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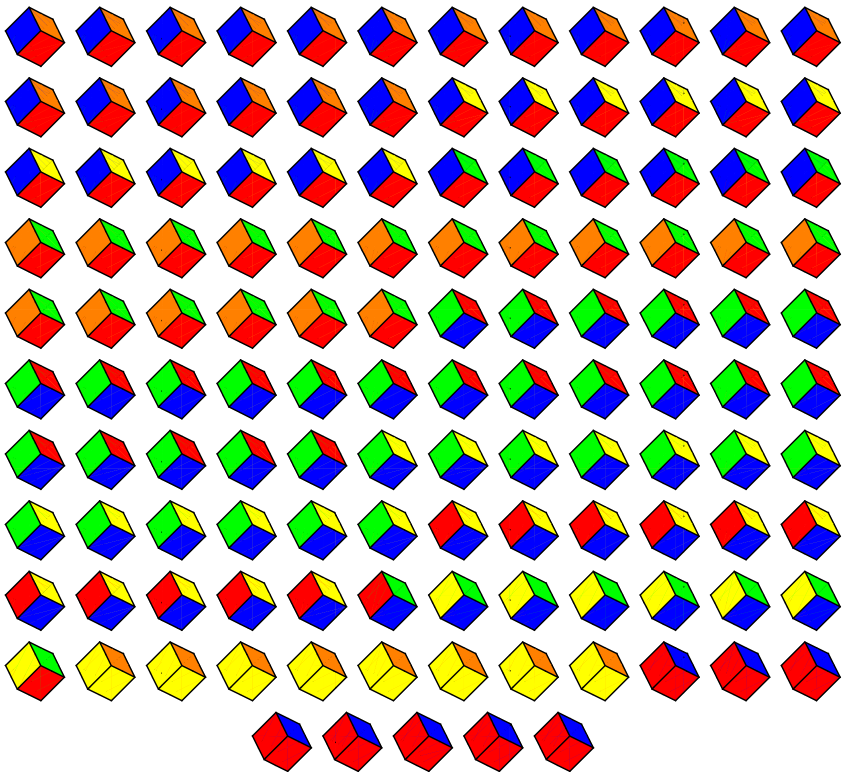

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M500 Winter Weekend 2020

The **thirty-ninth M500 Society Winter Weekend** will be held over

Friday 10th – Sunday 12th January 2020,

venue to be confirmed. Details, pricing and a booking form will be available nearer the time. Please refer to the M500 web site.

m500.org.uk/winter-weekend/

Solution 285.5 – 64 cubes

This is very similar to Problem 274.5 – 27 cubes. There are 64 cubes, where each face is painted using one of the four colours red, blue, green, yellow. Moreover, the 64 cubes can be assembled in four ways to form either a red, blue, green or yellow $4 \times 4 \times 4$ cube. How can this be achieved?

What about n^3 cubes, n colours and n monochromatic $n \times n \times n$ cubes?

Chris Pile

For $n > 1$, each $n \times n \times n$ cube formed of n^3 cubes has 8 vertex cubes with 3 faces visible. Between each vertex there are $n - 2$ cubes forming each of 12 edges, with two adjacent faces visible. The central $(n-2)^2$ cubes on each face of the $n \times n \times n$ cube have only one face visible. The core of the $n \times n \times n$ cube consists of $(n-2)^3$ cubes, which are not visible. That is, every $n \times n \times n$ cube has:

8 vertex cubes (V) with 3 faces (surrounding the vertex) visible;
$12(n-2)$ edge cubes (E) with 2 adjacent faces visible;
$6(n-2)^2$ face cubes (F) with only one face visible;
$(n-2)^3$ cubes hidden internally (H).

Thus

$$n^3 = 8 + 12(n-2) + 6(n-2)^2 + (n-2)^3,$$

the number of visible faces is

$$24 + 24(n-2) + 6(n-2)^2 = 6n^2$$

and the total number of faces on n^3 cubes is $6n^3$. Therefore with n colours available we need to determine the painting scheme to produce n monochromatic $n \times n \times n$ cubes.

Faces visible	n	2	3	4	5	6	7	8	9	10
3	Vertex cubes	8	8	8	8	8	8	8	8	8
2	Edge cubes	0	12	24	36	48	60	72	84	96
1	Face cubes	0	6	24	54	96	150	216	294	384
0	Hidden cubes	0	1	8	27	64	125	216	343	512
Visible faces: $6n^2$		24	54	96	150	216	294	384	486	600
Total cubes: n^3		8	27	64	125	216	343	512	729	1000

In the diagrams that follow, each n^3 cube is shown as a vertical column of six faces with a letter denoting the colour. Where a cube has three faces the same colour the faces must surround a vertex. Where a cube has two faces the same colour the faces must be adjacent. The number at the top of the column shows how many of the n^3 cubes are painted the same. The supplementary diagrams show how the cubes are assembled to form each $n \times n \times n$ cube with monochromatic external faces.

$n = 2$
8
R
R
R
B
B
B

$n = 3$: 27 cubes						
1	1	1	6	6	6	6
R	R	B	R	B	G	R
R	R	B	R	B	G	R
R	R	B	R	B	G	B
B	G	G	B	G	R	B
B	G	G	B	G	R	G
B	G	G	G	R	B	G
a	b	c	d	e	f	g

	R	B	G	
V	a b d	a c e	b c f	8
E	f g	d g	e g	12
F	e	f	d	6
H	c	b	a	1

$n = 4$: 64 cubes			
8	8	24	24
R	G	R	G
R	G	R	G
R	G	B	Y
B	Y	B	Y
B	Y	G	R
B	Y	Y	B
a	b	c	d

	R	B	G	Y	
V	a	a	b	b	8
E	c	c	d	d	24
F	d	d	c	c	24
H	b	b	a	a	8

$n = 5$: 125 cubes											
18	12	6	18	23	13	11	1	6	1	8	8
R	R	R	R	B	B	B	B	B	R	Y	R
G	G	R	R	O	O	O	O	O	R	Y	R
B	B	B	O	G	G	R	R	Y	Y	Y	R
B	B	B	O	G	G	R	G	G	G	O	B
O	Y	G	G	R	Y	Y	G	G	G	O	B
O	Y	Y	Y	Y	Y	Y	G	G	G	O	B
a	b	c	d	e	f	g	h	j	k	l	m

