* to satisfy the four-colour conjecture, thereby hoping to prove that such maps cannot possibly exist, or alternatively to obtain so many restrictions on such a map that it can be constructed explicitly. Such a method is now known essentially as the method of *reducibility*, and it played an important part in the eventual proof of the four-colour conjecture. Using the method of reducibility, Franklin proved that the four-colour conjecture is true for maps with not more than twenty-five countries.

After this little real progress was made on either problem until the 1950s when several cases of the Heawood Conjecture were solved. By this time it had become clear that the method of proof used in the Heawood Conjecture would depend on the number of countries of the map, and in particular on its remainder when divided by twelve. Throughout the 1950s and 1960s the various different remainders were investigated and solved, and the Heawood Conjecture was finally proved in 1968. Various mathematicians contributed to its proof, but the bulk of the credit must go toG Ringel and J W T Youngs (see **[6]).**

Meanwhile, progress was being made on the four-colour conjecture. In the late 1960s various authors started to develop the theory of 'unavoidable sets', which are sets of configurations at least one of which must appear on any map. At the same time the method of reducibility had been developed further and a large number of 'reducible configurations' had been obtained. By combining these two methods, K Appel and W Haken eventually managed in 1976 to obtain an unavoidable set of nearly two thousand reducible configurations, and this was enough to prove the four-colour conjecture (see **[**1**] .)** Further details of the theory of unavoidable sets and the method of reducibility, and of the way in which they were combined by Appel and Haken, may be found in their *Scientific American* article [2],

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5. A B Kempe, 'On the geographical problem of the four colours' *Amer. J. Math.* 2 (1879), 193-200.
6. G Ringel, *Map Colour Theorem***,** Springer-Verlag, 1974.

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Ed - *Not totally unconnected with the above* (*and to fill this last bit of space*) *can you arrange eleven* identical *squares so that four colours are necessary to prevent two squares of the same colour from having any part of a border in common*? *The answer will appear in a later issue but, more interestingly, can you find such a configuration needing* less *than eleven*?

AD

TONY FORBES

I see from M500 58 that you are publishing advertisements. If this is still the case do you think you could include the attached?

The following items are for sale (postage extra) .

BOOKS

Nering, *Linear Algebra and Matrix Theory*, £5.50.
Kreider *et al. An Introduction to Linear Analysis*, £3.50.
Gisberg, *Fundamentals of Modern Physios*, £9.50.
Gillespie, *A Quantum Mechanics Primer*, £3.
Mosteller *et al, Probability with Statistical Applications*, £5.
Halmos, *Naive Set Theory*, £2.50.
Minskey, *Computation: finite and Infinite Machines*, £3.50.
Manjallan, *Introduction to Modem Mathematics*, £1.

CALCULATORS

CASIO: "PROfx-1": (Scientific, Programmable with 126 steps and magnetic card read/write; price includes ten magnetic cards and mains unit), £55.

PRINZTRONIC: "1500M" (Pocket-size, scientific), £5.

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DEAR EDITOR

FROM PUBLISHER - MARION STUBBS

Well, as an ignoramus in mathematics who happened to found the M500 SOCIETY + its appendage MOUTHS and lately MATES, I offer you my annual half-page - print it or not as your daring dares.

When *I* - the supreme M Stubbs - was i/c This Operation, then I printed the utmost RUBBISH, mathematically and otherwise, from anyone who wrote. This included the first piece of rubbish submitted by your good-editorial-self, when you were a raw, unpassed M202 student (*c* January 1974 or thereabouts) yet now you have gone totally post-graduate-style and print - in my view - only post-grad stuff. If students are not writing for you the simple elementary rubbish that they (and you) used to write for me, then I claim there is something wrong with your personality-style; so get personalised Ed!

Seriously folks. M500 was never founded by me to be the incomprehensible stuff that it now is, such that the British Library and its associated five other Copyright Deposit Libraries now legally demand free copies of it. I founded it as a friendly sort of thing in which people could make themselves known and thus be telephoned, via MOUTHS, by other people who realised that they were human beings and dimwits.

At the present time I am FED UP with the incomprehensible postgraduate erudition of M500. The only thing that gives me any pleasure in producing is the MOUTHS/MATES lists, which demands a lot of my physical energy with a hand-duplicator and a non-electric typewriter. I could wish M500 totally dead and buried. I understand not more than one third of any issue and I am a so-called BA (Open).

