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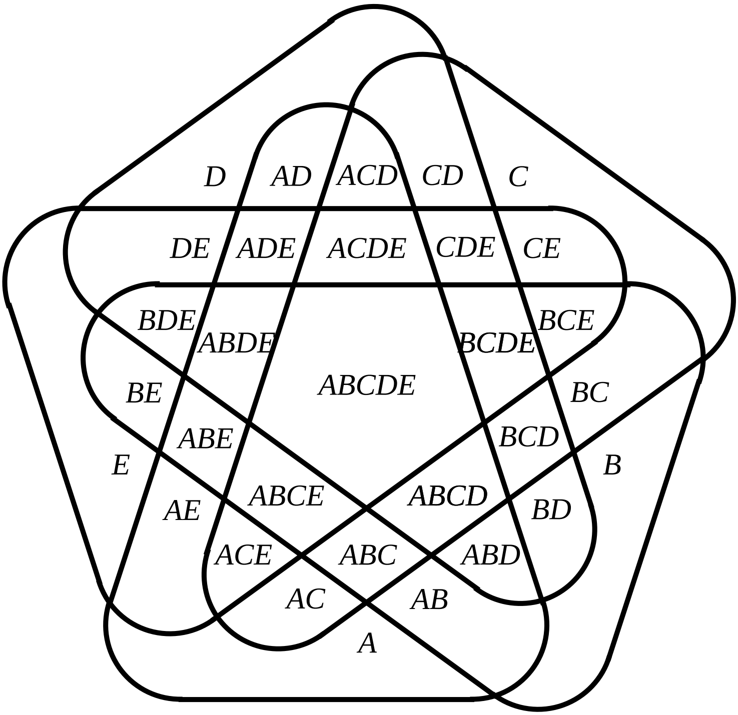


The M500 Society and Officers

The M500 Society is a mathematical society for students, staff and friends of the Open University. By publishing M500 and 'MOUTHS', and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching.

The magazine M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

MOUTHS is 'Mathematics Open University Telephone Help Scheme', a directory of M500 members who are willing to provide mathematical assistance to other members.

The September Weekend is a residential Friday to Sunday event held each September for revision and exam preparation. Details available from March onwards. Send SAE to Jeremy Humphries, below.

The Winter Weekend is a residential Friday to Sunday event held each January for mathematical recreation. Send SAE for details to Norma Rosier, below.

Advice to authors. We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to Tony Forbes, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. If you use a computer, please also send the file on a PC diskette or via e-mail. Camera-ready copy can be accepted if it follows the general format of the magazine.

Michael Falcus

With deep regret we have to inform you of the sudden death of M500 Committee member Michael Falcus.

Michael died of a heart attack in the early hours of 9 July, 2002. He was 53 years old. His death came as a shock to those who knew him; he was playing badminton the day before without any apparent ill effects.

Michael had been an active member of the Committee for a number of years, and he will be fondly remembered by regulars at the M500 Revision and Winter Weekends. As well as being an occasional contributor, he gave valuable help in the preparation and production of the M500 Magazine.

We would like to express our sympathy to Michael's family and friends. He leaves four children and five grandchildren; the youngest was born just two days after Michael died.

The Fibonacci series

Sebastian Hayes

A Fibonacci series is a recursive series which verifies the rule

$$u_{n+1} \;=\; u_n + u_{n-1}.$$

The simplest Fibonacci series, hereafter called F_n, is such a series with

$$u_1 \;=\; 1, \quad u_2 \;=\; 1.$$

It is probably the first recursive series to have been used in mathematics and is named after the medieval mathematician who discovered it, Leonardo of Pisa familiarly known as 'Fibonacci'. The first few terms are given below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377

Note that F_n does not need to be restricted to $\mathbb{Z}^+$ though it almost always is. For 'negative' terms we have the following.

n	−7	−6	−5	−4	−3	−2	−1	0
F_n	13	−8	5	−3	2	−1	1	0

If n is odd, $F_{-n} = F_n$, if n is even, $F_{-n} = -F_n$, and $F_0 = 0$.

Relation between the golden section and the Fibonacci series. At first sight there is no particular relation between the golden ratio, $\phi = (1+\sqrt{5})/2$, and the series 1, 1, 2, 3, 5, 8, 13, ... However, repeated application of the defining equation $\phi^2 = \phi + 1$ gives

$$\begin{aligned}\phi^2 &= 1+\phi,\\ \phi^3 &= \phi(1+\phi) \;=\; 1+2\phi,\\ \phi^4 &= 2+3\phi,\\ \phi^5 &= 3+5\phi,\\ &\ldots,\end{aligned}$$

with general term

$$\phi^n = F_{n-1} + F_n\phi, \qquad \text{(i)}$$

where F_n signifies the nth term of the Fibonacci series. In other words, the successive terms of the Fibonacci series appear as coefficients in the expansion of the powers of ϕ. (This result is easily proved by induction.)

Nor is this all. If we take the ratio of successive terms of the Fibonacci series, i.e. 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, ..., we find that this series converges to $1/\phi$, or alternatively, the ratio of a term to its immediate precedent, 1/1, 2/1, 3/2, 5/3, ..., converges to ϕ. Why is this?

Convergence of Fibonacci series to ϕ. As in many other situations, the most efficient way of proving this particular result, namely the convergence of 1/1, 2/1, 3/2, 5/3, ... to ϕ, is to take the general case—though this is certainly not how it was discovered in the first place.

Any series which verifies $u_{n+1} = u_n + u_{n-1}$ is a Fibonacci series, not just F_n, where $u_1 = 1$, $u_2 = 2$. Consider, then, the series defined by the relation

$$u_n = a\phi^{n-1} + b(-1/\phi)^{n-1},$$

where a and b are constants. Such a series is a Fibonacci series because

$$u_{n+1} = a\phi^n + b(-1/\phi)^n$$

and, adding, we obtain

$$\begin{aligned}&a\phi^{n-1}(1+\phi) + b(-1/\phi)^{n-1}(1-1/\phi)\\ &= a\phi^{n+1} + b(-1/\phi)^{n+1} \qquad (\text{because } 1+\phi=\phi^2 \text{ and } 1-1/\phi = 1/\phi^2)\\ &= u_{n+2}.\end{aligned}$$

This relation holds whatever values we give to a and b. As n increases without bound the second part of u_n, namely $b(-1/\phi)^n$, and the second part of u_{n+1}, namely $b(-1/\phi)^{n+1}$, both go to zero and so we only need to take into account the first parts $a\phi^{n-1}$ and $a\phi^n$. Thus, as n increases without limit, the ratio u_{n+1}/u_n goes to $\phi^n/\phi^{n-1} = \phi$.

Amazingly, for all series which can be expressed in this manner, the ratio of a term to the preceding term converges to the golden section. Moreover, each ratio will be alternately greater and lesser, or lesser and greater, and approach closer each time to ϕ. For example, take, purely at random, the numbers 7 and 19. The sum is 26; 19/7 is about 2.7—larger than ϕ; 26/19 is about 1.36—somewhat smaller than ϕ; 26 + 19 = 45 and 45/26 gives us 1.73…, already quite close to ϕ and greater than it; 45 + 26 gives 71, and 71/45 is about 1.58, a little less than ϕ. The series is obviously homing in to some intermediate value and a few more computations will establish the limiting value as 1.618 to three decimal places. Other starting points may take longer to converge but in every case the limiting ratio is ϕ.