Cube	Vertex (8)	Edge (36)	Face (54)	Hidden (27)
Red	m	c d g k	a b e h	f j l
Blue	m	a b c	e f g h j	d k l
Green	h j k	e f	a b c d	g l m
Yellow	l	b f g	c d e j k	a h m
Orange	l	a d	e f g h j	b c k m

$n = 6$: 216 cubes : see page 4

$n = 7$: 343 cubes																
45	15	30	15	15	30	15	15	60	30	30	15	8	8	4	4	4
R	R	R	R	R	R	B	B	B	B	B	G	R	G	P	Y	Y
G	G	O	O	O	Y	Y	Y	W	R	R	B	R	G	P	Y	Y
G	G	O	O	O	Y	Y	Y	W	R	R	B	R	G	P	Y	Y
B	P	P	P	W	W	W	P	P	P	P	P	B	O	W	W	P
B	P	P	P	Y	O	O	O	O	O	Y	Y	B	O	W	W	P
Y	Y	Y	W	G	G	G	G	G	W	W	W	B	O	W	W	P
a	b	c	d	e	f	g	h	j	k	l	m	n	p	q	r	s

Cube	Vertex (8)	Edge (60)	Face (150)	Hidden (125)
Red	n	k l	a b c d e f	g h j m p q r s
Blue	n	a m	g h j k l	b c d e f p q r s
Green	p	a b	e f g h j m	c d k l n q r s
Yellow	r s	f g h	a b c e l m	d j k n p q
Orange	p	c d e	f g h j k	a b l m n q r s
Pink	q s	b c d	h j k l m	a e f g n p r
White	q r	j	d e f g k l m	a b c h n p s

$n = 8$: 512 cubes													
24	72	72	48	24	72	72	24	24	48	8	8	8	8
R	R	R	R	B	B	B	B	B	B	B	G	Y	W
G	G	G	G	O	O	O	O	O	B	B	G	Y	W
Y	Y	Y	Y	P	P	P	P	P	G	B	G	Y	W
S	W	W	W	W	S	S	S	S	G	R	O	P	S
B	O	P	S	S	Y	W	G	R	R	R	O	P	S
B	O	P	S	S	Y	W	G	R	R	R	O	P	S
a	b	c	d	e	f	g	h	j	k	l	m	n	p

Cube	Vertex (8)	Edge (72)	Face (216)	Hidden (216)
Red	l	j k	a b c d	e f g h m n p
Blue	l	a k	e f g h j	b c d m n p
Green	m	h k	a b c d	e f g j l n p
Yellow	n	f	a b c d	e g h j k l m p
Orange	m	b	e f g h j	a c d k l n p
Pink	n	c	e f g h j	a b d k l m p
White	p	g	b c d e	a f h j k l m n
Sepia	p	d e	a f g h j	b c k l m n

$n = 9 : 729$ cubes																				
63	21	84	84	42	42	21	42	21	21	63	84	63	21	21	8	8	8	4	4	4
R	R	R	R	R	B	B	B	B	B	B	B	G	G	G	R	G	Y	W	L	S
G	G	O	Y	P	P	W	W	W	S	S	L	R	R	B	R	G	Y	W	L	S
G	G	O	Y	P	P	W	W	W	S	S	L	R	R	B	R	G	Y	W	L	S
B	S	S	S	S	S	S	P	P	P	P	P	P	W	W	B	O	P	S	W	L
B	L	L	L	L	L	L	O	Y	Y	Y	Y	Y	Y	Y	B	O	P	S	W	L
W	W	W	W	G	G	G	G	G	G	O	O	O	O	O	B	O	P	S	W	L
a	b	c	d	e	f	g	h	j	k	l	m	n	p	q	r	s	t	x	y	z

Cube	Vertex (8)	Edge (84)	Face (294)	Hidden (343)
Red	r	n p	a b c d e	f g h j k l m q s t x y z
Blue	r	a q	f g h j k l m	b c d e n p s t x y z
Green	s	a b	e f g h j k n p q	c d l m r t x y z
Yellow	t	d	j k l m n p q	a b c e f g h r s x y z
Orange	s	c	h l m n p q	a b d e f g j k r t x y z
Pink	t	e f	h j k l m n	a b c d g p q r s x y z
White	x y	g h j	a b c d p q	e f k l m n r s t z
Sepia	x z	k l	b c d e f g	a h j m n p q r s t y
Lilac	y z	m	b c d e f g	a h j k l n p q r s t x

$n = 6 : 216$ cubes					
8	8	8	48	48	96
R	G	O	R	Y	R
R	G	O	R	Y	B
R	G	O	B	O	G
B	Y	P	B	O	Y
B	Y	P	G	P	O
B	Y	P	G	P	P
Vertex			Edge		Face

As each cube has six faces, the painting and assembly is straightforward to give 8 vertex cubes, 48 edge cubes and 96 face cubes of each colour. The hidden cubes consist of 16 vertex cubes (4 unseen colours) and 48 edge cubes (3 unseen colours). See Nested monochromatic cubes, later.

$n = 10 : 1000$ cubes														
96	96	96	96	96	96	96	96	96	96	8	8	8	8	8
R	B	G	O	Y	P	W	S	L	T	R	G	Y	W	L
R	B	G	O	Y	P	W	S	L	T	R	G	Y	W	L
W	R	R	R	R	Y	Y	Y	Y	L	R	G	Y	W	L
L	L	L	G	G	G	G	W	W	W	B	O	P	S	T
T	T	T	T	O	O	O	O	S	S	B	O	P	S	T
S	S	B	B	B	B	P	P	P	P	B	O	P	S	T

The case $n = 10$ is straightforward as the ten different colour vertex cubes can be arranged in $5 \times 8 = 40$ cubes of the 1000, leaving 960 cubes, which is 10 times the number of edge cubes of each colour and 5/2 times the number of face cubes of each colour. The 10th colour is T, Tan.