This is my half page for the year 1979 and I give you my due warning that unless some of you M101, M201 and M202 (or their equivalents) begin to write again for M500 then it will die the death, and I will personally rejoice that it does so. It is the case, I must add, that if M500 dies in its present incomprehensible form, there is no need to worry about MOUTHS and MATES - I will see to it tflat they continue as they always have done. It is up to you as an individual, junior person in the OU undergraduate system to make M500 what you understand.

By heck, we used to have such humourous guys around who were not afraid to go into print with absolute howlers - like dear old Hugh Mclntyre who made a mathematical error every time he wrote anything at all. People were not scared in those days. Are you 1979 students a different breed from the original pioneers of this journal, or are you just put off by the level of 'erudition' which you see? I repeat - I have a BA (Open) and I understand less than one third of any current M500. So wake up at the back there and WRITE SOME RUBBISH! I, as a BA (Open), wish to read some rubbish for a change, thank you!

By the way, your Membership Secretary, currently too overworked with all you new members to write, agrees with my general sentiments. She says that she understands very little of M500 these days. It is really, honestly, up to you, John Smith/Agnes Jones, to put this right. Write rubbish, please?

Ed - *half a page* ?
****

MAZE

ROY BOLTON

The editorial in M500 58 about mazes reminded me of an exercise I was set some twelve or so years ago as part of training in computer programming.

I established an algorithm which seemed to be capable of solving all mazes. I was no mathematician then, and neither will I ever be (I'm partway through M101) - and so I used to invite people to design mazes that my program could not solve; the fact the program did solve all mazes was no proof in the maths sense.

My algorithm was based on someone else's idea - and was simply as follows. The maze runner is told to walk and turn at random and, at the same time, to draw a chalk line alongside her on her right. Then at any later position if a chalk mark was discovered on her left the maze runner was expected to turn round and retrace her steps until she could continue on unchalked road. At any junction, if chalk marks were sen on left and right, it implied that the road was a blind alley (chalk marks on the way in and on the way out).

Translated into computer terms, I considered the maze to be a matrix of squares, like a chessboard where every junction between squares was a potential wall position.

My program (in its simplest form) let the maze runner proceed in a straight line and then turn right when it crashed into a wall. At each position travelled through, the maze runner marked the square to say 'I've been here before'. The maze runner was not allowed into a position where it had trod before *except* hen escaping from a blind alley; and when escaping (i.e. backtracking) out of a blind alley, the maze runner erected notional walls behind her to signify that the alley was indeed blind.

If my memory serves me right these rules alone will allow the maze runner to find the target.

When the program was demonstrated with any given maze pattern the player was invited to choose a start point and a target point. Provided that a path really existed between start and target the program would always find it (eventually!)

A lack of time, talent and mathematical 'knowhow' prevented me from experimenting with algorithms for finding the shortest path. I recall concluding that shorter paths could be found on subsequent runs but that one could not guarantee finding the shortest path. It seemed to me that it was a trial and error business - and the objective then becomes how to minimize the number of trials before discovering (with certainty) the shortest path between start and target points!

****

THAT OUSA REFERENDUM

JIM EZECHIEL

With the latest MOUTHS list was enclosed a slip saying that the OUSA referendum had resulted on a 'No' majority and that another referendum on the point would be held next year.

Surely this must be unconstitutional - even if we do not have a constitution.

What happens if there is a 'Yes' majority next year? Will there be another chance in 1981 for the members to change there mind? Or will annual referendums be held only until a 'Yes' is recorded?

I do not suggest that one vote should decide the matter for all time but the membership having once expressed its opinion, surely the matter should be shelved for a respectable period - say five years.

****

Is a semiconductor a man who asks you to pass halfway down the bus?

PROBLEMS

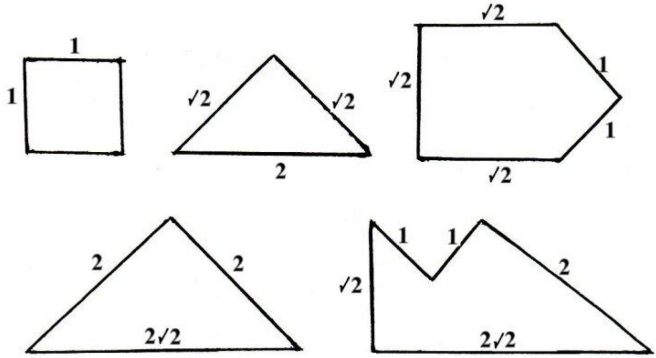
COMPILED AND EDITED BY

JEREMY HUMPHRIES

At last I sit down to write.