We can produce the basic Fibonacci series, F_n, by setting

$$a = \frac{\phi^2}{\phi^2+1}, \qquad b = \frac{1}{\phi^2+1}.$$

When we fit these values of a and b into $a\phi^{n-1} + b(-1/\phi)^{n-1}$ and give n the values 1, 2, 3, …, the familiar 1, 1, 2, 3, 5, 8, 13, … results.

General formula for expansion of powers. We now consider the conditions for integer solutions to the equation

$$\frac{a}{\phi^m} + \frac{b}{\phi^s} = \frac{1}{\phi^r}.$$

where $r = 0, 1, 2, \ldots, s$ and m are integers and $s > m$. Then $a\phi^{s-m} + b = \phi^{s-r}$,

$$a(F_{s-m}\phi + F_{s-m-1}) + b = \phi^{s-r}$$

(since $\phi^n = F_n\phi + F_{n-1}$),

$$aF_{s-m}\phi + b + aF_{s-m-1} = \phi^{s-r}.$$

Therefore $aF_{s-m} = F_{s-r}$, $b + aF_{s-m-1} = F_{s-r-1}$,

$$a = \frac{F_{s-r}}{F_{s-m}}, \qquad b = F_{s-(r+1)} - aF_{s-(m+1)}. \tag{ii}$$

A theorem concerning Fibonacci numbers states that F_m/F_n is integral only if n divides m—unless $n = 1, 2$, since F_1 and F_2 are both unity. Thus we must have $s - m = 1, 2$ or $s - m$ divides $s - r$.

Various useful rules can be derived from this. For example, if we make r, m and s successive positive integers, calling them n, $n+1$ and $n+2$, we have

$$\frac{a}{\phi^{n+1}} + \frac{b}{\phi^{n+2}} = \frac{1}{\phi^n}.$$

Using the formula (ii),

$$a \;=\; F_{s-r} \;=\; F_{n+2-n} \;=\; F_2 \;=\; 1$$

and

$$\begin{aligned} b &= F_{s-(r+1)} - aF_{s-(m+1)} \\ &= F_{n+2-(n+1)} - F_{n+2-(n+1)+1} \\ &= F_1 - F_0 \;=\; 1. \end{aligned}$$

Thus

$$\frac{1}{\phi^n} \;=\; \frac{1}{\phi^{n+1}} + \frac{1}{\phi^{n+2}}. \tag{iii}$$

For example, $\dfrac{1}{\phi^2}$ is the sum of $\dfrac{1}{\phi^3}$ and $\dfrac{1}{\phi^4}$.

Expansion of 1 *in two powers of* $1/\phi$. Another useful formula is obtained by setting r in the general equation at zero and having m and s as successive integers. We have to find a and b in $a/\phi^n + b/\phi^{n+1} = 1/\phi^0 = 1$. Using formula (ii),

$$\begin{aligned} a &= \frac{F_{s-r}}{F_{s-m}} = \frac{F_{n+1-0}}{F_{n+1-n}} = \frac{F_{n+1}}{F_1} = F_{n+1}, \\ b &= F_{s-(r+1)} - aF_{s-(m+1)} \;=\; F_{n+1-1} - F_0 \;=\; F_n. \end{aligned}$$

So

$$\frac{F^{n+1}}{\phi^n} + \frac{F_n}{\phi^{n+1}} \;=\; 1, \tag{iv}$$

which gives the only solution for the 'expansion' of 1 in terms of two consecutive powers of $1/\phi$. For example, if we want 1 in terms of $(1/\phi)^4$ and $(1/\phi)^5$, the only possibility is

$$\frac{F_5}{\phi^4} + \frac{F_4}{\phi^5} \;=\; 1, \quad \text{or} \quad \frac{5}{\phi^4} + \frac{3}{\phi^5} \;=\; 1,$$

and there will be $F_{n+1} + F_n = F_{n+2}$ individual terms in all.

If the powers are not consecutive, there may be other combinations. Returning to the basic equation $a/\phi^m + b/\phi^s = 1/\phi^r$ and setting $r = 0$ we have, by (ii),

$$a \;=\; \frac{F_s}{F_{s-m}}, \qquad b \;=\; F_{s-1} - aF_{s-m-1},$$

where $s > m$.

What are the conditions for positive integer solutions? If a is to be greater than 0, as previously mentioned, either $s-m$ must be 1 or 2, or $s-m$ must divide s. And b will be greater than 0 whenever $F_{s-1} > aF_{s-m-1}$. If $s = m+1$, both these conditions are satisfied. If $s = m+2$, that is, we are dealing with the equation $a/\phi^n + b/\phi^{n+2} = 1$, then

$$a = F_{n+2}, \quad b = F_{n+1} - F_{n+2}F_1 = -F_n < 0.$$

For example, setting $s = 6$, $m = 4$, we have $a = F_6/F_1 = 8$, $b = F_5 - 8F_1 = -3$.

For other values of s and m ($s > m$), whether b is an integer or not depends on the parity of m and s. Since $b = F_{s-1} - aF_{s-m-1}$, where $s > m$ and a, assumed to be an integer, is F_s/F_{s-m}, we require

$$F_{s-1} > \frac{F_s}{F_{s-m}} F_{s-m-1}, \quad \text{or} \quad \frac{F_{s-1}}{F_s} > \frac{F_{s-m-1}}{F_{s-m}}.$$

Now the ratio of successive terms of F_n, 1/1, 1/2, 2/3, 3/5, …, has as limiting value $1/\phi$ and, not only this, each successive ratio F_n/F_{n+1} is alternately greater than and less than $1/\phi$. Thus all terms $F_{\mathrm{odd}}/F_{\mathrm{even}}$, or F_{2n-1}/F_2n, such as 1/1, 2/3, 5/8, …, are greater than all terms $F_{\mathrm{even}}\ F_{\mathrm{odd}}$, or F_{2n}/F_{2n+1}, such as 1/2, 3/5, …. Thus, for example, if s is even and m is even with s, $m > 0$ and $s > m$, $\dfrac{F_{s-1}}{F_s} > \dfrac{F_{s-m-1}}{F_{s-m}}$; e.g. for $s = 8$, $m = 3$ we have

$$\frac{F_5}{F_6} = \frac{5}{8} > \frac{F_2}{F_3} = \frac{1}{2}$$

and we have the solution $a = 4$, $b = 5 - 4F_2 = 1$; hence $4/\phi^3 + 1/\phi^6 = 1$.

Expansion of other powers of $1/\phi$. We can expand other powers in the same way. Thus

$$\frac{1}{\phi^2} + \frac{1}{\phi^3} = \frac{1}{\phi}, \quad \frac{2}{\phi^3} + \frac{1}{\phi^4} = \frac{1}{\phi}, \quad \frac{3}{\phi^4} + \frac{2}{\phi^5} = \frac{1}{\phi}$$

and for $1/\phi^2$ we have

$$\frac{1}{\phi^3} + \frac{1}{\phi^4} = \frac{1}{\phi^2}, \quad \frac{2}{\phi^4} + \frac{1}{\phi^5} = \frac{1}{\phi^2},$$

so the same Fibonacci numbers keep reappearing. The general formula is

$$\frac{F_{n-r+1}}{\phi^n} + \frac{F_{n-r}}{\phi^{n+1}} = \frac{1}{\phi^r}, \qquad r = 0, 1, 2, \ldots. \tag{v}$$

If $r = n-1$, we obtain $1/\phi^n + 1/\phi^{n+1} = 1/\phi^{n-1}$, or formula (iii). Thus, for example, choosing $n = 4$ we have

$$\ldots, \quad \phi^2 = \frac{13}{\phi^4} + \frac{8}{\phi^5}, \quad \phi = \frac{8}{\phi^4} + \frac{5}{\phi^5}, \quad 1 = \frac{5}{\phi^4} + \frac{3}{\phi^5},$$

$$\frac{1}{\phi} = \frac{3}{\phi^4} + \frac{2}{\phi^5}, \quad \frac{1}{\phi^2} = \frac{2}{\phi^4} + \frac{1}{\phi^5}, \quad \frac{1}{\phi^3} = \frac{1}{\phi^4} + \frac{1}{\phi^5},$$

$$\frac{1}{\phi^6} = \frac{1}{\phi^4} - \frac{1}{\phi^5}, \quad \frac{1}{\phi^7} = \frac{-1}{\phi^4} + \frac{2}{\phi^5}, \quad \ldots.$$

Expansion of 1 *for negative powers.* Formula (v) also works for negative powers. For example, set $n = -2$. Then $F_{-1}/\phi^{-2} + F_{-2}/\phi^{-1} = \phi^2 - \phi = 1$ and, more generally, making $n > 0$, $F_{-(n-1)}\phi^n + F_{-n}\phi^{n-1} = 1$.

