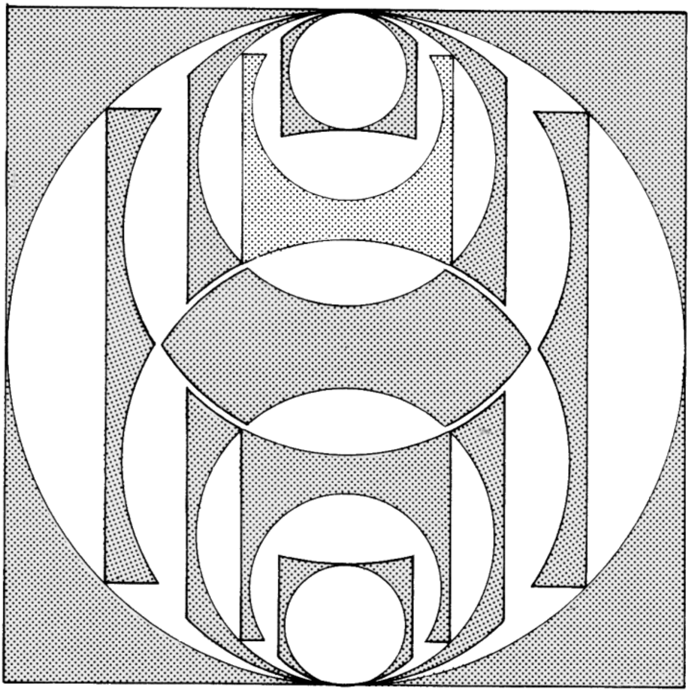
M500 60



M500 is a publication of the M500 Society for Open University mathematics students, staff and friends. It is designed to alleviate student academic isolation by providing a forum for publication of the mathematical interests of members.

Articles and solutions are not necessarily correct but invite criticism and comment. Anything submitted for publication of more than about 600 words will probably be split into instalments.

MOUTHS is a list of names, addresses, telephone details and data on previous and present 0U courses of its voluntary members, by means of which private contacts may be made to share 0U and general mathematical interests or to form self-help groups by telephone or correspondence.

MATES is a special list of the subset of MOUTHS members who have explicitly volunteered for their MOUTHS data to be distributed to M500 members in closed institutions such as prisons and special hospitals.

The views and mathematical abilities expressed in M500 are those of the authors and do not necessarily represent those of either the Editor or the Open University.

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ANYTHING SENT TO ANY OFFICER OF THE SOCIETY WILL BE CONSIDERED FOR POTENTIAL PUBLICATION IN THE MAGAZINE, UNLESS OTHERWISE SPECIFIED.

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The cover design was drawn by Lawrence Seaton, adapted from an original glass engraving by Peter Shute.

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TABLET IM 55357

CHRIS PILE

On a recent trip to Baghdad I came across a mathematical 'tablet' in the Iraq Museum, together with the English translation (copy below). This is one of many such tablets dating from the Babylonian period (*c* 1850 BC) some 1300 years before Pythagoras. The actual tablet is approximately 3½ inches by 2½ inches by 1¼ inches thick. The numbered lines in the translation correspond to the lines of the cuneiform inscription.

My approach to 'solving' the triangles was first to work out the lengths and then calculate the areas. However the text starts with areas and calculates lengths, presumably

$$\text{Area } ABD = \left(\frac{3}{5}\right)^2 \times 1350 = 486 \;\therefore \text{Area } ADC = 864.$$

$$\text{Area } AED = \left(\frac{3}{5}\right)^2 \times 864 = 311.04 \;\therefore \text{Area } EDC = 552.96.$$

$$\text{Area } EDF = \left(\frac{3}{5}\right)^2 \times 552.96 = 199.0656 \;\therefore \text{Area } EFC = 353.89.$$

The procedure for lines 6, 7, 8 appears to be

$$\frac{BD^2}{\text{Area } ABD} = \frac{AB^2}{\text{Area } ABC} = \frac{45^2}{½ \times 45 \times 60} \;\therefore BD^2 = 1/60 \times 45 \times 2 \times 486 = 729.$$

Lines 9,10 give $AD = \frac{\text{Area } ABD}{½BD}$.

Things go wrong from line 12 onwards, but it would appear that the same process is repeated for *AED* and *ADC*, i.e. AE^2/Area $AED = AD^2$/Area ADC $\therefore$ $AE^2 = 1/48 \times 36 \times 2 \times 311.04 = 466.56$. This last number appears in the Arabic of line 14 but not in the English translation.

There are obvious mistakes in the last few lines and these are common to both the Arabic and English versions. i.e. $AE = \sqrt{466.56} = 21.6$ (not 21.36).

The tablets were discovered at Tell Hamal on the outskirts of Baghdad. Here, it is claimed, was founded the world's most ancient academy, which was particularly concerned with mathematical studies.

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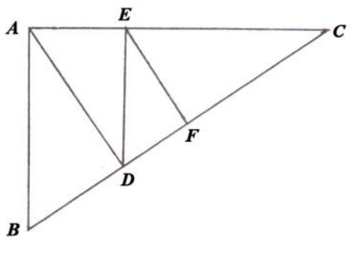

1. A triangle (*ABC*) 60 is the length (*AC*), 75 is the 'long' length (*BC*), 45 is the upper width (*AB*).

2. 1350 is the total area. From 1350 the total area 486 is upper area. (Area of *ABD*).

3. 311.04 is the next area (*AED*). Around 199 is the third area (*EDF*).

4. (and) around 353 is the lower area (*EFC*).

5. What is the upper length (*AD*, the ‘segment’ length (*ED*) the lower length (*EF*) and the perpendicular (*FC*)?

6. When you perform the operation take the reciprocal of 60 (= *AC*) and multiply it by 45.

7-8. 45 you see (get. Multiply 45 by 2 and 90 you get. Multiply 90 by 486 the upper area (*ABD*) and you get 729. What is the square root of 729. 27 is the square root.