For $n \geq 10$, the number of hidden cubes is greater than half of the total.

Since $6(n-2) = M$ is a common factor of the number of edge cubes and the number of face cubes ($E = 2M$, $F = (n-2)M$) it seems appropriate to use this multiple when attempting to minimize the number of groups of cubes painted in the same way. For some values of n, the arrangement is straightforward (not necessarily optimum?), and for some values of n, the arrangement is particularly difficult (e.g. $n = 5$, $n = 7$). In some cases the $(n-2)^3$ internal cubes can be arranged to be an $(n-2) \times (n-2) \times (n-2)$ cube that is monochromatic, suggesting a series of 'nested' monochromatic cubes!

Nested monochromatic cubes

As n increases, the number of possible arrangements of painting and assembly of the n^3 cubes also increases. Hence there is scope for investigating how other criteria can be satisfied. The arrangement for $n = 6$ is nicely symmetric. Each of the six $6 \times 6 \times 6$ monochromatic cubes can have the outer shell of cubes removed to reveal a $4 \times 4 \times 4$ cube that is monochromatic, which in turn can be 'peeled' to reveal a $2 \times 2 \times 2$ monochromatic core. Using the previous $n = 6$ diagram and colours, we can obtain the following nested arrangements.

$6 \times 6 \times 6$ cube	$4 \times 4 \times 4$ cube	$2 \times 2 \times 2$ cube
Red or Blue	Yellow Orange Pink	Orange or Pink Green or Yellow Green or Yellow
Green	Orange Pink	Red or Blue Red or Blue
Yellow	Red Blue	Orange or Pink Orange or Pink
Orange or Pink	Red Blue Green	Green or Yellow Green or Yellow Red or Blue
152 cubes	56 cubes	8 cubes

Critical sets and sudoku puzzles

Tony Forbes

Seeing Reinhardt Messerschmidt's solution in M500 **288** of Problem 284.2 – 13 cards (If a standard pack of 52 playing cards is shuffled and dealt into 13 piles of four, is it always possible to select one card from each pile so that the chosen cards consist of 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A, not necessarily of the same suit?) should remind me and other sudoku fans that critical sets play an important role in the solution of sudoku puzzles.

We start with a typical puzzle that could have appeared in a popular newspaper or magazine. Well, not quite. The squares which would normally be left blank have been filled in with the symbols 1, 2, ..., 9 to clearly indicate that before you start solving the puzzle the correct numbers in these cells for the unique solution are completely undetermined.

123 456 789	123 456 789	1	123 456 789	123 456 789	9	123 456 789	7	123 456 789
9	6	123 456 789	5	123 456 789	123 456 789	123 456 789	123 456 789	8
123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789
4	5	123 456 789	6	123 456 789	123 456 789	123 456 789	123 456 789	3
8	123 456 789	2	123 456 789	9	123 456 789	6	123 456 789	4
6	123 456 789	123 456 789	123 456 789	123 456 789	2	123 456 789	5	7
123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789
7	123 456 789	123 456 789	123 456 789	123 456 789	4	123 456 789	8	5
123 456 789	8	123 456 789	3	123 456 789	123 456 789	1	123 456 789	123 456 789

Attempt to solve the puzzle strictly according to the following algorithm, which I refer to as *the critical-set strategy*.

(1) Look at a region (a row, a column or one of the nine 3×3 boxes into which the array is partitioned), R, and try to find a *critical set* in R. This is a set of k cells in R such that collectively they contain exactly k distinct numbers, $d_1, d_2, \ldots, d_k$, say. Then cross out $d_1, d_2, \ldots, d_k$ wherever they appear in the remaining cells of R.

(2) Repeat (1) until the array is stable.

For example, you might notice that in the top row, each of the cells in the 3rd, 6th and 8th columns contains just one number, 1, 9 and 7 respectively. Thus $R =$ top row, $k = 3$, and the critical set is $\{1, 9, 7\}$, where, for simplicity, we refer to a cell by the symbol(s) contained in it. So you remove all occurrences of 1, 9 and 7 from the remaining 6 cells of R. Similarly, in the left column there is a critical set, $\{9, 4, 8, 6, 7\}$, of size 5 and hence you can eliminate 9, 4, 8, 6 and 7 from the 1st, 3rd, 7th and 9th rows in the left column. But be warned that it is not always as simple as I appear to be suggesting. You might need to make use of critical sets where each cell has more than one element.

The size k can be anything from 0 to 9. However, 0 and 9 are vacuous cases. Although you can always find critical sets of sizes 0 and 9, there is not a lot you can do with them. So for practical purposes, you should restrict k to $1 \le k \le 8$. If you ever find a set of k cells that covers fewer than k distinct symbols, you should give up and start again because something somewhere has gone badly wrong. On the other hand, if you follow the procedure correctly, you will solve the puzzle. As far as I am aware, the vast majority of sudoku puzzles published in newspapers, even ones described as 'hard', or 'fiendish', will eventually yield to the critical-set strategy. Nevertheless there do exist puzzles that don't.

Now try the critical-set strategy with this next one.

123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	8	123 456 789	123 456 789	123 456 789
123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	7	8	9
123 456 789	8	9	1	123 456 789	123 456 789	123 456 789	2	123 456 789
6	123 456 789	123 456 789	123 456 789	4	123 456 789	123 456 789	5	123 456 789
123 456 789	9	8	123 456 789	6	123 456 789	4	3	123 456 789
123 456 789	2	123 456 789	123 456 789	3	123 456 789	123 456 789	123 456 789	1
123 456 789	1	123 456 789	123 456 789	123 456 789	6	5	7	123 456 789
3	7	2	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789
123 456 789	123 456 789	123 456 789	9	123 456 789	123 456 789	123 456 789	123 456 789	123 456 789

When stability has been achieved, and provided you (or I) have made no mistakes, you will finish with the array on the next page.