For odd n the above formula gives $-F_{n-1}\phi^n + F_n\phi^{n-1} = 1$ and for n even, $F_{n-1}\phi^n - F_n\phi^{n-1} = 1$. This is more conveniently expressed in the form

$$F_{n+1} - F_n\phi = \frac{(-1)^n}{\phi^n}.$$

Thus, for example, $5 - 3\phi = 1/\phi^4$.

Families of ϕ. The number ϕ has been defined as the solution to the equation $x^2 - x - 1 = 0$. This solution we could call ϕ_1. The solution to the equation $x - 1 = 0$, namely 1, may be termed ϕ_0. Continuing in the same way, we term ϕ_2 the solution to the equation $x^3 - x^2 - x - 1 = 0$, and so on. The approximate value of ϕ^2 is 1.839. In this way we can define a whole family of solutions, with ϕ_n converging to 2 as n goes to infinity. (Because $2^n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} + 1$, so that the difference $2^n - \phi_n^n$ is always 1 and phi_n is very nearly $(2^n)^{1/n}$ for large n.)

Equation for the expansion of 1 *in three terms.* Leaving this aside for the moment, we examine solutions in integers to

$$\frac{a}{\phi^m} + \frac{b}{\phi^s} + \frac{c}{\phi^t} = \frac{1}{\phi^r}.$$

For the simplest case, take $r = 0$, and m, s and t successive integers, that is, $a/\phi^n + b/\phi^{n+1} + c/\phi^{n+2} = 1$, or $a\phi^2 + b\phi + c = \phi^{n+2}$. Using formula (i), this provides the conditions $a + c = F_{n+1}$, $a + b = F_{n+2}$. There are $F_{n+3} - a$ terms in all.

By taking various values of $n = 0, 1, 2, \ldots$, we can see if it is possible to expand 1 in terms of three successive powers n, $n+1$, $n+2$. If we are to have three non-zero solutions for a, b and c, n must be at least 2, since $F_1 = 1$, $F_2 = 1$. For $n > 1$ it is always possible to expand 1 in three consecutive non-zero powers of $1/\phi$ and there are in fact $n - 1$ ways of doing this. For given n there will be $n + 1$ possible values for a but $a = 0$ for one of these and $c = 0$ for another.

More generally, if we consider the equation

$$\frac{a}{\phi^n} + \frac{b}{\phi^{n+1}} + \frac{c}{\phi^{n+2}} = \frac{1}{\phi^r}$$

we obtain the conditions $a+c = F_{n+1-r}$, $a+b = F_{n+2-r}$, with $r = 0, 1, 2, \ldots$. There will be $n - 1 - r$ positive integer solutions for given n and r. Some examples are tabulated on the next page.

The golden angle. A plant functions as a whole: there would be no point in individual leaves or branches fighting amongst themselves for a greater share of light or warmth. Assuming the recurrent production of buds, and eventually branches, around a rigid, upright trunk, is there an 'ideal angle' to adopt for branch production? (By 'angle' I mean fraction of a turn between one branch and the next: in the technical literature this is called the *divergence*.)

There are two conflicting requirements: on the one hand we want a fairly equal spacing of branches around the trunk, otherwise the tree will be lop-sided, but we must avoid having one branch falling exactly above another and thus blocking its light. If we choose a very small angle, say a thirty-fifth of a whole turn, we satisfy the second requirement but not the first; if we choose a large fraction like a half we get a balanced tree but all the branches on top of each other.

To model this mathematically we ignore difference in height and consider a disc with a full turn, 360 degrees, set at unity. We can limit ourselves to angles of divergence of less than 1/2, since greater angles are the same taken the other way round and the eventual pattern will be a mirror image.

We mark in a radius representing branch 1. To get a reasonably equal spacing we can agree that the next radius, representing branch 2, should fall within the middle third or, since we are only considering angles of divergence less than 1/2, should lie between 1/3 and 1/2 (of a turn). Thus $1/3 < d < 1/2$. We require a value of d which will never give rise to overlapping and which will, if possible, continue to cut every gap within the middle third.

$$\frac{a}{\phi^n} + \frac{b}{\phi^{n+1}} + \frac{c}{\phi^{n+2}} \;=\; \frac{1}{\phi^r}.$$

$r = 0, \quad a + c = F_{n+1}, \quad a + b = F_{n+2}$

n	Conditions	Solutions	Expansion of 1
$n = 0$	$a + c = 1,\ a + b = 1$	$a = 1,\ c = 0,\ b = 0$	1
		$a = 0,\ c = 1,\ b = 1$	$1/\phi + 1/\phi^2$
$n = 1$	$a + c = 1,\ a + b = 2$	$a = 0,\ c = 1,\ b = 3$	$2/\phi^2 + 1/\phi^3$
		$a = 1,\ c = 0,\ b = 1$	$1/\phi + 1/\phi^2$
$n = 2$	$a + c = 2,\ a + b = 3$	$a = 0,\ c = 2,\ b = 3$	$3/\phi^3 + 2/\phi^4$
		$a = 1,\ c = 1,\ b = 2$	$1/\phi^2 + 2/\phi^3 + 1/\phi^4$
		$a = 2,\ c = 0,\ b = 1$	$2/\phi^2 + 1/\phi^3$
$n = 3$	$a + c = 3,\ a + b = 5$	$a = 0,\ c = 3,\ b = 5$	$5/\phi^4 + 3/\phi^5$
		$a = 1,\ c = 2,\ b = 4$	$1/\phi^3 + 4/\phi^4 + 2/\phi^5$
		$a = 2,\ c = 1,\ b = 3$	$2/\phi^3 + 3/\phi^4 + 1/\phi^5$
		$a = 3,\ c = 0,\ b = 2$	$3/\phi^3 + 2/\phi^4$
$n = 4$	$a + c = 5,\ a + b = 8$	$a = 0,\ c = 5,\ b = 8$	$8/\phi^5 + 5/\phi^6$
		$a = 1,\ c = 4,\ b = 7$	$1/\phi^4 + 7/\phi^5 + 4/\phi^6$
		$a = 2,\ c = 3,\ b = 6$	$2/\phi^4 + 6/\phi^5 + 3/\phi^6$
		$a = 3,\ c = 2,\ b = 5$	$3/\phi^4 + 5/\phi^5 + 2/\phi^6$
		$a = 4,\ c = 1,\ b = 4$	$4/\phi^4 + 4/\phi^5 + 1/\phi^6$
		$a = 5,\ c = 0,\ b = 3$	$5/\phi^4 + 3/\phi^5$