9. 27 is the width (*BD*) of the triangle (*ABD*). Halve 27 and 13.50 you get. Take the reciprocal of 13.50.

10. and multiply it by 486 (the area of *ABD*). 36 you get (which is) the length (*AD*) opposite to 45 the width (*AB*.

11. Come back and subtract 27 the length (*BD*) of the triangle (*ABD*) from 75.

12. 48 is the quantity left. Take the reciprocal of 48 and get 1,15. Multiply 1,15 by 36.

13. (and) 45 you get. Multiply 45 by 2 and you get 1,0,30. Multiply 1,0,30 by 5,0,11 : 2,24.

14. (and 7,0,46 : 33,36 you get. What is the square root of 7,0,46 : 33,36.

15. 21.36 is the square root. 21.36 is the width (*AE*) of the second triangle (*ADE*).

16. Halve 21.36 and you get 10.48. Take the reciprocal of 10.48 and by ... 2 ... .

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ONE LINE PROOFS

LEN WOOD

I was interested to read the *One line proofs* article by Bob Margolis in M500 52. I have no solution to the question “what constitutes a proof”, merely a few thoughts.

The question of what is obvious seems to be relevant. If a written proof can be elaborated sufficiently so that the progression from every step to the next is 'obvious' (whatever that means) then isn't the progression from step n to step $n + 2$ also 'obvious'? If so then this would seem to point roughly in the direction of a conclusion that if a proof can be sufficiently elaborated it can then be progressively reduced to a one line proof!

Until a rigorous definition of ‘proof’ is obtained perhaps the following rule of thumb could be used for OU TMAs and exams? “In any written proof the progression from any step to the next must only depend on mathematical knowledge assumed to be possessed by the average (or every?) student at the beginning of the current course plus what has been proved in previous TMAs”. This begs the question of skipped TMAs but I have no suggestion for that. Perhaps another member has?

INFINITY

J. BEDDARD

After reading Sid Finch's article *Potential Concepts of Infinity and Space* I should like to suggest two articles which may be of interest to enthusiastic readers.

a) *Non-Standard Analysis,* Abraham Robinson (North Holland, 1970); this is an advanced text with historical significance.

b) ‘A new perspective on infinity’ by L. A. Steen, N*ew Scientist* 9.11.78; the article discusses the significance of Robinson’s work compared to Newton and others. Describing how Robinson developed a special area of Mathematical Logic to extend finite space toward infinite space and in doing so working against the Newtonian idea of taking infinite space toward finite space.

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THE JOURNAL OF MATHEMATICAL MODELLING FOR TEACHERS

NORMAN LEES

The *Journal of Mathematical Modelling for Teachers* (*JMMT*) is a new journal whose first two issues were published last year. Its sponsors are the Department of Mathematics at the Cranfield Institute of Technology and the OU Mathematics Faculty. The aims of the journal are clearly expressed in the first issue:

> Emphasis will be placed on the setting up of the model, the interpretation of the model and the validation of the solution. The mathematics should be no more advanced than first year university, and preferably school mathematics.

There have been two articles on the background of mathematical modelling and its relationship to the real world. Three extended examples of this philosophy are presented by different authors –the setting up of temporary traffic lights; drying after a shower and car gearing. There are also shorter examples of drug therapy and stock control from M101 and population studies; and a number of book reviews.

Although aimed mainly at maths teachers in schools and colleges other readers will get a lot of ideas and stimulus from the journal. I found it very encouraging to see how much can be accomplished using simple maths – second level maths more than covers what has been done so far. Problem addicts will find a whole range of new material to try out.

The journal fills a need for readily available and straightforward information in this sadly neglected field which will become increasingly more important in the future.

The cost is £l for two issues. Information can be obtained from JMMT, Dept of Mathematics, Cranfield Institute of Technology, Cranfield, Bedford, MK **4** 3 OAL.

THE M500 PULL-OUT SUPPLEMENT ON CALCULATORS, CONTINUED

TONY BROOKS

I bought a SINCLAIR ENTERPRISE PROGRAMMABLE calculator in October 1978. In the months since then I have been extremely pleased with my purchase.

The calculator has an eight digit display in fixed point scientific or floating decimal notation. It has a full range of scientific functions: sin, cos, tan, arcsin, arccos, arctan, e^x, y^x, ln x, log x, antilog x, /x, 1/x, x^2 , +/–, degrees to radians and inverse. It also has two levels of brackets and seven independent three-function memories.

These functions alone would make the calculator a good buy but its best feature is its programming facility. It has an eighty step capability with fully merged functions (so functions requiring two or three key strokes count as only one program step). Program facilities include ‘goto’ and ‘goto if negative’.

One of the best features of the calculator is that it comes with a very comprehensive three volume software library covering statistics, maths, engineering and financial applications.

The Sinclair Enterprise shows a considerable improvement in keyboard operation over the earlier Sinclair Cambridge calculators which tended to be unreliable in this respect. The only criticism I have of the Enterprise is that it lacks a function for taking the integer or decimal part of a number. This can be done on the Enterprise but it requires several keystrokes (and program steps).

My Enterprise cost £24 in W. H. Smith's and for this you get the calculator plus program library, battery, and mains adaptor. Budget minded readers may like to know that the Enterprise has been advertized in the *New Scientist* for less than £21 including postage. I think it is an outstanding purchase at that price.

