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6407946920466964414730883494932671301032834945539014427271718483321388360701837312 $+ d$, $d = -1, 1, 5, 7$, are prime

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Solution 284.2 – 13 cards

> A standard pack of 52 playing cards is shuffled and dealt into 13 piles of four. Is it always possible to select one card from each pile so that the chosen cards consist of 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A, not necessarily of the same suit?

Reinhardt Messerschmidt

Such a selection is always possible. This is a consequence of the theorem below, which is a version of *Hall's marriage theorem*.

Suppose X, I are nonempty finite sets and $(A_i)_{i\in I}$ is a family of subsets of X. The family $(A_i)_{i\in I}$ satisfies *Hall's condition* if

$$|J| \;\le\; \left|\bigcup_{i\in J} A_i\right| \qquad \text{for every nonempty subset } J \text{ of } I.$$

A family $(x_i)_{i\in I}$ in X is a *system of distinct representatives* (SDR) for $(A_i)_{i\in I}$ if the x_i are distinct and $x_i \in A_i$ for every $i \in I$.

Note that if $(A_i)_{i\in I}$ satisfies Hall's condition and $i \in I$, then letting $J = \{i\}$ gives $1 \le |A_i|$; i.e. A_i is nonempty.

Theorem 1 *If X, I are nonempty finite sets and $(A_i)_{i\in I}$ is a family of subsets of X that satisfies Hall's condition, then $(A_i)_{i\in I}$ has an SDR.*

Application of the theorem to the original problem

A card can be viewed as an ordered pair consisting of a suit from the set $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and a rank from the set

$$X \;=\; \{2,\ 3,\ \ldots,\ 10,\ \mathrm{J},\ \mathrm{Q},\ \mathrm{K},\ \mathrm{A}\}.$$

Let $I = \{1, 2, \ldots, 13\}$ represent the set of piles that the cards have been dealt into. For every $i \in I$, let A_i be the set of distinct ranks in the i-th pile.

We will show that $(A_i)_{i\in I}$ satisfies Hall's condition. Suppose J is a nonempty subset of I, i.e. J represents a subset of piles. Since each pile has exactly four cards, the subset of piles has exactly $4|J|$ cards. Since the subset of piles has exactly $|\cup_{i\in J} A_i|$ distinct ranks and each rank can be paired with at most four suits, the subset of piles has at most $4|\cup_{i\in J} A_i|$ cards; therefore

$$4|J| \;\le\; 4\left|\bigcup_{i\in J} A_i\right|;$$

i.e. $(A_i)_{i \in I}$ satisfies Hall's condition.

It follows from the theorem that we can select one card from each pile so that the chosen cards have distinct ranks.

Proof of the theorem

The following proof is adapted from [1]. It uses induction on $|I|$.

Base case. Suppose $I = \{i_0\}$. Choose any $x_0 \in A_{i_0}$, then (x_0) is an SDR for $(A_i)_{i \in I}$.

Inductive case. Suppose $|I| > 1$ and the theorem holds for families of fewer than $|I|$ subsets. A subset J of I is *critical* if $\emptyset \subsetneq J \subsetneq I$ and

$$|J| \;=\; \left|\bigcup_{i \in J} A_i\right|.$$

Case 1. Suppose I does not have a critical subset. Choose any $i_0 \in I$ and $x_0 \in A_{i_0}$. Let $K = I - \{i_0\}$. For every $i \in K$, let $B_i = A_i - \{x_0\}$.

We will show that $(B_i)_{i \in K}$ satisfies Hall's condition. Suppose J is a nonempty subset of K. Since $(A_i)_{i \in I}$ satisfies Hall's condition and J is not a critical subset of I,

$$|J| \;<\; \left|\bigcup_{i \in J} A_i\right|.$$

Furthermore,

$$\bigcup_{i \in J} A_i - \bigcup_{i \in J} B_i \;\subseteq\; \{x_0\};$$

therefore

$$\left|\bigcup_{i \in J} A_i\right| \;\leq\; \left|\bigcup_{i \in J} B_i\right| + 1;$$

therefore

$$|J| \;\leq\; \left|\bigcup_{i \in J} B_i\right|;$$

i.e. $(B_i)_{i \in K}$ satisfies Hall's condition.

It follows by the inductive hypothesis that $(B_i)_{i \in K}$ has an SDR. This can be extended to an SDR for $(A_i)_{i \in I}$ by adding x_0.