If you look at page 0 you may see that I have a new address. My friend Rose and I have got married, and Fairgreen road is now my official hame.

On our recent travels in Wales Rose won herself a very nice chopping board by solving this puzzle in less than ten minutes. We had to buy the puzzle first, but it was only £1.50 - the boards were £4.



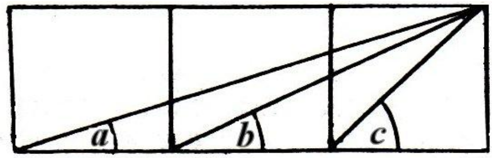
Here is a quote from *The Whetstone of Vitte* - Robert Recorde.

> And thei that bee dulle witted, and yet be instructed and exercised in it (Arithmeticke), though thei gette nothyng els, yet this shall thei all obtain, that thei shall bee moare sharpe witted than thei were before.

- so console yourselves.

And I must apologise to Fred Mosteller who became Fred Mostello during transcript in 58.

SOLUTION 57.4 *ANGLES* 57 Show that $a + b = c$ when thesquares are square.



I thought everybody would do this. The number of solutions is probably more than the number of contributors to M500.

Steve Ainley says:

My solution is sines. sin $a = 1/\sqrt{10}$; cos $a = 3/\sqrt{10}$; sin $b = 1/\sqrt{5}$; cos $b = 2/\sqrt{5}$; sin $c = 1/\sqrt{2}$; $\sin(a + b) = \sin a \cos b + \cos a \sin b = \frac{2+3}{\sqrt{5\sqrt{10}}} = \frac{1}{\sqrt{2}} = \sin c$.

And Tony Parry says:

tan $a = 1/3$; tan $b = 1/2$; tan $c = 1$. tan $a + b = (\tan a + \tan b)/(l - \tan a \tan b) = 1 = \tan c$.

If anybody wants to send any more proofs please do. Geometric ones would be nice. If there is any more interest I might even tell how this problem, in common with most M500 problems, is connected to the Fibonacci numbers.

SOLUTION 57.3 *GARMENTS* a) A can make 9 garments while B makes 5. 3 of A's garments earn the same profit as 7 of A's. What are the relative values of A and B to this employer?
b) Why do we add marks instead of multiplying them when determining a student's grade?

a) Tony Parry and Stene Ainley agree that A is worth 27 and B is worth 35.

b) Here Tony says that the multiplying method encourages consistency and is rather severe on anyone with a zero. eg:

student	scores	sum	arithmetic mean	product	geometric mean
A	10, 10, 5, 0, 0	25	5	0	0
B	9, 9, 5, 1, 1	25	5	405	3.3
C	7, 6, 5, 4, 3	25	5	2520	4.8
D	5, 5, 5, 5, 5	25	5	3125	5

The multiplying method would have the advantage of ensuring a serious attempt at every assignment.

Steve Ainley says that we do it for the same reason that a 90% good soup, a 100% good steak and a 10% good pudding is a better meal than 30% good of each. Addling says so: $\frac{200}{300} > \frac{90}{300}$ whereas multiplying doesn't: $\frac{90}{1000} < \frac{270}{1000}$.

SOLUTION 57.1 *SEQUENCES* 57 *$a = f(1), f(2), f(3), \ldots$ and $b = g(1), g(2), g(3), \ldots$ are two sequences of positive integers. Every positive integer is either in a or in b, but not in both. a and b are increasing.*

$\forall n,\ g(n) = f(f(n)) + 1$. *Find $f(240)$ elegantly.*

I got four answers to this. Two people said 388, one said 389 and one 384. Only the 388s, Bob Escolme and Steve Murphy, sent any details.

Steve begins by establishing several results including

1. $g(\mathrm{n}) = f(n) + 1$;
2. $g(g(n) = g(n) + f(n)$;
3. $f^{p}(g(n)) = g\,f^{p}(n) + a_p$, where $a_p = a_{p-1} + a_{p-2} + 1$, $a_1 = 1$, $a_2 = 2$;
4. $g\,f^{p}g\,f^{q}g(n) = g\,f^{p+q+4}(n) + 2 + a_{p+2} + a_{p+q+4}$.

The values of a_r (he goes on) are

r	1	2	3	4	5	6	74	8	9	...
a_r	1	2	4	7	12	20	33	54	88	...

with an obvious‘Fibonacci’ connection.