127	46	1367	23467	9	**8**	136	146	5
125	456	1356	23456	25	2345	**7**	**8**	**9**
57	**8**	**9**	**1**	57	34	36	**2**	346
6	3	17	278	**4**	1279	289	**5**	278
157	**9**	**8**	257	**6**	1257	**4**	**3**	27
4	**2**	57	578	**3**	579	689	69	**1**
9	**1**	4	238	28	**6**	**5**	**7**	238
3	**7**	**2**	458	158	145	1689	1469	468
8	56	56	**9**	127	12347	123	14	234

Although the puzzle is not solved, a situation has arisen where the cells of each region can be partitioned into *minimal* critical sets, i.e. sets that cannot be further partitioned into smaller non-empty critical sets. In the bottom row, for example, there are four such sets, $\{8\}$, $\{56, 56\}$, $\{9\}$ and $\{127, 12347, 123, 14, 234\}$ of sizes 1, 2, 1 and 5 respectively.[1] You might even notice that it is possible to construct a *system of distinct representatives* from the nine cells, $\{8, 5, 6, 9, 7, 2, 3, 1, 4\}$ for example, to give a solution to just the bottom row and which is also compatible with the bottom-left 3×3 box. However these numbers might not be valid for the whole array.

When a sudoku puzzle is solved all the minimal critical sets will have size 1, as you will have seen from the first example. Plainly this has not happened here, and if you want to complete the puzzle you will have to resort to some other means, brute force or, as a last resort, human ingenuity.

In the current array there are four regions that have minimal critical sets of size 7, and none with size greater than 7. They occur in column 4, column 5, and the 3×3 boxes at the top-left and bottom-right.

Decomposing a region into minimal critical sets can be quite tricky; so it is a good idea to get one's computer to help. Suppose the region, R, you want to partition has cells R_1, R_2, $\ldots$, R_9. Construct an incidence graph that has vertices $\{1, 2, \ldots, 9, R_1, R_2, \ldots, R_9\}$ by creating an edge $\{i, R_j\}$ whenever $i \in R_j$. Then decompose the graph into its connected compo-

[1] Recall that {56,56} here means {(cell at 9th row, 2nd column), (cell at 9th row, 3rd column)}; there really are two distinct elements.

nents. Conveniently, Mathematica can do this. The sizes of the connected components give, after dividing by 2, the sizes of the minimal critical sets. Mathematica can also solve the marriage problem, i.e. finding a system of distinct representatives for a set of sets $\mathcal{G} = \{B_1, B_2, \ldots, B_n\}$, that satisfies *Hall's marriage condition*,

$$\left| \bigcup_{B \in \mathcal{S}} B \right| \geq |\mathcal{S}| \quad \text{for every subset } \mathcal{S} \text{ of } \mathcal{G}. \tag{1}$$

Think of $B_i \in \mathcal{G}$, $i = 1, 2, \ldots, n$, as a list of boys each of which girl g_i is willing to put up with for a long time. The marriage theorem says that (1) is sufficient (and necessary) for a successful one-to-one assignment of boys to the girls $\{g_1, g_2, \ldots, g_n\}$.

Getting back to sudoku, here is a partly completed puzzle that has reached stability under the critical-set strategy and which took me a little while to find. It is noteworthy because there are eight minimal critical sets of size 9: four in the border of the array, two in the central cross and two in 3×3 boxes. It is not obvious (to me) how to complete the puzzle but I expect my computer program would make short work of it.

123456789	34578	12679	48	45678	57	279	23479	23479
24579	457	279	**1**	457	**3**	**8**	**6**	2479
34678	3478	67	**2**	4678	**9**	**1**	**5**	347
16	2	**8**	7	59	**4**	**3**	19	1569
13467	347	167	38	23589	25	25679	1279	125679
37	**9**	**5**	**6**	23	**1**	**4**	**8**	27
279	**6**	**4**	**5**	1237	**8**	279	12379	12379
2578	**1**	**3**	**9**	247	**6**	257	247	24578
25789	578	279	34	12347	27	25679	123479	123456789

This suggests a very interesting problem.

Find a sudoku puzzle where the critical-set strategy produces an array with at least 9 minimal critical sets of size 9. Or prove that no such thing exists.

Solution 284.3 – Characteristic polynomial

Show that a square matrix is a root of its characteristic polynomial. In other words, if M is an $n \times n$ matrix, and

$$P_M(x) \;=\; \det(xI_n - M) \;=\; \sum_{i=0}^{n} a_i x^i$$

is its characteristic polynomial, show that

$$P_M(M) \;=\; \sum_{i=0}^{n} (a_i I_n) M^i \;=\; 0_n,$$

where I_n is the $n \times n$ identity matrix, and 0_n is the $n \times n$ all-zeros matrix.

Reinhardt Messerschmidt

This result is known as the *Cayley–Hamilton theorem.* The following proof is adapted from [1].

Terminology and notation

Suppose A is an $n \times n$ matrix. The (i,j)-th entry of A will be denoted by A_{ij}. The *(i,j)-th minor* of A, i.e. the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i-th row and j-th column, will be denoted by $\operatorname{minor}_{ij} A$. The *adjugate* of A, i.e. the matrix whose (i,j)-th entry is

$$(-1)^{j+i} \det(\operatorname*{minor}_{ji} A),$$

will be denoted by $\operatorname{adj} A$.