Case 2. Suppose I has a critical subset K. The family $(A_i)_{i \in K}$ has fewer than $|I|$ elements, and satisfies Hall's condition because $(A_i)_{i \in I}$ does;

therefore $(A_i)_{i\in K}$ has an SDR by the inductive hypothesis. For every $i \in I - K$, let

$$B_i \;=\; A_i - \bigcup_{j\in K} A_j.$$

We will show that $(B_i)_{i\in I-K}$ satisfies Hall's condition. Suppose J is a nonempty subset of $I - K$. We have

$$\bigcup_{i\in J} B_i \;=\; \bigcup_{i\in J}\Bigl(A_i - \bigcup_{j\in K} A_j\Bigr) \;=\; \bigcup_{i\in J} A_i - \bigcup_{j\in K} A_j \;=\; \bigcup_{i\in J\cup K} A_i - \bigcup_{j\in K} A_j;$$

therefore

$$\Bigl|\bigcup_{i\in J} B_i\Bigr| \;=\; \Bigl|\bigcup_{i\in J\cup K} A_i\Bigr| - \Bigl|\bigcup_{j\in K} A_j\Bigr|.$$

Since $(A_i)_{i\in I}$ satisfies Hall's condition and K is a critical subset of I,

$$\Bigl|\bigcup_{i\in J\cup K} A_i\Bigr| \;\geq\; |J\cup K|, \qquad \Bigl|\bigcup_{j\in K} A_j\Bigr| \;=\; |K|;$$

therefore

$$\Bigl|\bigcup_{i\in J} B_i\Bigr| \;\geq\; |J\cup K| - |K| \;=\; |J|;$$

i.e. $(B_i)_{i\in I-K}$ satisfies Hall's condition.

It follows by the inductive hypothesis that $(B_i)_{i\in I-K}$ has an SDR. Joining the SDRs for $(A_i)_{i\in K}$ and $(B_i)_{i\in I-K}$ gives an SDR for $(A_i)_{i\in I}$.

References

[1] `http://homepages.warwick.ac.uk/~masgax/week10.pdf`
(accessed 28 October 2018).

Problem 288.1 – Matrix powers

Given a_1, b_1, c_1, d_1, let M be a 2×2 matrix defined by $M^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$, $n = 1, 2, \ldots$. Show that $b_n c_1 = b_1 c_n$ for $n = 1, 2, \ldots$.

Flippin' functions

Martin Hansen

When I first became a secondary school teacher of mathematics I rather liked the visual approach to writing down the inverse of a function. A typical GCSE examination question would begin by presenting a flow chart and ask what function it described. For example:

$$x \;\to\; +4 \;\to\; \text{Reciprocal} \;\to\; \times 5 \;\to\; f(x).$$

Readers of M500 will have no difficulty in realising that the function described by the diagram by flowing left to right as indicated by the arrows is

$$f(x) \;=\; \frac{5}{x+4}, \qquad x \neq -4.$$

Rather attractively, clever pupils were able to write down the inverse function by simply moving against the flow, remembering that the inverse of multiplication is division, and of addition is subtraction. Taking a reciprocal is self inverse. Thus:

$$f^{-1}(x) \;=\; \frac{5}{x} - 4, \qquad x \neq 0.$$

For some pupils it proved helpful to make them draw the reversed flowchart first;

$$f^{-1}(x) \;\leftarrow\; -4 \;\leftarrow\; \text{Reciprocal} \;\leftarrow\; \div 5 \;\leftarrow\; x.$$

My enthusiasm for this visual approach of obtaining the inverse of a function lessened when a colleague told me it failed as the functions became more complicated, and gave me, as an example,

$$f(x) \;=\; \frac{x}{x+1}, \qquad x \neq -1.$$

The problem is having an x in more than one place. The mainstream algebraic approach of, effectively, making x the subject of the formula seemed the only way to cope with such questions. Recently, however, when looking at this function with an A-Level class I began to wonder if I could, after all, get the flow charting method to work. Some simple, if mildly sneaky, algebra was employed to get the x into just one place;

$$\begin{aligned} f(x) \;=\; \frac{x}{x+1} \;&=\; \frac{x+1-1}{x+1} \\ &=\; \frac{x+1}{x+1} - \frac{1}{x+1} \\ &=\; 1 - \frac{1}{x+1}. \end{aligned}$$