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A TRIANGLE

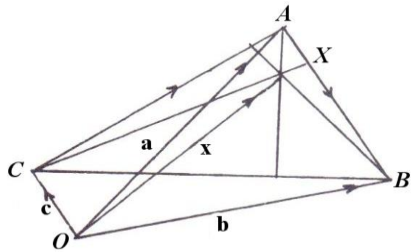
KEN MAY

[*Knowing how eager Senior Officers of* M500 *are to receive material for publication I decided to emerge from my cocoon to make my debut. On completing the disembalming process I thought that there might conceivably be a need for some worked examples in Vector Analysis applied to Euclidean Geometry and Trigonometry. With this in mind I wrote to dear Eddie whose reply was somewhat revealing. He said, “I can't think of anything more boring but I'm sure some people are dying to see them.” As the dying would have little interest in what I have to say I decided to crawl back into my cocoon never to emerge again. However, just before zipping myself in to work forevermore in the dark my thoughts went out to a few students who have found the subject somewhat bewildering and who might benefit from a few examples which are related to a familiar mathematical scene. So with confidence partially restored* (*Many things I might be but I am shattered to think I am a bore*) *I emerged once more into the real or perhaps unreal world. Eddie did ask me to make it funny. How, how to make Vector Analysis funny I simply do not know, but I am willing to have a go. However, as people’s humour varies* *considerably I make no apologies for not raising a laugh. As this is an experiment I will only submit one example this time. Its length you will appreciate has very little to do with mathematics - only lateral frivolities*.]

Prologue Our task is to prove using Vector Analysis that the perpendiculars from the vertices of a triangle to opposite sides meet at a point.

As every effort has been made to confuse the reader she will find it necessary to refer to the diagram below while wading through this mass of verbiage.

The Eternal Triangle



Take any triangle *ABC* and let it come gently to rest on a clean sheet of white paper. Let *O* be some innocent arbitrary fixed point in the plane of the paper. Now *O*, being naive, becomes interested in the eternal triangle and wishes to join up with apices *A*, *B* and *C*.

Marrying these with *O* gives rise to a set of position vectors namely $\overrightarrow{\boldsymbol{OA}} = \mathbf{a}$, $\overrightarrow{\boldsymbol{OB}} = \mathbf{b}$ and $\overrightarrow{\boldsymbol{OC}} = \mathbf{c}$. This sends the triangle into a spin (a flat one) in that the sides of the triangle are considered as vectors $\overrightarrow{\boldsymbol{AB}}$, $\overrightarrow{\boldsymbol{BC}}$ and $\overrightarrow{\boldsymbol{CA}}$. Now taking stock of the situation so far we discover that *O*'s interest in *A*, *B* and *C* has created three more triangles. This goes to show how complicated life can become. However, considering each in turn proves an advantage as they give rise to

$$\overrightarrow{\boldsymbol{CA}} = \mathbf{a} - \mathbf{c}, \quad \overrightarrow{AB} = \mathbf{b} - \mathbf{a} \quad \text{and} \quad \overrightarrow{BC} = \mathbf{c} - \mathbf{b}$$

Now to make matters worse a further point *X* becomes involved in the affairs of the other points. *O* being a rather inquisitive point now takes an interest in *X* and so by considering triangles *CXO* and *OXB* we are now endowed with vector relations

$$\overrightarrow{\boldsymbol{CX}} = \mathbf{x} - \mathbf{c} \quad \text{and} \quad \overrightarrow{\boldsymbol{XB}} = \mathbf{b} - \mathbf{x}$$

But $\boldsymbol{X}$ has a very special quality in that $\overrightarrow{\boldsymbol{CX}}$ is orthogonal to $\overrightarrow{\boldsymbol{AB}}$ and $\overrightarrow{\boldsymbol{XB}}$ is orthogonal to $\overrightarrow{\boldsymbol{CA}}$. By the definition of dot products this implies that

$$\overrightarrow{\boldsymbol{AB}} \cdot \overrightarrow{\boldsymbol{CX}} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{x} - \mathbf{c}) = \mathbf{b} \cdot \mathbf{x} - \mathbf{a} \cdot \mathbf{x} - \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} = 0$$
$$\overrightarrow{\boldsymbol{CX}} . \overrightarrow{\boldsymbol{XB}} = (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{x}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{x} + \mathbf{c} \cdot \mathbf{x} = 0.$$

Subtracting the second from the first gives

$$\mathbf{b} \cdot \mathbf{x} + \mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{x} = 0 \qquad (\text{Note } \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{b})$$
$$\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) - (\mathbf{c} - \mathbf{b}) \cdot x = 0$$
$$\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) - \mathbf{x} \cdot (\mathbf{c} - \mathbf{b}) = 0$$
$$(\mathbf{a} - \mathbf{x}) \cdot (\mathbf{c} - \mathbf{b}) = 0.$$

But $\mathbf{a} - \mathbf{x} = \overrightarrow{\boldsymbol{XA}}$ and $\mathbf{c} - \mathbf{b} = \overrightarrow{BC} \Rightarrow \overrightarrow{XA} \cdot \overrightarrow{\boldsymbol{BC}} = 0 \Rightarrow \overrightarrow{XA}$ is orthogonal to $\overrightarrow{\boldsymbol{BC}}$ and the rest follows, thus establishing that this is one triangle that didn't come to a sticky end.

CONTINUED FRACTIONS AND PELL'S EQUATION

JOHN READE

It is interesting to see that several of the problems in recent issues of M500 have boiled down to finding integer solutions of the equation

$$x^2 - Ny^2 = \pm 1$$

(Pell's equation). Also that there has been some correspondence about continued fractions, since there is an algorithm for solving Pell's equation which involves the use of continued fractions.

I should like if I may to depart slightly from the notation for a continued fraction used by previous correspondents. I would like to denote e.g. the fraction

$$1 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}$$

by $1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}$. In this notation the continued fraction for e.g. $\frac{1+\sqrt{5}}{2}$ is $1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}$ ... and the succesive approximations to $\frac{1+\sqrt{5}}{2}$ given by terminating the fraction after a finite number of steps are 1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, ... which are the ratios of the consecutive terms in the Fibonacci sequence.