A property of the adjugate of a matrix

For every i, j,

$$(A(\operatorname{adj} A))_{ij} \;=\; \sum_{k=1}^{n} A_{ik} (-1)^{j+k} \det(\operatorname*{minor}_{jk} A).$$

If $i = j$, then the right-hand side of this equation is the determinant expansion of A by its j-th row. If $i \neq j$ and A' is the matrix obtained from A by replacing its j-th row with its i-th row, then the right-hand side is the determinant expansion of A' by its j-th row. Since the i-th and j-th rows of A' are equal, we have $\det A' = 0$. It follows that

$$A(\operatorname{adj} A) \;=\; (\det A)I.$$

Proof of the Cayley–Hamilton theorem

Applying the above property of the adjugate of a matrix to $xI - M$,

$$\begin{aligned}(xI - M)\operatorname{adj}(xI - M) \;&=\; \det(xI - M)I \\ &=\; P_M(x)I \;=\; \left(\sum_{i=0}^{n} a_i x^i\right)I \;=\; \sum_{i=0}^{n} x^i (a_i I).\end{aligned}$$

Each entry of $\operatorname{adj}(xI - M)$ is a polynomial in x of degree at most $n - 1$. This implies that there exist matrices $B_0, B_1, \ldots, B_{n-1}$ such that

$$\operatorname{adj}(xI - M) \;=\; \sum_{i=0}^{n-1} x^i B_i;$$

therefore

$$\begin{aligned}(xI - M)\operatorname{adj}(xI - M) \;&=\; \sum_{i=0}^{n-1} (x^{i+1} B_i - x^i M B_i) \\ &=\; x^n B_{n-1} + \sum_{i=1}^{n-1} x^i (B_{i-1} - M B_i) - M B_0;\end{aligned}$$

therefore

$$\sum_{i=0}^{n} x^i (a_i I) \;=\; x^n B_{n-1} + \sum_{i=1}^{n-1} x^i (B_{i-1} - M B_i) - M B_0.$$

This matrix equation is equivalent to a system of n^2 scalar equations. In each scalar equation, the left-hand side and the right-hand side are polynomials in x of degree at most n; therefore we may equate the coefficients of x^i for $i = 0, 1, \ldots, n$. Equating coefficients in each scalar equation is equivalent to equating coefficients in the matrix equation; therefore

$$\begin{gathered} a_0 I \;=\; -M B_0, \qquad a_1 I \;=\; B_0 - M B_1, \\ a_2 I \;=\; B_1 - M B_2, \quad \ldots, \\ a_{n-1} I \;=\; B_{n-2} - M B_{n-1}, \qquad a_n I \;=\; B_{n-1}. \end{gathered}$$

Multiplying both sides of the i-th equation from the left by M^{i-1} for $i = 1, 2, \ldots, n+1$,

$$\begin{aligned} a_0 I &= -MB_0, \\ a_1 M &= MB_0 - M^2 B_1, \\ a_2 M^2 &= M^2 B_1 - M^3 B_2, \\ &\ldots, \\ a_{n-1} M^{n-1} &= M^{n-1} B_{n-2} - M^n B_{n-1}, \\ a_n M^n &= M^n B_{n-1}. \end{aligned}$$

The left-hand sides of these equations sum to $P_M(M)$, and the right-hand sides sum to 0_n.

References

[1] T. M. Apostol, *Calculus, Volume II*, 2nd ed., John Wiley & Sons, 1969.

Problem 289.1 – Odd divisor sum

Let

$$S(m,n) \;=\; \frac{1}{n^2} \sum_{k=n+1}^{2^m n} q(k), \qquad m,n = 1,2,\ldots,$$

where the function $q(k)$ is defined by

$$q(k) \;=\; \begin{cases} k & \text{if } k \text{ is odd}, \\ q(k/2) & \text{otherwise}; \end{cases}$$

i.e. $q(k)$ is the largest odd divisor of k. (i) Compute $S(1,1)$ and show that

$$S(m+1,1) \;=\; 4S(m,1) + 1.$$

(ii) Show that in general $S(m,n)$ is an integer that is (somewhat amazingly (at least in the opinion of me (TF))) independent of n, and hence obtain a closed formula for $S(m,n)$.

This seems to be a generalization of a puzzle that appeared in *Chalkdust*, issue 5.

Solution 284.4 – Factorial square

For positive integers p and q, define

$$F(p,q) \;=\; \frac{1}{q!}\prod_{j=1}^{p} j!.$$

(i) Show that $F(4n,2n)$ is always a square, or find a counter-example. For instance,

$$F(12,6) = 2!\,3!\,4!\,5!\;7!\,8!\,9!\,10!\,11!\,12! = 42050558739087360000^2.$$

(ii) With the exception of $F(1,1)=1$, show that $F(p,q)$ cannot be a square if p is odd. Or find another example.

(iii) Show that $F(14,9)$ and $F(18,7)$ are squares.

(iv) Prove the following, or find a counter-example. Apart from $F(14,9)$ and $F(18,7)$, if $F(2n,q)$ is a square, then $n-1 \le q \le n+1$.

Tommy Moorhouse

I initially thought of an oblique approach to this problem, but the 'brute force' route proved simpler. Writing out the full expression for $\prod_{k=1}^{4n} k!$ we have

$$\begin{aligned}
&(4n)!(4n-1)!(4n-2)!(4n-3)!\cdots 3!2!1!\\
&\;=\; 4n(4n-1)!^2(4n-2)(4n-3)!^2\cdots 4(3!)^2 2(1!)^2\\
&\;=\; 2(2n)2(2n-1)2(2n-2)\cdots 2((4n-1)!(4n-3)!\cdots(3!)(1!))^2\\
&\;=\; 2^{2n}(2n)!((4n-1)!(4n-3)!\cdots(3!)(1!))^2.
\end{aligned}$$

Dividing by $(2n)!$ we see that $F(4n,2n)$ is a square.

As a byproduct of my initial approach I found the identity

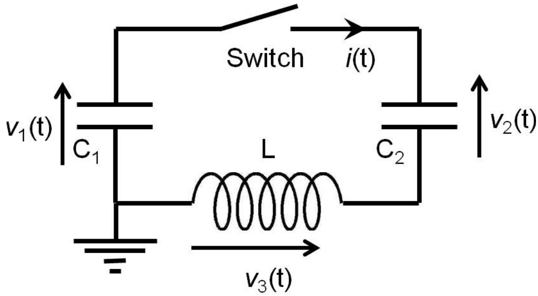
$$\sum_{k=1}^{N}\left[\frac{k}{m}\right] \;=\; \left[\frac{N}{m}\right]\left((N+1)-\frac{m}{2}\left(\left[\frac{N}{m}\right]+1\right)\right).$$

Here square brackets $[x]$ indicate the integer part of x. Some tidy formulae for $m \le 7$ are presented in the reference. Can you prove the identity in the form shown above?