A flowchart followed:

$$x \;\rightarrow\; +1 \;\rightarrow\; \text{Reciprocal} \;\rightarrow\; \text{Flip sign} \;\rightarrow\; +1 \;\rightarrow\; f(x).$$

$$x \;\Rightarrow\; x+1 \;\Rightarrow\; \frac{1}{x+1} \;\Rightarrow\; -\frac{1}{x+1} \;\Rightarrow\; -\frac{1}{x+1}+1,$$

which could be reversed:

$$f^{-1}(x) \;\leftarrow\; -1 \;\leftarrow\; \text{Reciprocal} \;\leftarrow\; \text{Flip sign} \;\leftarrow\; -1 \;\leftarrow\; x,$$

$$x \;\Rightarrow\; x-1 \;\Rightarrow\; -(x-1) \;\Rightarrow\; \frac{1}{-(x-1)} \;\Rightarrow\; \frac{1}{-(x-1)}-1$$

and finally some algebraic manipulation:

$$\begin{aligned} \frac{1}{-(x-1)}-1 &= \frac{1}{1-x}-1 \\ &= \frac{1}{1-x}-\frac{1-x}{1-x} \\ &= \frac{1-(1-x)}{1-x} = \frac{x}{1-x} \end{aligned}$$

got to the answer that I knew, from the algebraic approach, was correct;

$$f^{-1}(x) = \frac{x}{1-x}, \qquad x \neq 1.$$

I'm going to conclude with a challenge.

- Can this 'foot in the door' be developed to tackle the 'worst case' GCSE inverse function questions which are of the form of a Möbius transformation
$$f(x) = \frac{ax+b}{cx+d}$$
but with a, b, c and d restricted to being integers, and $x \in \mathbb{R}$, $x \neq -d/c$?

- What other awkward functions can be handled, keeping in mind that to have an inverse, a function must be one-to-one?

- For example, what about $f(x) = x^3 + x$?

Solution 283.2 – Another determinant

This is very similar to Problem 282.5, except that the answer is very different. Compute

$$\mathrm{L}(n,\lambda) \;=\; \det \begin{bmatrix} n & n-\lambda & \dots & n-\lambda & n-\lambda \\ n-\lambda & n & \dots & n-\lambda & n-\lambda \\ \dots & \dots & \dots & \dots & \dots \\ n-\lambda & n-\lambda & \dots & n & n-\lambda \\ n-\lambda & n-\lambda & \dots & n-\lambda & n \end{bmatrix},$$

where n is the number of rows (or columns) in the matrix.

Tommy Moorhouse

That useful identity again. This problem can be solved in exactly the same way as Problem 282.5 using the identity

$$\det e^A \;=\; \exp(\operatorname{Tr} A).$$

This time we set $L(n,\lambda) = \lambda\mathbb{I} + (n-\lambda)\mathbb{U}$ where $\mathbb{I}$ is the identity matrix and $\mathbb{U}$ is the matrix with all its entries equal to unity. This is our candidate for e^A. Formally taking logarithms we get

$$\begin{aligned} A &= \log\left(\lambda\mathbb{I} - (n-\lambda)\mathbb{U})\right) \\ &= \log\lambda\mathbb{I} + \log\left(\mathbb{I} - \left(1 - \frac{n}{\lambda}\right)\mathbb{U}\right) \;=\; \log\lambda\mathbb{I} - \sum_{k=1}^{\infty}\frac{1}{k}\left(1-\frac{n}{\lambda}\right)^k \mathbb{U}^k. \end{aligned}$$

Now, $\mathbb{U}^k = n^{k-1}\mathbb{U}$ for $k \geq 1$ giving

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{1}{k}\left(1-\frac{n}{\lambda}\right)^k \mathbb{U}^k &= \frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}\left(n\left(1-\frac{n}{\lambda}\right)\right)^k \mathbb{U} \\ &= -\frac{1}{n}\log\left(1 - n\left(1-\frac{n}{\lambda}\right)\right)\mathbb{U}. \end{aligned}$$

We have now determined that

$$A \;=\; \log\lambda\mathbb{I} + \frac{1}{n}\log\left(1 - n\left(1-\frac{n}{\lambda}\right)\right)\mathbb{U}$$

so taking the trace gives

$$\operatorname{Tr} A \;=\; n\log\lambda + \log\left(1 - n\left(1-\frac{n}{\lambda}\right)\right).$$

Taking the exponential gives the required determinant as

$$\det L(n,\lambda) \;=\; \exp(\operatorname{Tr} A) \;=\; \lambda^n(1-n) + n^2\lambda^{n-1}.$$

Peter Fletcher

The matrix determinant lemma (see e.g. `https://en.wikipedia.org/wiki/Matrix_determinant_lemma`) states that if $\mathbf{A}$ is an invertible $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are length n column vectors, then

$$\det\left(\mathbf{A} + \mathbf{u}\mathbf{v}^{\mathrm{T}}\right) = \left(1 + \mathbf{v}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}\right)\det(\mathbf{A}).$$