For every integer N which is not a perfect square there is a continued fraction expansion for $\sqrt{N}$ in the form

$$a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\,\ldots$$

where a_0, a_1, a_2 , ... are positive integers. This can be calculated as follows. For example take $\sqrt{7}$. Write

$$\begin{aligned}\sqrt{7} &= 2 + \sqrt{7} - 2 && 2 < \sqrt{7} < 3\\ &= 2 + \frac{3}{\sqrt{7}+2} \\ &= 2 + \cfrac{1}{\cfrac{\sqrt{7}+3}{3}} && \frac{4}{3} < \frac{\sqrt{7}+2}{3} < \frac{5}{3}\\ &= 2 + \cfrac{1}{1 + \cfrac{\sqrt{7}-1}{3}}\end{aligned}$$

etc. The complete fraction is

$$\sqrt{7} = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\ldots$$

this pattern repeating itself indefinitely. For the general N we always have a_0 = the integer part of $\sqrt{N}$, and the a_n's repeat themselves after a certain point, say $1/(a_1+)1/(a_2+)\ldots 1/(a_n+)$ is repeated. The last a_n before the recurrence starts is always equal to $2a_0$. Also the sequence $a_1\ a_2\ \ldots\ a_{n-1}$ is always symmetric (palindrome).

To obtain the solution of Pell's equation one computes the incomplete fraction obtained by stopping *one step before* recurrence starts in the continued fraction for $\sqrt{N}$. Suppose this incomplete fraction is x/y. Then x, y are the smallest solutions of Pell's equation.

For example the smallest solution of $x^2 - 7y^2 = \pm 1$ is obtained by computing $2 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1} = \frac{8}{3}$ (start from the right hand end). And indeed one readily verifies that $8^2 - 7 \cdot 3^2 = 1$.

All the other solutions of Pell's equation are obtained by taking powers of $x - y\sqrt{N}$ where x, y are the smallest solutions, e.g. $(8 - 3\sqrt{7})^2 = 64 - 48\sqrt{7} + 63 = 127 - 48\sqrt{7}$ giving the second solution $x = 127$, $y = 48$.

Two possibilities occur. If the smallest solution gives $x^2 - Ny^2 = 1$, then all the other solutions are of this equation and $x^2 - Ny^2 = -1$ is insoluble for this N. e.g. $\sqrt{7}$ is of this type. The other case is where there the smallest solution gives $x^2 - Ny^2 = -1$. In this case the solutions obtained by taking successive powers of $x - y\sqrt{N}$ are alternately solutions of $x^2 - Ny^2 = 1$ and $x^2 - Ny^2 = -1$. e.g. $\sqrt{2}$ is of this type.

The fraction for $\sqrt{2}$ is $1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}$ ... so we obtain $x = y = 1$which solves $x^2 - 2y^2 = -1$. The next solution is $x = 3$, $y = 2$ which solves $x^2 - 2y^2 = 1$, etc.

The continued fraction for $\sqrt{61}$ is

$$7 + \left(\frac{1}{1+}\frac{1}{4+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{4+}\frac{1}{1+}\frac{1}{14+}\right)$$

the expression in brackets recurring, which gives

$$29718^2 - 61 \cdot 3805^2 = -1$$

$1766319049^2 - 61 \cdot 226153980^2 = 1$. These are the smallest solutions of $x^2 - 61y^2 = \pm 1$.

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WHAT ABOUT THE SOCIOLOGISTS?

ARTHUR SMITH

Most students undergo disenchantment. I hope this is not too sour. The interdisciplinary promise turns out to be history of maths and wittering about the responsibility of science. Students have to take two foundation courses, apparently to produce a synthesis which dons aren't capable of.

Anything in M courses relevant to humanities is sketchy, optional and unsupported. If you want to use your mathematical knowledge there's nowhere to go in D courses or anything like. T341 (Systems Modelling): "unlikely to stretch the mathematical abilities". U201 (Risk): "basic arithmetical operations ... warm humanistic study". We all know what that means! Ask, and it is assumed you mean business maths or forecasting techniques.

School economics (i.e. Benham) gave multidimensional indifference curves as the all-singing model. In M201 you find functionals, duals, and individual linear programming. Pity the non-numerate What are the simplifications? Is cash medium significant? Many requests, even to specialists in duality, produced one title for follow-up, and that was a wild goose chase.

It’s dodgy to translate “persons A and B are distinct” into modelling by linearly independent vectors for their values on some particular. But underneath the prestigious summation signs “cost benefit” depends on the special case where A and B are on one axis, a simplification handicapped by conflicting with facts. The same goes for engineering by linear programming. So do D's distinguish by models? No; “Micro-economics” sprawls into this type of programming. Social Science takes the kudos of being superficially mathematical, and goes huffily humanistic if you ask questions. And where's your PhD?

M341 was disappointing. (Nicknames: A Difficult Course in Intermediate Statistics; Angela Dean in Your Own Home). Bayes was virtually optional, though its relation to humanities is possibly “it must be jelly ’cos jam don’t shake like that”. I hoped for little more than hints on modelling; but after being told in the first unit that probability was modelling, that was that. I expected a general discussion on approaches, with for example the state of the art on “computational complexity”, which does define randomness. Not a mention!

M202 with its philosophical interest has made way for M203, “especially suitable for students primarily interested in engineering and science”.

It turned out to be foundations of algebra. The new half credit, which we guessed was the dropped computability component, is more algebra.

I hoped for drill in set theory to help reading of books on probability, and insight into the peculiarity of ordinal arithmetic on which rigorous social science is said to be founded. Halmos was virtually optional, like Minsky. I got little on metrics. Can computational complexity lead to arithmetical probability? Meanwhile arts students into the philosophy discipline had to chat on mathematical objects; for which M202 would have been a drawback.