Let us put $n = 1$ in (4) and choose values for a_{p+2} and a_{p+q+4} which together give about 240. a_3 and a_{11}, giving $p = 1$ and $q = 6$ are easily seen. Whence

$$gfgf^{6}g(1) = gf^{11}(1) = 2 + a_3 + a_{11} = g(1) + 238 = 240 \qquad (5)$$

Now $fg\,f^{6}g(1) = g\,f^{7}g(1) + 1$ (from 3)
$= f9g(1) + 2$ (definition of g)
$= g\,f^{9}(1) + 2 + a_9$ (from 3)
$=2 + 2 + 88 + 92.$ (6)

(5) and (6) together show $g(92) = 240$, and from (1) $f(92) = 240 - 92 = 148$ so that $f(240) = f(g(92)) = g(92) + f(92) = 388$.

Steve concludes: Unfortunately I still cannot prove what must be the best result, namely $f(n)$ = integer part $\left[\frac{n}{2}\left(1+\sqrt{5}\right)\right]$.

Bob Escolme begins by working out a few values:

n	1	2	3	4	5	6	7	...
f	1	3	4	6	8	9	11	...
g	2	5	7	10	... .			

He assumes $g(n) = f(n) + n$, Vn and proceeds: We see $f^{2}(n) = f(n) + n - 1$, from which we can produce sequences of n, $f(n)$, $f^{2}(n)$, $f^{3}(n)$, ... when we are given the first two terms. In the table given sequences I and II come from the previously calculated values. The choice of starting values for sequences III to VIII is determined by the desire to have the end values of successive sequences ever closer to 240.

In each sequence, for a given starting value n, we must calculat $f(n)$, the second term. In III for instance we know $f(9)$ is 14 and $f(11) = 17$ (from I and II) so $f(10)$ is 15 or 16. But $f(6) = 9$ so $g(6) = 15$. Therefore $f(10)$ is 16. Similarly we proceed for subsequent sequences.

See next page for table.

I	6	9	14	22	35	56	90	145	234	
II	7	11	17	27	43	69	111	179	289	
III		10	16	25	40	64	103	166	268	
IV			15	24	38	61	98	158	255	
V				23	37	59	95	153	247	
VI					36	58	93	150	242	
VII						57	92	148	229	386
VIII								149	241	389
IX									240	388

SOLUTION 57.2 *FUNCTIONS* 57 *$\mathbb{N}$ is the set of positive integers and $f : \mathbb{N} \to \mathbb{N}$. Prove that $f(n + 1) > f(f(n))\ \forall n \in \mathbb{N} \Rightarrow f(n) = n, \forall n \in \mathbb{N}$.*

Steve Murphy and Tony Parry sent similar solutions to this. Here is Tony:

Let l be the least member of $f(\mathbb{N})$. Since $f(n + 1) > f(f(n))$, $f(n+1) \neq l$, $\forall n \in \mathbb{N}$ $\therefore$ $f(1) = l$.

Restrict f to $\mathbb{N} - \{1\}$. Let m be the least member of $f(\mathbb{N} - \{1\})$.

Then similarly $f(2) = m$.

Eventually $f(1) < f(2) < f(3) < ... < f(n) < f(n+1) < ....$. (1)

From (1) we have: A) $f(p) < f(q) \Leftrightarrow p < q$; and

B) $f(n) \geq n\ \forall n \in \mathbb{N}$, since if $\exists\ k$ such that $f(k) < k$ there would be no room below k to fit in all the distinct images of $f(j)$, $j < k$.

Now $f(n+1) > f(f(n))$, $n \in \mathbb{N}$. $\therefore$ $n + 1 > f(n)$
(by A) $\therefore n \geq f(n)$.

But from B, $n \leq f(n)$.

$\therefore$ $f(n) = n$, $\forall n \in \mathbb{N}$.

PROBLEM 59.1 *SEQUENCE* 59 Tony Parry

Tony Parry is a new member and says he doesn't know if we've done this one previously. Well - I don't know either. I'm still looking for files mislaid in the move. But I don't remember it.

A sequence is defined

$$\left.\begin{aligned} u_1 &= x \\ u_{r+1} &= x^{u_r} \end{aligned}\right\} x \in \mathbb{R}.$$

For what values of x does this sequence converge?

PROBLEM 59.2 *FIBONACCI* 59 STEVE MURPHY

Steve sends a problem which is reputed to have been solved by Fibonacci.

Find a number which is the square of a rational number when it is either increased by five or decreased by five.