We can write

$$L(n, \lambda) = \det\left(\lambda \mathbf{I}_n + \mathbf{u}\mathbf{v}^{\mathrm{T}}\right),$$

where $\mathbf{u}$ is a length n column vector of 1s and $\mathbf{v} = (n - \lambda)\mathbf{u}$. Then we have

$$\mathbf{v}^{\mathrm{T}}\mathbf{u} = (n - \lambda)(1\ \ 1\ \ 1\ \cdots\ 1)(1\ \ 1\ \ 1\ \cdots\ 1)^{\mathrm{T}} = (n - \lambda)n.$$

Now we can write

$$\begin{aligned} L(n, \lambda) &= \left(1 + \mathbf{v}^{\mathrm{T}}\left(\frac{\mathbf{I}_n}{\lambda}\right)\mathbf{u}\right)\det(\lambda \mathbf{I}_n) \\ &= \left(1 + \frac{1}{\lambda}\mathbf{v}^{\mathrm{T}}\mathbf{u}\right)\lambda^n = \left(\lambda + \mathbf{v}^{\mathrm{T}}\mathbf{u}\right)\lambda^{n-1} = \big(\lambda + (n - \lambda)n\big)\lambda^{n-1} \\ &= \big(n^2 - (n-1)\lambda\big)\lambda^{n-1} = n^2\lambda^{n-1} - (n-1)\lambda^n. \end{aligned}$$

Stuart Walmsley

This is very similar to Problem 282.5. Both problems can be solved at once using my solution to Problem 261.6 Determinant Equation, which appeared in M500 **263**. Problem 282.5 concerned the determinant

$$\Lambda(n, \lambda) = \det \begin{bmatrix} \lambda & -1 & \dots & -1 & -1 \\ -1 & \lambda & \dots & -1 & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & \dots & \lambda & -1 \\ -1 & -1 & \dots & -1 & \lambda \end{bmatrix}.$$

Using the symmetry of the determinant as described in the solution to 261.6,

$$\Lambda(n, \lambda) = (\lambda - n + 1)(\lambda + 1)^{n-1}.$$

This readily adapted to the present situation. To get a more general form consider the mapping $\lambda \to x,\ -1 \to a$. Then the new determinant, $D(n, x)$ say, is given by

$$D(n, x) = (x + (n-1)a)(x - a)^{n-1}.$$

The determinant under consideration in this problem $L(n,\lambda)$ is given by the mapping $x \to n,\ a \to n-\lambda$, leading to

$$L(n,\lambda) \;=\; (n+(n-1)(n-\lambda))(n-n+\lambda)^{n-1},$$

which simplifies to

$$L(n,\lambda) \;=\; n^2\lambda^{n-1} - (n-1)\lambda^n.$$

For example:

$$\begin{aligned} L(2,\lambda) &= 4\lambda - \lambda^2, \\ L(3,\lambda) &= 9\lambda^2 - 2\lambda^3, \\ L(4,\lambda) &= 16\lambda^3 - 3\lambda^4. \end{aligned}$$

Two specific forms may be noted: $L(n,n) = n^n$, $L(n,0) = 0$. Both these results can, however, be deduced from the unexpanded form using simple properties of determinants.

Problem 288.2 – Matrix

Mohammed Mehbali

For n a positive integer, construct a $3n \times n^3$ matrix M_n as follows. Let I_n denote the $n \times n$ identity matrix.

The first n rows of M_n is I_n with each column repeated n^2 times.

For the next n rows, take n copies of I_n and repeat each column n times.

For the last n rows, take n^2 copies of I_n.

Thus when $n=2$ and $n=3$ we have

$$M_2 = \begin{bmatrix} 11110000 \\ 00001111 \\ 11001100 \\ 00110011 \\ 10101010 \\ 01010101 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 111111111000000000000000000 \\ 000000000111111111000000000 \\ 000000000000000000111111111 \\ 111000000111000000111000000 \\ 000111000000111000000111000 \\ 000000111000000111000000111 \\ 100100100100100100100100100 \\ 010010010010010010010010010 \\ 001001001001001001001001001 \end{bmatrix}.$$

Prove that the rank of M_n is $3n-2$.

Solution 283.1 – Two integer equations

> Given positive integers m and n, show that the number of solutions in non-negative integers of
>
> $$x_1 + x_2 + \cdots + x_n \;=\; m - 1 \tag{1}$$
>
> is equal to the number of solutions in non-negative integers of
>
> $$x_1 + x_2 + \cdots + x_m \;=\; n - 1. \tag{2}$$
>
> Construct a one-to-one mapping between the solution sets.