The main thing I’ve learnt is how narrow dons are, to the extent of branches of their own subject and ramifications of their own specialization. I doubt if the more and more on less and less really leaves them with less time than laymen.

Investigations have shown that I couldn’t stomach D courses, which I joined the OU for. Can anyone put me on the track of some real social science?

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His brain, trained by long years of high living and plain thinking, had become too subtle, too refined an instrument for arithmetic.

Stephen Leacock, Lord Oxhead’s Secret

BRITISH SOCIETY FOR THE HISTORY OF MATHEMATICS

TONY CRILLY

The British Society for the History of Mathematics (formed in 1971) exists to provide a forum for all interested in the history and development of mathematics and related disciplines. Members come from a broad cross-section of those interested in mathematics; from colleges of education, polytechnics, the professions, schools and universities and other areas.

The Society's usual custom is to organise three meetings a year, including one weekend conference. The conference this year is being held at Pembroke College, Oxford; September 14-16. It will be called

CONFERENCE on Geometry and Physical Theory (1630-1930).

Addresses will include: *The Work of Christiaan Huygens* by Dr E Aiton; *Geometry and Mathematical Physics in Napoleonic France* by Dr I Grattan Guinness; *Projective Geometry in the Early Nineteenth Century* by Dr G T Kneebone; *Riemann's discovery of nonEuclidean Geometry* by Dr D Gillies; *Squared Paper: Geometry for Engineers at the end of the Nineteenth Century* by Dr W H Brock and *Einstein and the Geometrisation of Physics* by Professor C W Kilminster.

Members of the Society receive, where possible, summaries of papers read at meetings. They may also receive the international journal *Historia Mathematica* at a specially reduced rate.

Officers of the Society are elected at the Annual General Meeting held in December. Further details can be obtained from

Tony Crilly, Secretary, Middlesex Polytechnic,
Queensway Enfield Middlesex EN3 4SF.

If you wish to join the Society send £2 (or £9.50 if you wish to subscribe to *Historia Mathematica)* together with your name and address printed clearly to

Dr Frank Hickman, Honorary Treasurer, Mathematics Department, Polytechnic of the South Bank, Borough road, London SE1 OAA.

To attend the Conference, send £5 not later than August 1st to Tony Crilly at his above address. Don't forget to give your name and address and explain exactly what the money is for. There is not enough room in M500 to print an application form, and anyway, who wants to destroy a nice magazine by cutting bits out of it?

The complete conference fee covers accommodation and full board in Oxford for the duration of the Conference - Friday evening dinner to Sunday luncheon. It also covers expenses for visiting speakers and is non-profit making.

The complete Conference fee will likely be £25.

(There is also a possibility of visiting the Museum of the History of Science on Saturday afternoon, September 15th. But numbers are strictly limited to 20 and places will be allocated on a first come - first served basis which is rather a disadvantage for readers of M500. Still, you can always ask.)

PROBLEMS *COMPILED AND EDITED BY JEM HUMPHRIES*

Many people have written to me about abstemious and facetious, including Bob Bertuello who called them problem 58.6 because they had no number.

What happens is that every so often I send Eddie a particularly challenging personal problem, and if he can't do it it sometimes finds its way into the magazine. This is such a one. Nothing can be done about it, I'm afraid.

The remarkable thing about the words is that the letters which are not in alphabetical order are consonants. Eddie's list has a similar property.

Bob says he has found a word with the reverse property but it is uncomplimentary and he would rather not disclose it.

Steve Ainley has discovered the difference between a wise-cracking Indian and a teetotal westerner. One is a facetious subcontinental and the other is an abstemious unoriental.

Angus MacDonald, Elizabeth Segell and Howard Parsons also sent stuff on this. Howard wonders if people who lived before the invention of religion were ante-pious, and also if there are any words ending in -endous besides trem, stup and horr. (From such questions, Lord def.)

He is also looking for short sentences containing all twenty-six letters. One - made up by Claude Shannon the world's brainiest unicyclist is

> Squdgy fez, blank jimp crwth vox

which is what a Persian says to his short hat as he pulls it over his ears to shut out the delicate voice of a Welsh musical instrument. Most of the aforementioned people sent me sentences containing strings of has's. Steve Ainley effectively disposed of all the others by pointing out that the winner of the competition to get most has’s in a row will probably be someone who, wherever the runner up has has has has has, wherever the runner up has has has has has has has, wherever the runner up has has has ... .

At this point those of you who agree with Howard that we are getting a long way from maths can turn to solution 58.5, where we discuss three points that are sometimes uncollinear.

In 58 I asked for comments on the solutions to 56.4:

> a) *divide a unit square by two straight lines into parts so that by taking one or more contiguous parts you can get an area of* $1/k$, $2/k$, $3/k$, ..., $(k-1)/k$, 1, *for* k *as large as possible*
>
> b) *same with three straight lines, or four, or ... .*

Steve Ainley, who set the problem, says that the second solution a joint effort by Angus MacDonald and me “is very interesting but forgets that the parts taken must be contiguous, that is, yield a connected piece”. He continues:

“Dividing orthogonally by n lines does make a teasing puzzle itself when we specify contiguous pieces. For $n = 2$ the first pattern in figure 1 is best giving $k = 9$. For $n = 3$ the second pattern, giving *n* = 28, isn't bad. For n = 4 we must consider patterns three and four."