$r = 1, \quad a + c = F_n, \quad a + b = F_{n+1}$			Expansion of $1/\phi$
$n = 4$	$a + c = F_4 = 3$	$a = 0,\ c = 3,\ b = 5$	$5/\phi^5 + 3/\phi^6$
		$a = 1,\ c = 2,\ b = 4$	$1/\phi^4 + 4/\phi^5 + 2/\phi^6$
		$a = 2,\ c = 1,\ b = 3$	$2/\phi^4 + 3/\phi^5 + 1/\phi^6$
		$a = 3,\ c = 0,\ b = 2$	$3/\phi^4 + 2/\phi^5$

$r = 2, \quad a + c = F_{n-1}, \quad a + b = F_n$			Expansion of $1/\phi^2$
$n = 4$	$a + c = F_3 = 2$	$a = 0,\ c = 2,\ b = 3$	$3/\phi^5 + 2/\phi^6$
		$a = 1,\ c = 1,\ b = 2$	$1/\phi^4 + 2/\phi^5 + 1/\phi^6$
		$a = 2,\ c = 0,\ b = 1$	$2/\phi^4 + 1/\phi^5$

Any irrational number would, of course, rule out overlapping. If a branch is to cover a previous branch, we have $nd = N+d$, i.e. n applications of the angle of divergence give rise to N full turns plus the original angle. Then $d = N/(n-1)$, which is a rational number. If d is not rational, we have a contradiction and so there can be no overlapping.

The problem can be rephrased thus: If we can find an angle d *which cuts every gap in the same ratio as it cuts the first gap and it cuts the first gap (the whole disc) in a ratio between* $1/3$ *and* $1/2$, then this will provide a satisfactory solution to the spacing problem. This gives

$$\frac{x}{1-x} \;=\; \frac{1-2x}{x}, \quad \text{or} \quad x^2 - 3x + 1 \;=\; 0,$$

with solutions

$$\begin{aligned} x &= \frac{3+\sqrt{5}}{2}, \quad \text{or} \quad 1 + \frac{1}{2}\,(1+\sqrt{5}) \;=\; 1+\phi \;=\; \phi^2, \\ x &= \frac{3-\sqrt{5}}{2}, \quad \text{or} \quad 1 + \frac{1}{2}\,(1-\sqrt{5}) \;=\; 1-\frac{1}{\phi} \;=\; \frac{1}{\phi^2}. \end{aligned}$$

We choose the second solution, $1/\phi^2$, since we want an angle less than $1/2$, but the first angle—which is $1/\phi^2$ less than three whole turns and so is equivalent to an anti-clockwise angle of $1/\phi^2$—gives exactly the same distribution going the other way. In fact, $1/\phi^2$ is a little less than 0.382 and so $1/3 < 1/\phi^2 < 1/2$, as required. Its value is about 137.5 degrees.

The number $1/\phi^2$ is known as the *golden angle* since it divides the disc into two parts whose areas are in the ratio $\phi : 1$. Why is this? Because the remaining angle is $1 - 1/\phi^2 = 1/\phi$. The ratio of the areas, the greater to the lesser, is thus $1/\phi : 1/\phi^2 = \phi : 1$.

We obtain the following partition after each successive cut.

1. $1/\phi^0$
2. $1/\phi^2, 1/\phi^1$
3. $1/\phi^2, 1/\phi^2, 1/\phi^3$
4. $1/\phi^4, 1/\phi^3, 1/\phi^2, 1/\phi^3$
5. $1/\phi^4, 1/\phi^3, 1/\phi^4, 1/\phi^3, 1/\phi^3$
6. $1/\phi^4, 1/\phi^3, 1/\phi^4, 1/\phi^3, 1/\phi^4, 1/\phi^5$
7. $1/\phi^4, 1/\phi^4, 1/\phi^5, 1/\phi^4, 1/\phi^3, 1/\phi^4, 1/\phi^5$
8. $1/\phi^4, 1/\phi^4, 1/\phi^5, 1/\phi^4, 1/\phi^4, 1/\phi^5, 1/\phi^4, 1/\phi^5$

Note the following features: (a) every row adds to 1; (b) every row contains three types of entries at most; (c) if the row number is a Fibonacci number, there are only two types of entries (or one for row 1); (d) the indices are successive integers.

Since every natural number n either is a Fibonacci number or falls between two successive Fibonacci numbers, we can number the rows: row $1 = F_1$, row $2 = F_2$ and, generally, row $N = F_{r+2} + r$, where r takes the values 0, 1, 2, 3, ..., $F_{n+1} - 1$. For example, row 10 would be row $F_{4+2} + 2 = F_6 + 2 = 8 + 2$, with $n = 4$, $r = 2$. Row N in the above is then given by

$$\frac{F_{n+1} - r}{\phi^n} + \frac{F_n + r}{\phi^{n+1}} + \frac{r}{\phi^{n+2}},$$

where n takes the values 0, 1, 2, ...

The golden angle series—assuming it continues in the same way—can be defined recursively by the double rule: (1) Row 1 = 1, and (2) every row repeats the entries of the previous row except that (one of) the largest interval(s) is split in golden section. Thus 1 produces $1/\phi$ and $1/\phi^2$ and more generally $1/\phi^n$ produces $1/\phi^{n+1}$ and $1/\phi^{n+2}$. The resulting series is precisely that which results from giving a the various values 0, 1, 2, ..., $F_{n+1} - 1$ in the table but starting with the highest value for a instead of zero.

The golden angle series thus seems to be the same as the series which gives all the integral solutions to the expansion of 1 in three consecutive powers of $1/\phi$ taking successive values of n (and excluding repetitions). Since such a series will only ever have three different sized intervals at any one stage and will always cut the largest available interval in golden section (i.e. between 1/2 and 1/3) this solves the 'even spacing' problem.

However, the above falls well short of a rigorous proof since it is by no means obvious that using $1/\phi^2$ as a constant angle will give this distribution *no matter how long we continue.* What follows is simply a plausibility argument which can (possibly) be made into a rigorous proof.

It is easy to show (simply by drawing) that up to a certain point we have the consecutive power distribution. Take row 8. Since 8 is a Fibonacci number there are only two types of sections, those of size $1/\phi^4$ and those of size $1/\phi^5$. Moreover, row 8—like every row—must split up neatly into two (composite) sections of value $1/\phi$ and $1/\phi^2$ respectively. We have in fact

$$\frac{5}{\phi^4} + \frac{3}{\phi^5} \;=\; 1 \;=\; \left(\frac{3}{\phi^4} + \frac{2}{\phi^5}\right) + \left(\frac{2}{\phi^4} + \frac{1}{\phi^5}\right) \;=\; \frac{1}{\phi} + \frac{1}{\phi^2}.$$

Now radius 8 has just been drawn in so that the section radius 7 to radius 8 is $1/\phi^2$. The next radius is thus going to fall within the $1/\phi$ part of the disc and one of the sections $1/\phi^4$ or $1/\phi^5$ is going to be split into x and y. Moreover, since the new radius (radius 9) is once more dividing up the disc into $1/\phi^2$ and $1/\phi$, x and y will not be in the same section; x (say) will be in the $1/\phi^2$ part and y in the $1/\phi$ part.

It will be the section $1/\phi^4$, the larger of the two basic sections, that will be split. Why? Suppose the contrary. We then have to 'make up' $1/\phi^2$ from $3/\phi^4+1/\phi^5$ and x (y being excluded since it goes into the other part). Since $x < 1/\phi^5 < 1/\phi^4$ this is impossible. If we take $2/\phi^4$ and reject $1/\phi^5$, x is not big enough to make up $1/\phi^2$, and the same goes if we take $1/\phi^4+1/\phi^5$. And we must include x.