Tony Forbes

The number of solutions to (1) is $\binom{n+m-2}{n-1}$. To see this, carry out the following procedure, where $m = 26$ and $n = 29$ in the example.

Write down $n + m - 2$ copies of the symbol `I` and add parentheses.

`(IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII)`

Choose $n - 1$ `I`s and change them to commas. The number of ways to make this choice is $\binom{n+m-2}{n-1}$.

`(,I,,I,II,,,I,I,,II,,II,III,,I,,IIIIIII,,,,,,,,,II,I,I)`

Now interpret the n not necessarily non-empty blocks of `I`s in this list as n Roman numerals to represent x_1, x_2, …, x_n. (Recall that nothing (i.e. the absence of something) denotes 0 in the Roman system, and I suppose you could gather the `I`s in the usual way using `V`, `X`, `L`, `C`, `D`, `M`, but I don't see how that helps—if m is large and n is small, say $m > 10^{100}$ and $n < 100$, you are merely trading a lot of `I`s for a lot of `M`s.) There are

$$n + m - 2 - (n - 1) \;=\; m - 1$$

`I`s and hence

$$x_1 + x_2 + \cdots + x_n \;=\; m - 1.$$

Thus we have constructed a solution of (1). Clearly the process is reversible. To get the corresponding solution of (2), interchange `I` $\Leftrightarrow$ comma.

I am wondering if this theorem was known to Roman mathematicians.

Discovery in mathematics

Sebastian Hayes

How does one discover a theorem of interest? There seems to be no lack of inventive minds amongst the contributors to this magazine, so it would be interesting to hear some of them tell us how they arrived at their discoveries. As far as I can make out, there seem to be three main procedures: *the inductive method*, *the deductive method* and *the exploratory or playful method*.

By 'inductive method' I do not mean mathematical induction as such but simply the standard procedure of scanning a mass of data and trying to discern some underlying structure or pattern. One then investigates to see whether the pattern keeps on recurring and if it does, one attempts to show that this feature is bound to persist—but this is, strictly speaking, part of proving not discovery. Most theorems in Number Theory seem to have been discovered in this way: I doubt if anyone ever deduced from first principles that $F_{2n} = F_n^2 + F_{n-1}^2$; almost certainly Lucas and others noted that $5^2 + 3^2 = 34$ and $13^2 + 21^2 = 610$ and went on from there. Even such a prestigious theorem as the Prime Number Theorem apparently originated in the scanning of data (allegedly Gauss, as a teenager, conjectured that $\pi(x)$ varied with $x/(\log x)$ after examining a recently published list of primes).

The 'natural' movement of the human mind is from the particular to the general, the concrete to the abstract, which is why the inductive approach comes much more easily to most people. By concentrating on the logical aspects modern mathematicians have certainly tightened up the subject but at the cost of completely alienating the general public. This is far from being a good thing even for mathematics itself: it means that people for whom aesthetic considerations are uppermost avoid mathematics like the plague—the exact reverse of the situation that prevailed during the Renaissance and baroque eras.

The deductive method usually proceeds either by generalizing some known result, or by applying it to a particular case where it yields unexpected consequences. This magazine has printed several articles in recent years which are generalizations of the Fibonacci Sequence and one gathers from M500 **205** that Dennis Morris is currently involved in generalizing the hyperbolic functions, themselves generalizations of the trigonometric functions.

The starting point for an extension must itself already have some generality—a single numerical case is of little or no value. However, the starting point mustn't be too general: I don't think any contemporary pure mathematician ever sat down of an evening with the Axioms of von Neumann Set Theory in order to see what new theorems he or she could deduce.

An example of the opposite process, particularizing, is Pascal considering the expansion of $(a + b)^n$ and setting $a = 1$, $b = 1$ thus showing that the total number of possible combinations of n objects taking r at a time is 2^n. (This includes the choice of not making a selection at all.) Setting some variable or variables at unity or at a multiple or submultiple of π seems to be a standard stratagem that has yielded a surprisingly rich harvest of theorems. One would like to hear of other 'tricks of the trade' but modern textbooks are surprisingly coy on the subject—I have yet to come across a chapter, let alone a whole book, entitled *How to Devise or Discover Interesting Theorems*.

The third method is not really a method at all: it is basically just messing about and seeing what comes up. *Homo sapiens* is, thank God, also *Homo ludens*. Leonhard Euler, the most prolific mathematician of all time—his works run to seventy-five large volumes—played around with mathematical formulæ as children play with toys (or did before the advent of computers). What would happen if we did this? Or this? And then that?