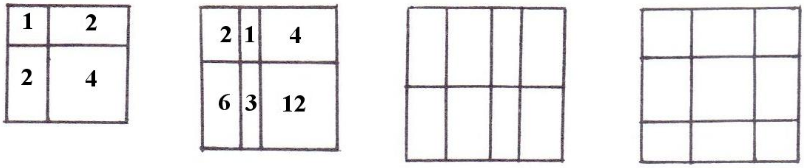


***Figure* 1**

Steve Murphy also pulled me up on the contiguous matter. He went on to say that if contiguous could mean 'sharing at least one point' then the best for *n* = 2 would be

1	4
2	8

giving *k* = 15, so it is apparent from Steve Ainley's answers that he intends contiguous to mean sharing a line. Steve Murphy continues:

"Three lines produce seven regions; these are shown on the left in figure 2. We can represent contiguity by the diagram on the right, which perhaps makes it easier to see. Now we try putting

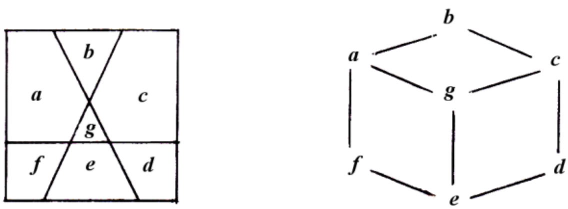
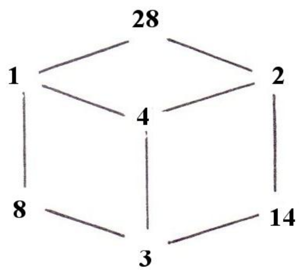


***Figure* 2**

numbers in place of the letters so that we can find chains which add to 1, 2, 3, ... , *k* –1, *k* for the biggest *k* we can get. The best I've managed so far is *k* = 6 0. Unfortunately I have no system and



can't be sure that 60 is maximum, though I think it is. Moreover I don't know if a square can be divided by three lines to give these proportions. Obviously this puzzle should go in the 'too hard' category."

I agree that this division looks unlikely. Can anyone find a value of k for which the subdivision of the square into seven parts can be achieved, or show that there isn't one.

Rosemary Bailey and Tony Forbes wrote about the solutions to 56.3: *Is there an integer x such that* $6x^2 - 105$ *is a square*? Rosemary found two mistakes in the solutions. $6x^2 - 105 \neq 3(x - 35)$ and $6x^2 - 105$ is divisible by 9 (so long as x is not divisible by 3).

She sends a solution of her own:

> Odd squares $\equiv 1 \bmod 8$. If x even $6x^2 - 105 \equiv 7 \bmod 8$.
> If x odd $6x^2 - 105 \equiv 5 \bmod 8$. $\therefore$ $6x^2 - 105$ is not square.

Tony discovered the second error which was his anyway and sent a correct solution similar to Rosemary's. John Reade also sent solutions to both parts of 56.3.

A little grouse. Some of you - I don't know who (or why) - affect illegible signatures. Don't you want me to know who you are?

I will pass on the best of the limericks sent for 58.1 to Eddie so that he may use one from time to time to fill the odd corner. Here is one, which Brian Stewart tells me is from a 1908 *Punch*:

> $112 - b$
> $+ 79 - c$
> (where $b = 7$
> $c = 11$)
> 173.

SOLUTION 58.2 582 *NUMBERS* *Show* $x^2 - 7$ *cannot be a cube*.

Tony Forbes sent this, and sent me this solution.

$$y^2 = x^2 - 7 \Rightarrow x^2 + 1 = y^2 + 8 = (y + 2)(y^2 - 2y + 4).$$

Now if y is even it follows that $x^2 \equiv -1 \pmod 8$ which is impossible. Putting $y = 2z + 1$ we obtain $x^2 + 1 = (y + 2)(4z^2 + 3)$. This means that there is a prime, q, where $q \equiv 3 \pmod 4$ such that $x^2 + 1 \equiv 0 \pmod q$.

Therefore (see J Gray, *Gauss* XI, M500 57 9) we have

$$(-1)^{(q-1)/2} \equiv 1 \pmod q.$$

This is not so.

From Steve Murphy I got a similar one:

$x^2 \equiv 0, 1 \pmod 4$ $\therefore$ $x^2 - 7 \equiv 1, 2 \pmod 4$. Now $y^3 \equiv 0, 1, 3 \pmod 4$. Thus if $x^2 - 7 = y^3$ it follows that $y^3 = 1$ which implies $y \equiv 1 \pmod 4$. i.e. $y = 4k + 1$ for some k.

Now write $x^2 + 1 = (y+2)(y^2 - 2y + 4)$

$= (4k + 3)(16k^3 + 3)$. Now $4k + 3$ must have a prime factor q of the form $4n + 3$, whence we deduce $x^2 + 1 \equiv 0 \pmod{q}$. But this is impossible - see J Gray again and M202.

Howard Parsons also solved this - by exhaustion - examining terminal roots.

At the end of his reasoning he said: "It is now necessary to check the last four figures of cubes of numbers ending in three or nine to show that one cannot get a satisfactory value for $x^2 - 7$. Will you take my word for it that it involves some eight pages of columns of figures?"

Yes, I will. Thanks for not sending them.

SOLUTION 58.3 583 *NUMBERS* *Show that* $n^2 + 4$ *is composite for* $n = 2, 3, 4, \ldots$.

This is another Tony Forbs problem which turns out to be, he says " ... much easier than I thought. All you have to do is use the formula $n^4 + 4 = (n^2 + 2 - 2n)(n^2 + 2 + 2n)$".

It was solved in a similar quick fashion by Steve Ainley, Rosemary Bailey, Angus MacDonald, Steve Murphy, Howard Parsons and Eddie.

Eddie and both Steves remarked that 5 divides $n^4 + 4$ unless 5 divides n, when it doesn't.

SOLUTION 58.4 *PACKING* *Show that* 41 *bricks of dimension* $1 \times 2 \times 4$ *can be packed in a*
$7 \times 7 \times 7$ *box. Is there a packing of* 42 *such bricks*?