Thus the section to be split is $1/\phi^4$. Since $1/\phi$ is the sum of $1/\phi^2$ and $1/\phi^3$ we have to make up $1/\phi^2$ and $1/\phi^3$ from $2/\phi^4 + 2/\phi^5 + x + y$, where $x, y < 1/\phi^5$ and x and y are not in the same section. We cannot take $2/\phi^4$ for the larger section because there would be no room for $x > 0$ but $1/\phi^5 > x$, while if we take $1/\phi^4$ this means $x = 1/\phi^6$ since $1/\phi^2 = 1/\phi^4+2/\phi^5+1/\phi^6$, which is the only available solution. Then $y = 1/\phi^4 - 1/\phi^6 = 1/\phi^5$ so that $1/\phi^4$ has been cut in golden section, as desired.

This clumsy method can (presumably) be generalized to make an inductive argument—but hopefully someone can come up with a neater proof of the identity of the $1/\phi^2$ series and the consecutive power series.

Problem 187.1 – Cosets

ADF

Let G be a group and let H be any subgroup of G. Let L be a set of representatives of the left cosets of H in G. Thus we have a partition of G into disjoint sets $\{xH : x \in L\}$.

Similarly, let R be a set of representatives of the right cosets of H. The partition of G, $\{Hx : x \in R\}$, is not necessarily the same as $\{xH : x \in R\}$.

Question: Is it always possible to choose $L = R$?

For obvious reasons only non-Abelian groups are relevant. Take, for instance, S_3, the symmetry group of the triangle,

$$\{1,\theta,\theta^2,\phi,\phi\theta,\phi\theta^2\}, \quad \theta^3 = \phi^2 = 1, \quad \theta\phi = \phi\theta^2, \quad \theta^2\phi = \phi\theta.$$

The left and right cosets of the subgroup $\{1,\phi\}$ are $\{\{1,\phi\},\ \{\theta,\phi\theta^2\},\ \{\theta^2,\phi\theta\}\}$ and $\{\{1,\phi\},\ \{\theta,\phi\theta\},\ \{\theta^2,\phi\theta^2\}\}$, respectively. Hence in this case we can choose $L = R = \{1,\theta,\theta^2\}$.

Solution 184.3 – Lake escape

> A young lady is trying to escape from a man who is pursuing her. She gets into a rowboat and rows to the centre of a circular lake. The man waits for her to come ashore. He can run four times as fast as she can row but she can outrun him once she is on dry land. How does the woman escape?
>
> What if the lake is not necessarily a circle?

Dick Boardman

Here is a not-too-serious solution. The lady can escape if the distance between her boat and the shore is less than a quarter of the distance the man must run round the circumference. At the start, if she rows straight for the shore to the point opposite the man, she must row one radius and the man must run π times the radius. However, $\pi < 4$; so this fails.

However, suppose she rows a curved track, always moving along a line exactly away from where the man is at any instant. If she follows this rule consistently, she will eventually reach the shore safely, having rowed some 2.25 radii. However, when she has rowed only 0.25 radii around the curved track, she will be closer to her nearest point on the shore than 1/4 of the distance than the man must run to reach the same point and may thus safely make a straight line dash for safety.

Note that since she is always moving directly away from the man he may change direction at any time and she can adjust her direction accordingly without altering the distance to go.

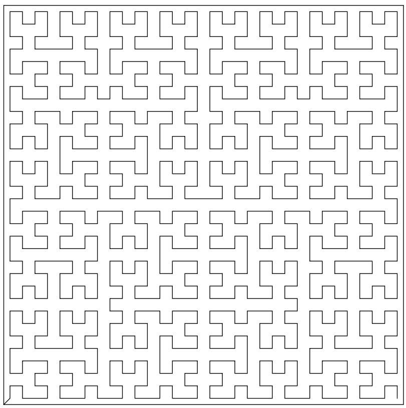
The path she must follow is the solution to a differential equation. This equation has (as far as I can see) no solution in terms of elementary functions and must be approximated to by numerical means. Let the radius of the lake be 1 distance unit and let the time unit be the time taken for her to row one distance unit. This makes her speed 1 and his speed 4.

I used Euler's numerical method. In this method, a very small time interval dt is chosen (I used 0.001 time units) and the lady is assumed to row in a straight line for this time interval and her direction is recalculated every dt units while the man runs at a constant speed around the edge. This would be a very inefficient method if pencil and paper and tables were used, the number of calculations needed for a given level of accuracy being very large. More accurate (and far more complicated) methods exist but, using a modern computer, the time taken to enter and check the simplest equations far exceeds the time needed to do the sums. To check the level of accuracy achieved, repeat the calculation with one tenth of the time interval. If the results agree to the required number of decimal places, OK.

With regard to the more general problem, the same strategy will work for many shapes of lake (row directly away from the man until the distance to the nearest point is less than a quarter of the distance he must run). This is because the circle is the curve which encloses the greatest area for a given perimeter and is thus the worst case for the lady. However, the lady must be very careful in her choice of lake. The simplest case where all methods fail is if there is a large island in the middle of the lake leaving only a narrow for her to row round. Assuming she cannot carry the boat over the island, she has no escape.

Even if there are no islands, there may be other problems. We are told that the lady can always outrun her pursuer on land. However, even the fastest lady can only have a finite speed advantage.

Imagine a square lake containing what would be an island were it not joined to the side by a single narrow isthmus. This island occupies most of the centre of the lake. Furthermore this island consists of a single narrow track in the shape of Peano's space filling curve. The poor girl now has a real problem. If she tries to land on the outside of the lake or near the isthmus she is caught. If she lands on the 'island' she has an infinite distance to run before she gets to the isthmus. Thus her pursuer need only pause until she has run well into the maze before running round his finite distance to the isthmus and waiting.



David Tansey gave a similar solution for the circle and showed that it applies at least to an arbitrary convex lake.

Solution 185.2 – Two streams

> Water from a tank flows through a perfectly insulated pipe and splits into two paths. One path goes directly to a tap at the kitchen sink marked 'cold'. The other enters a heating appliance and then proceeds to a tap marked 'hot'. On the other sides of the taps the two streams merge into a single outlet. Explain why the temperature of the water can be varied by adjusting the 'hot' and 'cold' taps.

David Aldridge

I am not 100 per cent certain this is a maths problem. It's more like an engineering one. This suits me as I am an engineer, not a mathematician. However, the answer can be investigated by applying the maths of a classic engineering equation.

Basically, each tap is open to atmospheric pressure, and they both pour into the kitchen sink. Switching on a tap creates a stream of water from the (constant pressure) tank outlet to the tap.

Aside no. 1. I note that the question states that the two streams merge into a single outlet on the other side of the taps, and this caused me some confusion. A tap consists of a valve and an outlet to atmosphere. You may be confusing the term tap with valve. In any case, I think that this may be the basis of a mistaken assumption. On a kitchen mixer tap you do not have the two streams joining together within the pipework prior to opening to atmosphere instead you have two distinct tap outlets positioned together (sometimes even one within the other) within the single tap unit. These allow the hot and cold flows to mix as they emerge from the unit hence the term. In such a case, the true situation remains as I have drawn it above, except the plug is simply the point at which the streams mix just below the tap outlet.