As far as I am concerned mathematics is not an ensemble of water-tight logical systems but more like a series of wild life reserves where strange plants and animals can not only be observed but actually bred or grown from seed. It is notable that many of the most inventive mathematicians were amateurs, e.g. Leibnitz, Fermat. In contemporary theatre and above all 'painting' (conceptual art), originality is so much the order of the day that any sort of rubbish is acceptable provided you are doing something that nobody has done before. But for some reason in mathematics we have the opposite set-up: rigour has well nigh stifled elegance and inventiveness. Mathematics was once one of the 'humanities'. I am not quite sure what the 'humanities' were, or were intended to be, but I assume the basic idea was that studying them did not just make you more learned but more 'human'. It would perhaps be going too far to claim that the humanistic approach to mathematics, i.e. treating it at once as a science, an art and a philosophy of nature, invariably produced a better type of person—one gathers that Newton was rather a nasty man—but there is no doubt in my mind that trying to turn someone into a logical machine is unlikely to improve human behaviour (or even judgement in real world situations which are all too often extremely messy). *Est in medio verum*—truth lies in the middle.

Solution 283.5 – Primes

Show that

$$\lim_{N\to\infty}\left(\prod_{\substack{p\le N,\ p\text{ prime}\\ p\equiv 1 \pmod 4}}\frac{p}{p-1}\right)\left(\prod_{\substack{p\le N,\ p\text{ prime}\\ p\equiv 3 \pmod 4}}\frac{p}{p+1}\right) = \frac{\pi}{4}.$$

Tommy Moorhouse

Forming a Dirichlet L-function We can recast the expression to be taken in the limit $N\to\infty$ in the form (where all products extend to $p\le N$)

$$\left(\prod_{\substack{p\equiv 1\\ \pmod 4}}\frac{p}{p-1}\right)\left(\prod_{\substack{p\equiv 3\\ \pmod 4}}\frac{p}{p+1}\right) = \left(\prod_{\substack{p\equiv 1\\ \pmod 4}}\frac{1}{1-p^{-1}}\right)\left(\prod_{\substack{p\equiv 3\\ \pmod 4}}\frac{1}{1+p^{-1}}\right).$$

In this form it is easier to see that the product is related to the Dirichlet series

$$L(s,\chi_2) = \prod_p \frac{1}{1-\chi_2(p)p^{-s}},$$

where χ_2 is the non-principal Dirichlet character modulo 4 (i.e. $\chi_2(1)=1$, $\chi_2(2)=0$, $\chi_2(3)=-1$, $\chi_2(4)=0$; see Chapter 6 of [Apostol]).

We will take the limit $N\to\infty$ from the outset, but take $s=1+\epsilon$ (say) to ensure that the Dirichlet L-function converges. Using the definition

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$$

we see that

$$L(s,\chi_2) = 4^{-s}\left(\zeta\left(s,\frac{1}{4}\right)-\zeta\left(s,\frac{3}{4}\right)\right).$$

The expression on the right hand side can be evaluated at $s=1$ because of cancellations between the terms of the ζ function, and we find

$$\begin{aligned}\lim_{s\to 1} L(1,\chi_2) &= \frac{1}{4}\left(\frac{1}{1/4}+\frac{1}{5/4}+\cdots-\frac{1}{3/4}-\frac{1}{7/4}-\cdots\right)\\ &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\\ &= \arctan 1 = \frac{\pi}{4}.\end{aligned}$$

Thus the original product is equal to $\pi/4$.

Alternative approach We could make use of the relation (Theorem 12.2 of [Apostol])

$$\Gamma(s)\zeta(s,a) \;=\; \int_0^\infty \frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx$$

to rewrite $L(s,\chi_2)$ as

$$\begin{aligned}\lim_{s\to 1} L(s,\chi_2) &= \frac{1}{4}\int_0^\infty \frac{e^{-x/4}-e^{-3x/4}}{1-e^{-x}}\,dx \\ &= \frac{1}{4}\int_0^\infty \frac{e^{-x/4}(1-e^{-x/2})}{1-e^{-x}}\,dx \\ &= \frac{1}{8}\int_0^\infty \frac{dx}{\cosh(x/4)} \\ &= \frac{1}{2}\arctan(\sinh(x/4))\Big|_0^\infty \\ &= \frac{\pi}{4}.\end{aligned}$$

The last equality may be found in standard texts, such as [Abramowitz & Stegun]. This leads to an interesting problem.