I don't know yet how to do this packing but since nobody sent it to me I will commence searching for it.

Steve Ainley and Howard Parsons sent me various packing of 40 bricks which are fairly straightforward. I suppose that the answer to the second part is no, because if it were yes we wouldn't have been asked for a 41-packing. This is Howard's less heuristic argument:

The volume of the box is 343 and the volume of the bricks is 336 so a 42-packing leaves an empty volume of 7.

A brick is $1 \times 2 \times 4$, if it lies flat it occupies 8 units in one layer; if it lies on its side it occupies 4 units in two layers; and if it stands up it occupies 2 units in four layers. Therefore any layer has a filled volume which is even, for any packing.

Therefore at least one unit is void in each layer.

The 42-packing has only seven voids. Therefore exactly one unit is void in each layer.

But this applies in all three orthogonal directions in the box.

Therefore no two voids can meet face to face.

But this is inevitable.

Therefore there is no 42-packing.

SOLUTION 58.5 *POINTS Three points are chosen at random on the circumference of a circle. What is the possibility that they are the vertices of an acute angled triangle*?

I was offered three different solutions for this. I remember Bertrand's paradox - what is the probability that a random chord in a circle is longer than the side of the inscribed equilateral triangle? - which has at least four different correct answers. Therefore I give the received solutions and invite comment.

Rosemary Bailey, Tony Forbes and Howard Parsons give ¼, . Rosemary says her first try involved the nasty symbols π and $\int$, so she was pleased to find this one:

The triangle is obtuse iff the three points *A, B, C* lie on a semicircle, when the obtuse angle is at the middle point. The diameter through *A* divides the circle into two semicircles. Prob(*A* is an acute corner of an obtuse angled triangle)

= Prob(*B* and *C* lie in the same semicircle)

= ½.

Each obtuse angled triangle has two acute corners so: Prob(*ABC* is obtuse angled)

= (½ + ½ + ½)/2.

Hence

Prob(ABC is acute angled) = ¼.

Tony says assume the first two points *A*, *B* have been chosen at random, (uniform distribution) and the angle between them is $\theta < \pi$. Then *ABC* is acute iff *C* lies in the arc vertically opposite to arc *AB* (see figure 4). The probability of this is θ/π. Clearly θ has uniform distribution on $[0, \pi)$, and hence Prob(*ABC* acute) = $\int_0^\pi \frac{\theta}{2\pi} \cdot \frac{d\theta}{\pi} = \frac{1}{4}$.

Howard's solution was similar to Tony's and he has a graph which shows the answer nicely.

Let P_1 be datum, and $\angle P_1P_2$ be θ, $\angle P_1P_3$ be ϕ. Triangle is acute iff P_3 is in the segment vertically opposite P_1P_2. Plot θ against ϕ. Points in the shaded area (figure 5) give acute angled triangle.

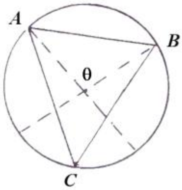


Figure 4

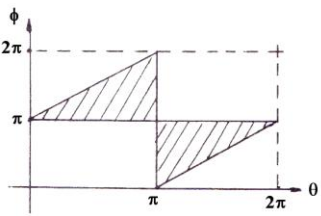


Figure 5

Angus MacDonald says ½.

Choose first point at random and draw diameter from this point. Choose second point and call the semicircle it is in *A*, and the other one *B*.

We get an acute angled triangle iff the third point is in *B*. Therefore probability is ½.

Angus wants to know if the probability of a right angled triangle is zero.

Yes it is, as Howard pointed out in his letter. Likewise I suppose the probability of any sufficiently special triangle, e.g. specific length, angle, area &c. is zero. Angus said he had tried unsuccessfully to set up a one–one correspondence between acute and obtuse angled triangles, and wondered if success would have automatically implied they were equiprobable.

No - I don't think so. It is not true in general that one–one correspondence implies equiprobability in infinite sets. Consider for instance the square with diagonal (0, 0) to (1, 1) in the plane.

Choose a point at random in the square.

The probability that it is on the x-axis is zero. Yet we can demonstrate a one–one correspondence between the points of the line segment [0, l] and the points of the square. Let a point in the square have coordinates (x, y) where $0 \le x \le 1$ and $0 \le y \le 1$. Suppose

$x = 0.a_1a_2a_3a_4 \ldots$

$y = 0.b_1b_2b_3b_4 \ldots$.

Combine these to produce the number $0.a_1b_1a_2b_2a_3b_3a_4b_4 \ldots$ which represents a unique point on the line [0, 1]. This operation can obviously be reversed so that a point on the line segment can be matched with a unique point in the square. e.g. $(x, y) = (0.123, 0.4567)$ gives 0.14253607 on the line segment. Similarly 0.9876543 on the line gives point (0.9753, 0.864) on the square. Therefore the line has the same number of points as the square. (Howard was also asking about this.) Note that the line and square don't have to be the same size - just use different 0, 1 scales for each one. Note also that we are not restricted to lines and squares; we can go by the obvious extension into higher dimensions.

Steve Ainley says ¾. He defines his random as - each point, independently is equally likely to be in any of the n different arcs of length $1/n$ for all $n \in \mathbb{N}$.

If second point is k distance from first ($k < ½$ for circumference of 1), then third point yields acute angled triangle when it falls in the longer $(1- k)$ arc.

So for each $k \in [0, ½)$ Prob(acute) $= 1 - k$. So overall probability P is

$$\frac{1}{½}\int_0^{½}(1-k)\,dk = \frac{3}{4}.$$

(I've just remembered that Steve Murphy sent me another letter about Sequences 57, in which he proved his conjecture that $f(n)$ = integer part of $\frac{1}{2}n(1+\sqrt{5})$.)