Reference Apostol, T., *Introduction to Analytic Number Theory*, Springer, 1976, Chapter 3, Ex 25, 26.

Solution 285.3 – A coil and two capacitors

This is like Problem 256.5 – Lost energy. Behold, a simple circuit containing a coil of L henrys and two capacitors of C farads each. The diagram represents the initial state, with 100 volts across C_1. Assume the wiring consists of perfect conductors. What happens when you close the switch?



Edward Stansfield

This hypothetical problem asks what happens when the switch is closed in the lossless circuit shown here. The capacitors $C_1 = C_2 = C$ farads and the inductance is L henrys. Prior to the switch being closed at time $t = 0$ the capacitor C_1 is charged to $v_1(0) = 100$ volts, capacitor C_2 has no charge and the current flow is zero. The initial conditions are thus:

$$v_1(0) \;=\; 100, \qquad v_2(0) \;=\; v_3(0) \;=\; 0, \qquad i(0) \;=\; 0.$$

At time t after the switch has been closed we have the following.

Transient response By Kirchhoff's Voltage Law we have

$$v_1(t) \;=\; v_1(0) - \frac{1}{C}\int_0^t i(\theta)d\theta \;=\; v_2(t) + v_3(t) \;=\; \frac{1}{C}\int_0^t i(\theta)d\theta + L\frac{di(t)}{dt}. \quad (1)$$

Simplifying (1) yields

$$Cv_1(0) \;=\; 2\int_0^t i(\theta)d\theta + LC\frac{di(t)}{dt}. \qquad (2)$$

Differentiating (2) once obtains

$$0 \;=\; 2i(t) + LC\frac{d^2 i(t)}{dt^2}. \qquad (3)$$

The general solution of (3) is

$$i(t) \;=\; \alpha\sin(\omega_0 t) + \beta\cos(\omega_0 t) + \gamma, \qquad (4)$$

where α, β and γ are constants. Differentiating (4) twice gives

$$\frac{d^2 i(t)}{dt^2} \;=\; -\,\omega_0^2(\alpha\sin(\omega_0 t) + \beta\cos(\omega_0 t)) \;=\; \omega_0^2(i(t) - \gamma). \qquad (5)$$

Substituting from (5) into (3) yields $\gamma = 0$, and $\omega_0^2 = 2/(LC)$. Since the voltage across a capacitor cannot change instantaneously, the voltage across the inductor L immediately after the switch is closed must be equal to the initial voltage across capacitor C_1. That is, the initial conditions at time $t = 0$ are

$$v_3(0) \;=\; v_1(0) \;=\; L\frac{di(0)}{dt} \;=\; \alpha L \omega_0.$$

Furthermore, since the current through an inductor cannot change instantaneously, the initial current in the circuit after the switch the switch is closed must also be zero, i.e. $i(0) = 0$, which shows that $\beta = 0$. Hence we have the particular solution of (4):

$$i(t) \;=\; \frac{v_1(0)}{L\omega_0}\sin(\omega_0 t), \qquad \text{where } \omega_0 \;=\; \sqrt{2/(LC)}.$$

In terms of the circuit components this is

$$i(t) \;=\; v_1(0)\sqrt{C/(2L)}\,\sin\left(t\sqrt{2/(LC)}\right).$$

Observe that in this case there is no 'steady state' in the traditional sense, since the current $i(t)$ is sinusoidal with a constant amplitude at frequency ω_0 radians per second. The voltages $v_1(t)$ and $v_2(t)$ across the two capacitors are necessarily also sinusoidal at the same frequency. Note that if the inductance L is zero, the frequency ω_0 is infinite.

To complete the analysis, the capacitor voltages are given by

$$\begin{aligned} v_1(t) \;&=\; v_1(0) - \frac{1}{C}\int_0^t i(\theta)d\theta, \qquad v_2(t) \;=\; \frac{1}{C}\int_0^t i(\theta)d\theta, \\ v_3(t) \;&=\; L\frac{di(t)}{dt}. \end{aligned}$$

The current integral is

$$\int_0^t i(\theta)d\theta \;=\; v_1(0)\sqrt{\frac{C}{2L}}\int_0^t \sin(\omega_0\theta)d\theta \;=\; \frac{C}{2}v_1(0)(1-\cos(\omega_0 t)).$$

Hence

$$\begin{aligned} v_1(t) \;&=\; \frac{1}{2}v_1(0)(1+\cos(\omega_0 t)), \qquad v_2(t) \;=\; \frac{1}{2}v_1(0)(1-\cos(\omega_0 t)), \\ v_3(t) \;&=\; v_1(0)\cos(\omega_0 t). \end{aligned}$$

Observe that, as expected, the average values are given by

$$\langle v_1(t)\rangle \;=\; \langle v_2(t)\rangle \;=\; v_1(0)/2, \qquad \langle v_3(t)\rangle \;=\; 0.$$

Energy transfer At any time t the total energy $E(t)$ stored in the two capacitors and the inductor is given by

$$\begin{aligned}
E(t) &= \frac{1}{2}\left(Cv_1^2(t) + Cv_2^2(t) + Li^2(t)\right)\\
&= \frac{1}{2}v_1^2(0)\left(\frac{C}{4}(1+\cos(\omega_0 t))^2 + \frac{C}{4}(1-\cos(\omega_0 t))^2 + \frac{C}{2}\sin^2(\omega_0 t)\right)\\
&= \frac{1}{8}v_1^2(0)C\left((1+\cos(\omega_0 t))^2 + (1-\cos(\omega_0 t))^2 + 2\sin^2(\omega_0 t)\right)\\
&= \frac{1}{8}v_1^2(0)C\left(2 + 2\cos^2(\omega_0 t) + 2\sin^2(\omega_0 t)\right) \;=\; \frac{1}{2}v_1^2(0)C.
\end{aligned}$$

This is the same as the initial energy stored on capacitor C_1 prior to the switch being closed, and as expected this implies that no energy is lost. It oscillates at frequency ω_0 between the two capacitors, with intermediate storage in the inductor.