Problem Using the above alternative methods to analyse the product

$$\left(\prod_{p\equiv 1 \bmod 6}\frac{p}{p-1}\right)\left(\prod_{p\equiv 5 \bmod 6}\frac{p}{p+1}\right),$$

or otherwise, show that

$$1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\cdots \;=\; \frac{\sqrt{3}\pi}{6}.$$

The integral

$$\int \frac{\cosh(x/6)}{1+2\cosh(x/3)}\,dx \;=\; \sqrt{3}\arctan\left(\frac{2\sinh(x/6)}{\sqrt{3}}\right)$$

(which you can check by differentiation) may be of use.

References

[Apostol] Apostol, T., *Introduction to Analytic Number Theory*, Springer, 1976.

[Abramowitz & Stegun] Abramowitz, M., and Stegun, I. A., *Handbook of Mathematical Functions*, Dover, 1972.

Solution 283.6 – Pistachio nuts

> A bowl contains n pistachio nuts and an equal number of empty half shells. You can't see what you are taking from the bowl, and may therefore get a nut or a half shell. A single pistachio nut supplies 10 kcal of energy, but 0.1 kcal is used in opening the shell and eating the nut. Also, 0.1 kcal is used in removing an object from the bowl, and another 0.1 kcal in replacing an object in the bowl.
>
> You can therefore devise a bizarre and pointless ritual of removing, opening and eating nuts, and replacing some or all of the half shells or unopened nuts in the bowl, that will ensure that you gain or lose no energy by consuming all the nuts. What is the simplest ritual, i.e. involving the fewest actions, that will produce this result for any value of n?

Ralph Hancock

I think this is the simplest solution. No one said you couldn't sort what you get out of the bowl. So it's utterly banal, involving no mathematical procedure more advanced than counting. But at least it leaves the table tidy.

1. Remove the entire contents of the bowl, one by one.
2. Replace all the half shells in the bowl, again one by one.
3. Remove all the half shells again.
4. Repeat steps 2 and 3 46 times.
5. Finally replace all the half shells in the bowl.
6. Shell and eat the nuts.
7. As you do this, return all the newly opened half shells to the bowl.

Problem 288.3 – Digit powers

Take a positive integer, n, and compute $P(n) = \sum_d d^d$, where d runs through the (decimal) digits of n. For example, recalling that $0^0 = 1$, $P(3210) = 27+4+1+1 = 33$. Show that $n = 1$ and $n = 3435$ are the only instances where $P(n) = n$. Or find another.

This is not dissimilar to 'Factorial digital invariants' by David Singmaster, M500 **187**, where he sums $d!$ instead of d^d. Here, the only fixed points of $n \mapsto \sum_{d,\ d \text{ runs through the digits of } n} d!$ occur at $n = 1$, 2, 145 and 40585.

Problem 288.4 – Two dice

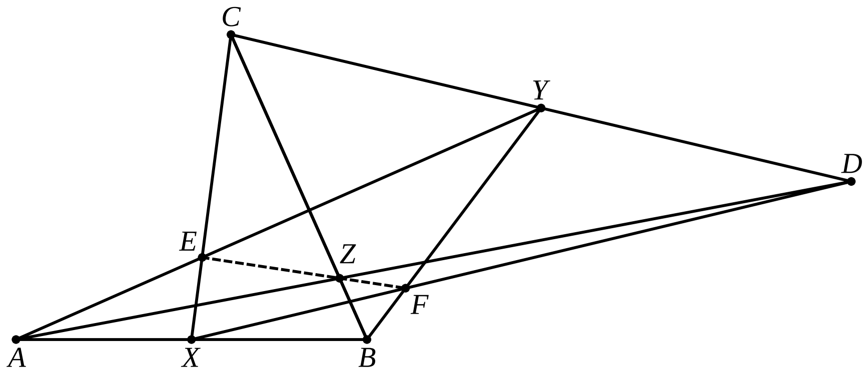
You go to the casino to play a game involving two dice and the redistribution of wealth. The casino's die has its faces marked 2,3,6,6,6,7, and yours is marked 1,1,5,5,9,9. At each turn the dice are thrown and the winner is whoever's die shows the higher number. The odds are $1:1$ and the loser pays the winner the agreed stake.

As usual, the game is biased in favour of the casino, as you can verify by examining all 36 possible outcomes (or otherwise). After a while you detect this bias and you refuse to play any more unless the game is made fairer.

You suggest throwing each die twice and taking the average of the two scores. For example, if the casino throws (6,7) and you throw (1,9), you lose because 6.5 is greater than 5. The casino, whose ultimate aim is to ruin you, agrees on the additional condition that it wins if the averages are the same—as would happen if the scores were (7,7) and (5,9). You agree and the game continues under the new rules. Was this wise?