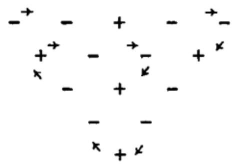
PROBLEM**.** 59.3 *STEINHAUS* EDDIE

Eddie found this problem engaging and has drawn my attention to it.

A Steinhaus triangle is made thus:

Make a row of n plus and minus signs. Under each like pair place a plus; under each unlike pair place a minus. Continue, to get a triangle of ^n(n + l) signs.

Prove that if the first row pattern is – – + – – + ... (two minuses, one plus, two minuses, one plus, ...) then the same pattern repeats itself when one traverses all of the entries in a clockwise spiral, as shown



PROBLEM 59.4 *MATRIX* 59 EDDIE

It's a long time since we had a matrix. Here is one, again passed to me by Eddie.

Let A be an $n \times n$ real matrix such that the main diagonal terms are zero and the others are ±1. Show that A is non-singular if n is even but can be singular for n odd.

PROBLEM 59.5 *THE PLANET KOPHIKKUP*

This is from an old edition of the magazine *Eureka*.

The recently discovered planet Kophikkup, which has the shape of a torus or doughnut, has now been colonised. There are four space ports and from each port there is a direct monorail line to each major city. The rails do not join or cross and, subject to that proviso the settlers have built the maximum number of major cities. How many?

Ed **-** *Have you ever noticed how, whenever mathematicians talk of a doughnut they mean a thing with a hole in whereas if you ask for a doughnut in your local baker's you will get a solid indigestible object filled with jam*?

EDITORIAL

Anyone working through Tony Parry's solution to Bob Escolme's problem on functions, 57.2, will notice that the condition $f(n + 1) > f(f(n)) \Leftrightarrow f(n) = n$ applies not only to f but also to n. In other words if there is a function f such that for some particular n $f(n) = (n)$ the result still applies. For instance set $f(n) = n^2$ then $f(f(n)) = f(n^2) = n^4$ and $f(n + l) = (n + 1)^2 = n^2 + 2n + l$.

Now $n^2 + 2n + 1$ is greater than n^4 only when n is equal to 0 or 1, and these are precisely the integer values at which $f(n) = n$.

Since $f(n + l) > f(f(n))$ for $n = 1$ and $f(n + l) < f(f(n))$ for $n = 2$ you may wonder whether if n is not restricted to $\mathbb{N}$ but is allowed to take any real value there is a point at which $f(n + l) = f(f(n))$. To find such a point set $n^2 + 2n + 1 = n^4$; i.e. find a root of the equation $x^4 - x^2 - 2x - 1 = 0$. But this factorizes as $(x^2 - x - 1)(x^2 + x + l)$ giving the roots $(\pm l \pm \sqrt{5})/2$. Come back filius Bonaccio, all is forgiven.

Sorry about the above. You will of course notice the astounding generalisation which begins at the end of line three and is offered without justification of any kind. Show me a counterexample - but I think you'll have to go outside powers.

But what I really wanted to say was that perhaps part of the reason why M500 has started coming out so erratically is that I tend to get sidetracked into enjoying myself, especially with the problems section; and part of the reason I hived off that section to Jem Humphries was to put myself out of that temptation. It didn't entirely work as I still have the task of typing out the material for publication - so it would be an enormous help if someone (with access to an IBM typewriter) could take the manuscript directly from Jem, type it, and send it to me to go in unseen. Any volunteers?

There are other reasons why M500 doesn't come, crisp and shining, dropping on your doormat on the tenth of almost every month any more. You can perhaps imagine some of them if you think hard, but one you might not think of is a bouncy, loud, new young female Kent named Tabitha Kathryn (or Tabbycat) which is causing delays and upheaval throughout the houshold.

If you have looked to our centrefold you will have noticed a lack of Gauss. This is because Jeremy Gray feels that twelve episodes is enough and although he will write things for us in the future he does not wish to get so involved again. Hence I was able to grab space to print Robin Wilson's article. I was originally going to give it the same space as Jeremy's; i.e. the four middle pages and run it in two issues but for various reasons I decided to have it complete in one go. One reason is that if you look at page four of the article you will see that it doesn't make sense without page five; another is that some people's subscriptions run out with this issue and I would hate them to miss the end.

Since I wrote the above our nice new house at 20 Statham Grove caught fire. A few things (including a large part of the roof) were destroyed; several more were damaged. That is the reason for the yet further delay in getting this out, and it also explains the dirty marks on some of the pages that I couldn't be bothered to re-do. Of course they might disappear in the printing and so you won't know what I'm talking about, but I thought you'd like to know.

Eddie Kent.