Aside no. 2. For the purposes of clarity, this type of mixer tap is mandatory because it is often fed by two very different pressure systems. The cold tap is often connected directly to the (usually high pressure) mains supply, while the hot tap is connected to the (lower pressure) hot water tank and ultimately to the house cistern. If the flows were able to mix in the pipework behind the tap outlet, you could have a situation where the cold water forces its way back up the hot water pipe and into the tank/cistern. This could cause flooding of the cistern. Having both outlets directed separately to atmosphere avoids this.

This situation can be assumed to be governed by Bernoulli's equation,

$$P + \frac{1}{2}\rho V^2 + \rho g h \;=\; \text{constant},$$

where P = pressure, ρ = density, V = velocity, g = gravitational constant and h=height.

This equation is fairly reliable for steady-flow situations when the fluid is non-viscous, incompressible and non-rotational. All of these can be assumed to hold true for water flow in a pipe system.

Now we can assume that the tank pressure P_1 is greater than atmospheric pressure. We can also assume that the tap is at the same height as the outlet of the tank, so there is no height change. Finally, we can assume that the velocity at the middle-bottom of the tank, V_1, is zero. We want to find V_2, the velocity at the tap. We have

$$P_1 \;=\; P_{\mathrm{atm}} + \frac{1}{2}\rho V_2^2.$$

By rearranging the equation, we can see that

$$V_2 \;=\; \frac{\sqrt{P_1 - P_{\mathrm{atm}}}}{\rho}.$$

If we assume that the tank is 1m high, $P_1 = P_{\mathrm{atm}} + \rho g h$, $g = 10\,\mathrm{m/s}$ and $\rho = 1000\,\mathrm{kg/m^2}$, we have a differential pressure of $P_1 - P_{\mathrm{atm}} = 10000\,\mathrm{Pa}$. This leads to a velocity of $V_2 = \sqrt{20} = 4.47\,\mathrm{m/s}$.

Now the interesting thing about this is that the velocity at the tap is completely unrelated to the diameter of the pipe the water has travelled along.

So what happens when we open the second tap?

Well, if you imagine you have a lemonade bottle full of water and you pierce a hole in the side near the bottom, you get a parabolic stream of water coming out at a (relatively) constant velocity. If you make a second hole at the same level on the other side (and keep bottle filled up) you get a second stream with exactly the same velocity.

The same is true here. When you open the second tap, water flows out of it at approximately the same speed at the water coming out of the first tap. That's what Bernoulli's equation has confirmed: that the *velocity* of the water at the outlet is constant.

If you have two outlets (taps) of the same size, switching on the second will *double* the total output flow rate.

To answer the question, then, it is clear that the flow passing through the heater does not change much when the second (cold) tap is opened. The hot water remains flowing from the tap at the same (hot) temperature, but the same amount of cold water is mixed in the sink and therefore the hot water is cooled to a reasonable level at the outlet (the plug hole?).

It seems to me that the assumption the question was based on (that a constant flow simply splits into two) was not right. Of course, in a real life domestic water system, the flow through a pipe is limited by other factors such as friction in the pipes, a lack of ability of the tank to maintain its pressure, etc. An example of this is when you are in an electric shower and someone turns on a tap.

- The flow rate to the shower reduces (because your water pressure is not high enough or stable enough to maintain both flows evenly).
- The constant-power heating element heats up the lower velocity (less mass per second) water to a higher temperature.
- You get scalded.

This may be the kind of situation that was envisaged by the writer, in that switching on one flow automatically affects another.

Dick Boardman

By the law of conservation of energy (heat inserted by the heater) = (heat out of tap) = (volume of water per second) multiplied by ((temperature out) − (tank temperature)). Thus (temperature out) − (tank temperature) = (heat inserted by heater per second) divided by (volume of water per second).

The problem compares two cases, all the water goes through the heater and a proportion of the water goes through the heater. If the volume of water and the heat inserted by the heater were the same in both cases then the output temperature would be the same and this is what your argument implicitly assumes.

However, the designer of the system, who wanted the temperature to vary, has arranged that the two quantities are not the same. Firstly, by adding a parallel path for the water to take, the total volume out must increase when the cold tap is turned on. Secondly, the rate of flow of heat is proportional to the temperature difference, and if the water in the heater is hotter, then less heat will be transferred.

Finally, the designer of the heater must make the heat input depend on the flow through the heater because otherwise, if the flow were very small, the system would explode, and this I suspect is the main reason.

Ralph Hancock

Problem 185.2 was of interest because I lived in a house where this happened. There was a gas geyser, and the amount of gas going into it was regulated by the water pressure. When both taps were on, the rapid flow along the main pipe caused a considerable drop in pressure, as per Bernoulli. This reduced the flow of water through the geyser, but the pressure sensitive valve reduced the gas supply by more than that, so the water was cooler as well as less copious. It didn't take much effort to deduce what was going on, because you could hear and see the flame decreasing.

But that is probably not the answer to the question.

Re: Problem 182.6 – n balls

> There are n balls in an urn, all of different colours. Remove two balls at random, paint the second of the pair to match the first and replace both balls. Repeat until all balls have the same colour. What is the expected number of turns?

Ted Gore

Not a solution but may be of interest. I use the notation D_n for the average number of draws of two balls that are required to ensure that all n balls are the same colour, and $D_{n,k}$ for the average number of draws required to reduce the number of colours from k to $k-1$. Thus

$$D_n = \sum_{k=2}^{n} D_{n,k}.$$

As we have already seen [M500 **184** 17], computer simulation indicates that

$$D_n = (n-1)^2.$$

Computer simulation also gives the following results for $D_{n,k}$.

n	D_n	$D_{n,2}$	$D_{n,3}$	$D_{n,4}$	$D_{n,5}$	$D_{n,6}$	$D_{n,7}$
3	4	3	1				
4	9	6	2	1			
5	16	10	10/3	5/3	1		
6	25	15	5	5/2	3/2	1	
7	36	21	7	7/2	21/10	7/5	1

In each case

$$D_{n,k} = \frac{n(n-1)}{k(k-1)} \tag{1}$$

and mathematical induction can be used to show that

$$\sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)} = (n-1)^2.$$

Unfortunately I can't prove (1).

Solution 185.4 – Two sins

When does $(\sin a)/(\sin b) = a/b$?

Jim James

Rearranging the given equation, we see that the problem reduces to one of establishing whether there exist $a, b \in \mathbb{R}$ such that $(\sin a)/a = (\sin b)/b$, where $a, b \neq 0$ and $a \neq \pm b$. Like many problems involving trigonometric functions, it helps to examine a graph of the function concerned. Figure 1 is the graph of $f(x) = (\sin x)/x$ in the interval $[-2\pi, 6\pi]$.

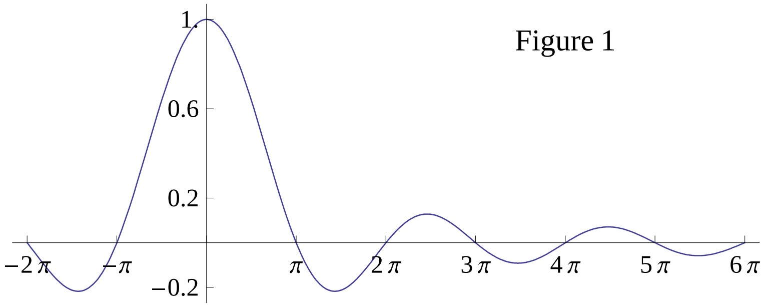


Figure 1

Several important points are immediately apparent as follows:

The graph appears to go through the point (0, 1). But f is not defined at 0. There is a nice geometric proof available, however, that $f(x)$ tends to a limit as x approaches zero, from above and below and that this limit is 1.