Problem 289.2 – Tetrahedron

A regular tetrahedron with side length $2\sqrt{2}$ has its centre at the origin and has vertices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$. Let $\mathbf{W}_i = (V_{i,x}, V_{i,y}, V_{i,z}, 1)$, where $\mathbf{V}_i = (V_{i,x}, V_{i,y}, V_{i,z})$, $i = 1, 2, 3, 4$. Show that $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ and $\mathbf{W}_4$ have magnitude 2 and are mutually orthogonal.

Problem 289.3 – Continued fractions

Prove the following continued fraction formulæ:

$$\frac{e+1}{e-1} \;=\; [2; 6, 10, 14, 18, \ldots], \qquad \frac{e^2+1}{e^2-1} \;=\; [1; 3, 5, 7, 9, \ldots].$$

Is there a nice expression that has continued fraction [1; 2, 3, 4, ...]?

A certain mathematician devised a theorem which meant he didn't have to pay for food in restaurants. What did he call it?

Later he came up with another theorem, which meant he didn't get to taste any of that restaurant food. What did he call that one?

— Sent by Jeremy Humphries.

Number fields from prime pairs

Tommy Moorhouse

Background This problem concerns properties of a certain family of number fields and makes use of basic properties of rings and fields. All the mathematical background can be found in standard texts, and I've suggested a couple in the references. Essentially a number field is an extension of the field of rational numbers $\mathbb{Q}$ by some irrational number satisfying a polynomial equation. For example, appending $\sqrt{2}$ (a solution of $t^2 - 2 = 0$) to $\mathbb{Q}$ gives a number field including the rational numbers and combinations such as $a + b\sqrt{2}$.

Integers A number field has an associated ring of integers $\mathfrak{O}$, just as $\mathbb{Q}$ has ring of integers $\mathbb{Z}$, the ordinary integers. In a general number field integers are those elements satisfying a monic polynomial equation (that is, the leading coefficient is 1) with coefficients in $\mathbb{Z}$. The ring of integers $\mathfrak{O}$ can contain some unlikely looking members. For example if the square-free integer m is congruent to 1 (mod 4) then $1/2 + \sqrt{m}/2$ is an integer.

The prime pairs number field Here we will consider a number field given by $\mathbb{Q}[\sqrt{p}, \sqrt{p+2}]$ where p and $p+2$ form a prime pair. You could show that this is the same field as $\mathbb{Q}[\sqrt{p} + \sqrt{p+2}]$, which we will call $\mathbb{Q}[\theta]$.

Integral basis An integral basis for the ring of integers in $\mathbb{Q}[\theta]$ is a subset of $\mathfrak{O}$ consisting of integers such that any integer in $\mathfrak{O}$ is uniquely expressible as a linear combination (over $\mathbb{Z}$) of the elements of the basis. Note that the $\mathbb{Q}$-basis $\{1, \theta, \theta^2, \theta^3\}$ (consisting of integers of $\mathbb{Q}[\theta]$) is not an integral basis because not all integers in $\mathbb{Q}[\theta]$ can be written as sums of powers of θ with integer coefficients, as we will see.

Some properties You are invited to explore the properties of $\mathbb{Q}[\theta]$. To start, find the minimum polynomial of θ and deduce that θ has four conjugates (including the trivial one). Now consider the following conjecture:

Conjecture $\mathbb{Q}[\theta]$ has integral basis

$$\left\{1, \theta, \frac{1}{2} + \frac{\theta}{2} + \frac{\theta^2}{4}, \frac{1}{2} + \frac{\theta}{4} + \frac{\theta^3}{8}\right\}$$

if $p \equiv 1 \pmod 4$, and

$$\left\{1, \theta, \frac{1}{2} + \frac{\theta}{2} + \frac{\theta^2}{4}, \frac{1}{2} + \frac{3\theta}{4} + \frac{\theta^3}{8}\right\}$$

if $p \equiv 3 \pmod 4$.

Prove this conjecture or find a counterexample. One way to proceed is to write out the monic polynomial satisfied by each listed basis element. Requiring that the coefficients of this polynomial be ordinary integers gives some conditions on the residue class of p (mod 4). This establishes that each basis element is an integer when p satisfies the conditions. You could try this out on a simpler number field such as $\mathbb{Q}[\sqrt{p}]$ to start with.

Further investigation You could explore the integers of this family of number fields, which offers some nice examples to work with. For example, is there an integral basis of the form $\{1, \phi, \phi^2, \phi^3\}$ (a monogenic basis) for some integer ϕ? What about number fields like $\mathbb{Q}[\sqrt{-p} + \sqrt{-(p+2)}]$?

References

S. Alaca and K. Williams, *Introductory Algebraic Number Theory*, Cambridge, 2004.
I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem* (3rd ed.), A. K. Peters, 2002.

Solution 285.6 – Two integrals

Show that

$$\int_0^1 \left(\frac{\cos(1/x)}{x} + (\sin x)(\log x)\right) dx \;=\; -\gamma$$

and that

$$\int_0^1 \left(2x\cos\frac{1}{x} - (\sin x)(\log x)\right) dx \;=\; \gamma + \cos 1 - \sin 1.$$

Recall that $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = \int_1^\infty\left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx = 0.57721566\ldots$.