PROBLEM 60.1 *CLOCKS* 60 STEVE MURPHY

On a recent M101 programme they showed us that we can tell if a clock with no figures is upside down by examining the relative positions of the hour and minute hands.

Would this still be true if the clock were reflected in a mirror?

What is the effect of identical hour and minute hands, either in the mirror case or the upside down case?

PROBLEM 60.2 *AGES* 60 ANGUS MACDONALD

(Another of those standbys. Angus knows the answer but would like someone to demonstrate the reasoning.)

When Bert was just one year younger than Bill was when Ben was half as old as Bill will be three years from now, Ben was twice as old as Bill was when Ben was one third as old as Bert was three years ago. But when Bill was twice as old as Bert, Ben was one quarter as old as Bill was one year ago. If Bert is over fifty, how old are they all.

PROBLEM 60.3 *DRINKS* 60 ANGUS MACDONALD

Messrs Beer, Brandy, Chianti, Cider, Port, Sherry and Whiskey hold a party at which all those drinks are available.

Each man samples three different drinks, but nothing which begins with his initial. After the party, three of the seven drunks under the table are dead. One of the drinks was poisoned.

a) The three dead had among them drunk all seven drinks.

b) The brandy drinker was married, refused sherry, and his name drink was drunk only by Mr Brandy.

c) Mr Whiskey did not drink port.

d) Only Messrs Port, Sherry and Whiskey were not married.

e) The port drinkers outnumber the whiskey drinkers by one.

f) The namesake of the only drink sampled by all three bachelors drank chianti. Mr Chianti was the only other to try the drink all three bachelors had sampled.

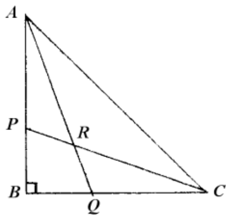
g) The murderer did not drink the name drink of any of his victims.

h) No one committed suicide.

Who poisoned what to kill whom?

PROBLEM 60.4 *TRIANGLES* 60

The figure shows a right-angled triangle *ABC*, with sides *AB*, *BC* each of length *s* cm. The points *P*, *Q* on *AB*, *BC* respectively are such that *PB* and *BQ* are both of length *t* cm. The lines *AQ* and *CP* meet in *R*. Find the area of the triangle *ARC* in terms of *s* and *t*.



PROBLEM 60.5 *TENNIS* 60

It is familiar that in games 'upsets of form' often occur: if *A* beats *B* and *B* beats *C* it may nevertheless happen that *C* beats *A*. Let us call a set {*A*, *B*, *C*} of three players whose results conform to this pattern a 'form breaking triple'.

Eight players participate in a tennis singles tournament, in which each player meets each of the other seven just once. At the end of the tournament each player receives a payment of w^2 pounds, where *w* is the number of matches he has won. The total amount paid out to the eight players is £106. How many form breaking triples are there? (There are no draws in tennis.)

(The above two problems came from Brian Stewart. They are from a set sent to Scottish schools last year by the Scottish Mathematical Council.)

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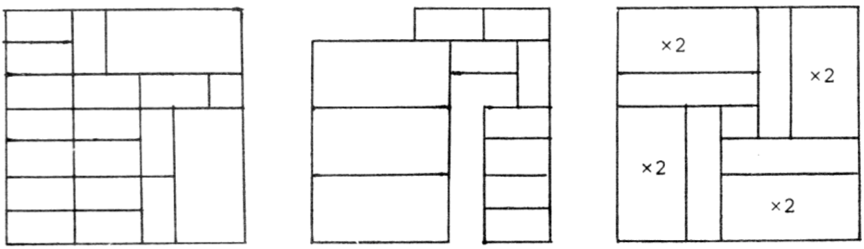
STOP PRESS - Here is a packing of the 41 bricks in the 7^2 box.



Layer 1 ***Layer 2*** ***Layer 3***

STOP STOP PRESS. There was no mention in 59 13 *about what to do with the pieces of puzzle. You could make them into a square. Also line* 59 15 5 *should be* $g(n) = f(n) + n$, obviously.

EDITORIAL

The reason I asked the question about getting consecutive has's which has been so effectively disposed of by Steve Ainley (page 10) was because of a sentence I once saw (and never forgot. I have become quite a bore with it though I can't find anyone to take it seriously.) The sentence was composed by Noam Chomsky. It goes: "Remarkable is the rapidity of the motion of the wing of the hummingbird". This can be transmuted through "The hummingbird's wing's motion's rapidity is remarkable" to

> The rapidity that the motion that the wing that the hummingbird has has has is remarkable.

Isn't that remarkable? For every extra noun you get another has!

(If you don't believe that it can work like that, consider this, from *Psychology of Communication* by George A Miller, 1967. "We booed the football team that played the team that brought the supporters that chased the girls that were in the park." i.e.: "The girls that the supporters that the team that the football team that we booed played brought chased were in the park.")

This issue is very much taken up with words, isn't it. I might as well tell you about the amulet that was dug up in Manchester last year. It is in the form of a cryptogram in the shape of a magic square. *Country Life* did an article on it on February 1st and I pass on some of the points made (though without the pictures) on the assumption that M500 readers don't visit their dentists very often.

The cryptogram is

```
R O T A S
O P E R A
T E N E T
A R E P O
S A T O R
```

which can be translated "Sator the sower guides the wheels with care". The letters can be rearranged as

```
          P
          A
          T
          E
          R
P A T E R N O S T E R
          N
          O
          S
          T
          E
          R
```

the first two words of the Lord's Prayer in the form of a cross. A and O are twice repeated, suggesting "I am the Alpha and Omega, the beginning and the ending, saith the Lord", *Rev* 18. The hypothetical explanation for the existence of the amulet is that it was one worn by a member of a group of Christians who wanted to be able to recognise one another without being detected by the Romans.

Eddie Kent.