The graph is symmetric about the y-axis. This results from $\sin -x = -\sin x$, so that $(\sin -x)/(-x) = (\sin x)/x$. This suggests that we can safely restrict our study of the properties of f to the case $x > 0$.

The graph is obviously periodic, passing through zero at each integer multiple of π. This property cannot be used to demonstrate that $(\sin a)/(\sin b) = a/b$, however, since it would imply division by zero, which is undefined.

Successive oscillations have decreasing amplitude as x increases. Consider an arbitrary point, a, that lies on the curve such that a is greater than the minimum close to $x = 1.5\pi$. A straight line drawn parallel to the x-axis through a will intersect at least one corresponding point, b, such that $0 < b < a$. The value of $f(b)$ at this point equals $f(a)$, thereby showing that there exist a and b, not multiples of π, such that $(\sin a)/a = (\sin b)/b$, $a \neq \pm b$. Note that since there are infinitely many real numbers in any interval, there are infinitely many values of a and b which satisfy the specified relationship.

This answers the question posed in the problem, but added strength to the argument can be given by providing an example of how, given a value of a, a corresponding value of b can be determined. A numerical interpolation appears to be called for and what better than the bisection method? It is simple, reliable, does not require the evaluation of derivatives and is easily programmed for a home computer (where, for a problem like this, computational efficiency is not an issue).

Thus for $a = 5.2\pi$, the graph above indicates that a corresponding value of b lies in the interval $(3.5\pi, 4.0\pi)$. Using Visual Basic 6.2 with double precision variables, we establish, in less than one second, the following result:

$$\begin{aligned} a &= 16.3662817986660, & b &= 12.1153176248526, \\ \sin a &= -0.5877852522924661, & \sin b &= -0.4359134541440360, \\ \frac{a}{b} &= 1.34839896934735, & \frac{\sin a}{\sin b} &= 1.34839896934735. \end{aligned}$$

Dick Boardman

David Singmaster is guilty of at least one sin by arranging the equation so as to divide by b and then saying that $b = 0$ is a solution [M500 **185** 22].

Apart from this, rearrange the equation to read $(\sin a)/a = (\sin b)/b = k$, say. If $(\sin a)/a = k$ has multiple solutions then any pair will be solutions to the original equation.

Look at the plot [opposite] of $(\sin x)/x$. A line parallel to the x-axis and fairly close to the axis will cut the plot many times until the oscillations are too small to reach it. Thus $(\sin x)/x = 0.1$ has solutions at $x \approx \pm 2.85234$, ± 7.06817 and ± 8.4232. Any pair, say $a = -2.85234$ and $b = 7.06817$, will be a solution.

Problem 187.2 – 29

Colin Davies

Find all solutions in integers n, a_0, a_1, ..., a_n and b of

$$29 \sum_{k=0}^{n} a_k 10^k \;=\; 10 \sum_{k=0}^{n} a_k 10^k + b(10^{n+2} + 1),$$

where $n \geq 1$, $1 \leq a_n, b \leq 9$ and $0 \leq a_0$, a_1, ..., $a_{n-1} \leq 9$.

Re: Problem 184.9 – States

What is the probability of winning a game of *Hangman* restricted to the names of USA states. Assume only one life. Assume also that you and your opponent always play sensibly.

Tony Forbes

Despite its kindergarten origin, analysis of the problem seems to be interesting and difficult. I do not have a solution except to observe that the answer is at least 0.5. (Choose at random from {NORTH DAKOTA, SOUTH DAKOTA}.) Indeed, it would not surprise me if the answer is exactly 0.5. However I am not an expert in game theory, assuming one needs to be for an adequate understanding of the problem.

To indicate the sort of difficulty that arises, suppose you write '_ _ _ _ _ _'. Your opponent reasons thus, 'ALASKA, HAWAII, KANSAS and NEVADA have a common letter, 'A'; hence 'A' looks like a good choice.' Therefore to frustrate him/her you would have chosen OREGON. However your opponent would surely see through this obvious play and perhaps choose 'N', common to KANSAS, NEVADA and OREGON. But of course you have already anticipated this attack and you opt for HAWAII or ALASKA. So 'A' might have been a good choice after all. ...

Now that we have explained the problem in more detail, this is what we really want you to do: Either

(i) find a strategy that enables you to win more than half the games (on average); or

(ii) prove that no such strategy exists.

And to save you reaching for a map of North America, here is a list of the states that cause trouble: Iowa, Ohio, Utah; Alaska, Hawaii, Kansas, Nevada, Oregon; Alabama, Arizona, Florida, Georgia, Indiana, Montana, Vermont, Wyoming; Arkansas, Colorado, Delaware, Illinois, Kentucky, Maryland, Michigan, Missouri, Nebraska, Oklahoma, Virginia; North Dakota, South Dakota.

Problem 187.3 – Square wheels

A car has square wheels. On what sort of road can you drive it and experience a smooth ride?

Problem 187.4 – Cots

Prove that $\cot\dfrac{\pi}{2n}\cot\dfrac{3\pi}{2n}\dots\cot\dfrac{(n-2)\pi}{2n} = \sqrt{n}$ for odd n.

Solution 184.8 – Four regions

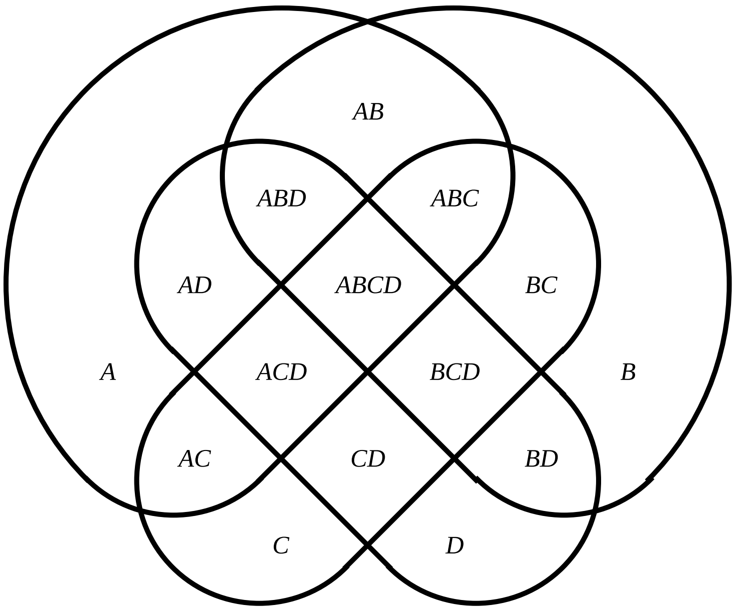
Draw a planar Venn diagram for four (or more) events.

Chris Pile

I drew the diagram, below, for four events, when I was involved in part-time teaching of statistics some years ago. The four closed curves give rise to 16 mutually exclusive regions.

I also offer a symmetrical diagram for five events which yields 32 mutually exclusive regions. I see no reason why this should not be extended. Each additional event must intersect with all the existing regions and so long as a continuous line can be drawn through all the regions a diagram is possible, but not necessarily very useful!

[I was very impressed with the beautiful symmetry of Chris's five-event diagram, so I put it on the front cover of this issue. An isosceles triangle with base $[(-1,0),\ (1,0)]$ and apex angle $36°$ is rotated about the point $(\phi/10,\ \sqrt{\phi})$, where $\phi = (\sqrt{5}+1)/2$ is the golden ratio. I am not aware of any deep theory relating to my choice of the centre of rotation; I just experimented until I found one which made the thing look pretty. — **ADF**]



Solution 185.1 – Three strings

> How do you time 105 minutes using three of the inflammable strings described in Problem 187.5, below.