Tommy Moorhouse

This problem can be solved using some classical results on the gamma function and its relatives. Our first step is to prove the

Lemma

$$I(x) \;\equiv\; \int_0^\infty t^{x-1}\sin t\, dt \;=\; \Gamma(x)\sin\frac{\pi x}{2}.$$

Proof The first step is to note that

$$\int_0^\infty e^{-at}a^{-x}da \;=\; \Gamma(1-x)t^{x-1}.$$

This is essentially a statement of the Laplace transform for powers of a variable. Now we use this expression for t^{x-1} and change the order of integration:

$$\begin{aligned}\int_0^\infty t^{x-1}\sin t\,dt &= \frac{1}{\Gamma(1-x)}\int_0^\infty\left(\int_0^\infty e^{-at}a^{-x}da\right)\sin t\,dt\\ &= \frac{1}{\Gamma(1-x)}\int_0^\infty da\;a^{-x}\left(\int_0^\infty e^{-at}\sin t\,dt\right)\\ &= \frac{1}{\Gamma(1-x)}\int_0^\infty da\frac{a^{-x}}{1+a^2}.\end{aligned}$$

The last line uses the Laplace transform for $\sin t$. Now substitute $u = a^2$ to get

$$I(x) = \frac{1}{\Gamma(1-x)}\int_0^\infty \frac{u^{-(1+x)/2}du}{2(1+u)} = \frac{\beta((1+x)/2,(1-x)/2)}{2\Gamma(1-x)}$$

where we have used the expression for the β function

$$\beta(y,x) = \int_0^\infty \frac{s^{x-1}ds}{(1+s)^{x+y}}.$$

Expanding the β function in terms of Γ functions we find

$$I(x) = \frac{\Gamma((1+x)/2)\Gamma((1-x)/2)}{2\Gamma(1-x)}.$$

This simplifies with the help of various Γ function identities, including

$$2\sqrt{\pi}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma\left(s+\frac{1}{2}\right), \qquad x\Gamma(x) = \Gamma(1+x)$$

and

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

to give the stated result.

Next consider

$$\frac{d}{dx}\left(\int_0^\infty t^{x-1}\sin t\,dt\right)_{x=1} = \int_0^\infty \log t\sin t\,dt.$$

Using the lemma above we easily see that this is $-\Gamma'(1) = -\gamma$.

Now we are ready to tackle the first integral set out in the problem:

$$J \;\equiv\; \int_0^1 \left(\frac{\cos(1/x)}{x} + \sin x \log x \right) dx.$$

Substitute $y = 1/x$ in the first term and note that

$$\int_0^1 f(x)dx \;=\; \int_0^\infty f(x)dx - \int_1^\infty f(x)dx$$

to arrive at

$$J \;=\; \int_1^\infty \frac{d}{dx}(\cos x \log x)\; dx + \int_0^\infty \log x \sin x \; dx.$$

The first term is an exact differential and can be taken to vanish, while the second term is $I'(1) = -\gamma$.

For the second integral (call it K) we look at the term

$$\int_0^1 2x \cos(1/x) dx.$$

Making the substitution $y = 1/x$ and relabelling we find by integration by parts:

$$\begin{aligned}
K \;&=\; \int_1^\infty \left(\frac{2}{x^3} \cos x + \sin x \log x \right) dx - \int_0^\infty \sin x \log x \, dx \\
&=\; \left[\frac{-1}{x^2} \cos x \right]_1^\infty - \int_1^\infty \frac{1}{x^2} \sin x \, dx + \int_1^\infty \sin x \log x \, dx - \int_0^\infty \sin x \log x \, dx \\
&=\; \left[\frac{-1}{x^2} \cos x \right]_1^\infty + \left[\frac{1}{x} \sin x \right]_1^\infty - \int_1^\infty \frac{1}{x} \cos x \, dx + \int_1^\infty \sin x \log x \, dx \\
&\quad - \int_0^\infty \sin x \log x \, dx \\
&=\; \cos 1 - \sin 1 + \gamma
\end{aligned}$$

using the previous result.

Problem 289.4 – Squares

Find all solutions in positive integers d and n of

$$n^2 \;\equiv\; n \pmod{10^d}.$$

Or, if you prefer, find all numbers which are the last d digits of their squares.

Things you can't buy in shops – II

Following on from the list in M500 **278**, here are a few more useful, everyday items you might consider enquiring about when your peaceful browsing in a shop is interrupted by an enthusiastic salesperson.

1. A door chime that can be heard above a noisy vacuum cleaner.
2. An anticlockwise corkscrew.
3. Fireworks where the explosive yield of each item is clearly indicated in microtons. Here, one is reminded that food items must be labelled to indicate their energy value in kilocalories. Curiously, the units are identical; 1 microton = 1 kilocalorie. The microton was originally intended to represent the energy yield of 0.000001 tons of TNT and is now standardized to 1000 calories. The actual yield of 1 gram of TNT can vary from about $4184 - 1500$ to about $4184 + 2500$ joules, depending on how it is detonated. The (thermochemical) calorie, or nanoton, is exactly 4.184 joules. Although we do of course have the larger multiples kiloton and megaton for grading nuclear bombs, I am not aware of the unscaled unit ton ever being used in this way.
4. A toaster which pops up as soon as the bread is actually toasted rather than after a number of seconds that has to be set on each and every occasion by an expert authority on bread and the charring thereof.
5. Leakproof batteries that do not leak.
6. An upright vacuum cleaner that is effective all the way to the wall.
7. A compact umbrella that will survive intact after being turned inside-out by a strong wind.
8. A pocket scientific calculator that supports the mod function (in addition to having at least 12 digits precision; cf M500 **278**).

Just as someone said that the greatest insult to the scientist by the computer programming language C is the lack of a to-the-power-of operator, I (TF) claim that the greatest insult to the number theorist by the commonly available scientific calculator, whether or not it has at least 12 digits precision, is the lack of the ability to do integer calculations modulo something.

Problem 289.5 – Cubic coefficients

A cubic $x^3 + ax^2 + bx + c$ has a double root at $x = \alpha$ and another root at $x = \beta$. Show that the coefficients a, b and c are all real if and only if both α and β are real. Or find a counter-example.

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04.01

Problem 289.6 – 52 Cards

I take a standard deck of 52 playing cards, randomized and placed on the table face down. One by one I turn over the cards. Before a card is turned over I invite you to guess what it is. Assuming you play intelligently, how many do you expect to get right?

The game is played again, but this time you are to guess only the rank of the card. The suit is irrelevant. Again, how many do you expect to get right?

Front cover 125 coloured cubes to make any one of 5 monochromatic $5 \times 5 \times 5$ cubes according to the specification in the table for $n = 5$ on page 2 of Chris Pile's article. (It works better when viewed in colour.)