Problem 288.5 – Lines

In the picture, X is the midpoint of AB, Y is the midpoint of CD, E is the intersection of AY and CX, F is the intersection of BY and DX, and Z is the intersection of AD and BC. Show that Z is on EF.



Problem 288.6 – Icosahedron in a cylinder

What is the smallest radius of a cylinder into which you can insert a regular icosahedron with edge length 1?

Problem 288.7 – Number representation

Here's a chance for an M500 reader to make some money. All you have to do is decide the following conjecture.

> Every integer greater than 7 can be written in the form
>
> $$p + 2^k + (1 + (n \bmod 2)) \cdot 5^m,$$
>
> where p is an odd prime, k and m are nonnegative integers, and $2^k + (1 + (n \bmod 2)) \cdot 5^m$ is squarefree.

Here, *squarefree* means not divisible by the square of a prime, and (n mod 2) means 0 if n is even, 1 if n is odd. The conjecture is saying that every sufficiently large even number can be expressed the sum of a prime and a squarefree integer which is itself the sum of a power of 2 and a power of 5. Clearly one is being reminded of Goldbach's conjecture: every sufficiently large even number can be expressed the sum of a prime and a squarefree integer which is itself a prime. However, squarefree numbers which are not necessarily prime are much easier to deal with than primes, and so it is possible that the problem might be attackable. The conjecture for odd numbers is the same except that we must multiply the power of 5 by 2. For example,

$$\begin{aligned}
8 &= 3 + 2^2 + 5^0, \\
9 &= 3 + 2^2 + 2 \cdot 5^0, \\
10 &= 3 + 2^1 + 5^1, \ldots, \\
20000000001 &= 7792968719 + 2^5 + 2 \cdot 5^{14}, \ldots.
\end{aligned}$$

For the last one, you can verify that 7792968719 is prime and that

$$20000000001 - 7792968719 = 2 \cdot 3049 \cdot 2001809.$$

You might want to look up `https://oeis.org/A304081`, where it states that Zhi–Wei Sun, the originator of the problem, is offering \$2500 for a proof or \$250 for a counter-example. It also states that the conjecture has been verified up to $2 \cdot 10^{10}$. (Actually $2 \cdot 10^{10} + 1$; see above.)

Problem 288.8 – Binomial coefficient sum

Show that

$$\sum_{i=0}^{k} \binom{k}{i}^2 \binom{n+2k-i}{2k} = \binom{n+k}{k}^2.$$

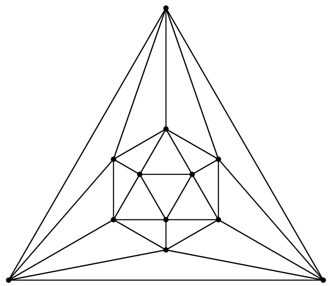
Problem 288.9 – Chain

A chain of length $2C$ is suspended between the tops of two vertical poles of height H, $0 \le H \le C$ and just grazes the ground at its centre. How far apart are the poles?

A large company was and possibly still is offering $H = C = 40\,\mathrm{m}$, as a test for job applicants. Presumably, mathematical skills are irrelevant and one's employment prospects are determined only by how long it takes to discover that this special case is trivial.

Problem 288.10 – 12-vertex 5-regular graphs

Tony Forbes



A graph is *d-regular* if every vertex has d neighbours. A graph is *planar* if you can draw it on a sheet of paper without any lines crossing. (I leave it to you to transform this vague explanation into a precise definition.) Suppose graph G has 12 vertices and is 5-regular. Show that if G is planar then it must be the vertex-edge graph of the icosahedron.

If you go to

`http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#CRG`,

you will find a lot of information about d-regular graphs. In particular, there are 7849 5-regular graphs with 12 vertices (7848 connected, 1 not), which can be downloaded in compressed form. So one way to solve the problem is to test each of the these graphs for planarity. However, I am hoping there is a proof that avoids a lot of computation.

More stuff wanted Once again we are running out of substantial articles. Please get writing! May I (TF) especially urge those of you who are of the opinion that much of M500 is incomprehensible to send us mathematical stuff that you and other readers of M500 can readily understand.

Fortunately we do have plenty of solutions to M500 problems. But to make sure there is something to print in M500 **289** I have retained material relating to problems that have appeared in M500 **285** and beyond. They provide an essential backstop against possible shortages.

Contents

04.01

Front cover Four magnificent 10132-digit primes,

$$667674063382677 \times 2^{33608} + d, d \in \{-1, 1, 5, 7\},$$

discovered by **Peter Kaiser** in February 2019. A new world record and the first example of a prime quadruplet breaking the 10000-digit barrier.