Andrew Pettit

Identify the strings as A, B and C. Light strings A and B at one end and C at both ends. When string C is fully burned, one hour has passed and strings B & C have one hour left to burn. At this point light the unlit end of string B. It will be fully burned after a further half an hour. Now light the unlit end of string C. It will be fully burned after a further quarter of an hour. 1 hour + 30 minutes + 15 minutes is 105 minutes.

Also solved by **Ralph Hancock**, **Peter Fletcher** and **John Hudson**. Obviously the problem cries out for generalization. So we ask …

Problem 187.5 – Strings

You have an unlimited number of pieces of string and a very large box of matches.

A piece of string takes exactly two hours to burn from end to end. That is, if you set light to one end of the string, the flame will reach the other end precisely two hours later. However the flame does not necessarily travel along the string at a constant speed.

Determine the set, S, of intervals that can be timed exactly by setting light to a finite number of pieces of string in a sequential manner—as described in Solution 185.1, above. We have already seen that S includes $\{0.25, 0.5, 1, 1.5, 1.75, 2\}$.

Problem 187.6 – Iteration

Peter Griffiths

Heron's iteration formula for the square root of A takes the form

$$a \;\to\; \frac{1}{2}\left(\frac{A}{a} + a\right).$$

What is the more general iteration formula for any rational nth root of A?

Contrary to the opinion in many textbooks, this has nothing to do with calculus, but everything to do with finding the arithmetic means of estimates.

Letters to the Editors

Re: Problem 183.4

Dear Tony,

I was a little disappointed that you did not have more response to this problem. The solution by Basil Thompson in M500 **185** is fine, but there are other ways. Here is a way to show the result using the the concept of curvature, which involves the second derivative. The curvature of $y(x)$ at the point (x, y) is given by

$$\frac{y''(x)}{(1 + y'(x)^2)^{3/2}}.$$

Question: Find the values of t for which $\cosh x < \exp(tx^2)$ for all x.

Consider the two curves: $y = \cosh x$ and $y = \exp(tx^2)$. If $t < 0$ the inequality always fails, and if $t > 0$ the inequality succeeds for sufficiently large x. Hence we are only concerned about the behaviour of the functions for small x.

Sketching the curves show that both are concave from above and meet at minima when $x = 0$ and $y = 1$. In the region of the minima, the curve $y = \exp(tx^2)$ must be inside the curve $y = \cosh x$, otherwise the curves will cross for some $x > 0$. This requires that the curvature at $(0, 1)$ for $y = \exp(tx^2)$ must be greater than that for $y = \cosh x$.

For $y = \cosh x$: $y' = \sinh x$, $y'' = \cosh x$ and $y''(0) = 1$, giving a curvature of 1. For $y = \exp(tx^2)$: $y' = 2tx \exp(tx^2)$, $y'' = (2t + 4t^2x^2) \exp(tx^2)$ and $y''(0) = 2t$, giving a curvature of $2t$. Hence $2t \geq 1$; that is, $t \geq 1/2$.

John Bull

M500 185

Dear Tony

I have to say that I thought that problem 184.4 (Three real numbers) was complex when I first read it—especially after I had seen how you had modified the numbers. Following the reference from the contents page of issue 185 I now realize that it is imaginary!

Slightly less frivolously: Surely there is a false assumption in Problem 185.2 (Two streams) because the rate of flow of the water is increased as a result of opening the cold tap. The evidence is that the sink fills up more quickly with both taps on.

Regards,

Andrew Pettit

Sorry about that. Such are the dangers of recycling!—**ADF**

Observations on Problem 182.7

On p. 15 of M500 **184** ADF asks, 'For which integers p, q is it the case that

$$\sum_{k=1}^{p} \cos a_k = \sum_{k=1}^{p} \sin a_k = 0 \quad \Rightarrow \quad \sum_{k=1}^{p} \cos q a_k = \sum_{k=1}^{p} \sin q a_k = 0$$

for any real numbers a_1, a_2, $\ldots$, a_p?'

If we translate it to mechanics, we see that the problem concerns systems of particles in equilibrium since, in order to have static equilibrium, we must have the algebraic sum of the moments equal to zero in two mutually orthogonal directions. We can assume a coefficient of unity before each cos and sin, so we have n particles of unit mass lying on the circumference of a circle with unit radius.

Some readers may find it more suggestive to imagine n particles 'balancing' rather than adopting the 'path' approach. Note that it is always possible to arrange n particles around a circumference and obtain equilibrium since for any regular polygon with one vertex set at $(l, 0)$ the condition

$$\sum \cos a_r = \sum \sin a_r = 0$$

is fulfilled. If we fix one particle at $(l, 0)$ and only allow a position to be occupied by a single particle, the condition is equivalent to making the cosines of $n-1$ particles sum to -1. For a system of three particles there is only one possible configuration and, since $k \cdot 0 = 0$ for all k, the first particle stays there for ever. To recover the initial configuration we need only reflect in the x-axis, which, in terms of multiples of the angles, means—as Jim James usefully suggests—that we must have a multiple $\equiv 1$ or 2 (mod 3).

For other sets of points it might be possible to meet the conditions (zero sum of sines and cosines) *without* recovering the initial configuration. Also, for other values of p there will generally be an unlimited number of starting positions; e.g. for four points we only need the angles 0, pi, θ and $\pi + \theta$, with arbitrary θ, forming a cyclic parallelogram.

Finding other sets of values for the angles apart from $360/n$ means relaxing the requirement that the polygon be regular—but still keeping the requirement that all the vertices lie on the circumference of a circle. One wonders whether some of the properties of regular polygons will carry over to these irregular inscribed polygons in much the same way as certain properties of squares or rectangles carry over to cyclic quadrilaterals.

Sebastian Hayes

Problem 187.7 – Task

Tony Forbes

This will bring a smile to the face of anyone who has used commercial computer software. You have a Very Important Task, T, to perform on your computer. Normally the task takes t seconds to complete. However, in any interval of duration one second while it is running T will fail with probability p. When T fails it has to be started again from the beginning.

What is the expected total time for a successful completion of the task?

Problem 187.8 – Seven events

Tony Forbes

Inspired by Chris Pile's Venn diagram for five events [this M500, p. −1] I am wondering if it is possible to do the same for $p = 7$, 11, 13, ...

Take $p = 7$ for instance. Draw a closed curve representing event A and rotate it through multiples of $2\pi/7$ about a suitable point to obtain six new closed curves representing events B, C, ..., G. Do this in such in such a manner that you create precisely 127 regions representing every non-empty subset of $\{A, B, \ldots, G\}$. Or prove that it can't be done.

Note that this construction won't work for $p = 6$, or 4, or any p that does not satisfy $2^p \equiv 2 \pmod{p}$. So p must be either a prime or a pseudoprime to base 2.

Factorial digital invariants

David Singmaster

For a decimal number N, let $S(N)$ be the sum of the factorials of the digits, excluding leading zeros. At a recent meeting, a colleague stated that this function has a cycle: $(169, 363601, 1454)$.

There are five numbers such that $S(N) = N$, namely, 0, 1, 2, 145 and 40585. (Zero is a bit exceptional; since leading zeroes are excluded, the sum is empty!) The largest number for which $S(N) > N$ is 1999999 and there are 208907 such numbers. Consequently the problem of finding all the cycles of this function is finite. In fact, there are two other cycles, both 2-cycles: (871, 45361) and (872, 45362).

ADF—We have already published similar investigations involving $n \to n^k$ instead of $n \to n!$. Are there other interesting functions to analyse?

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