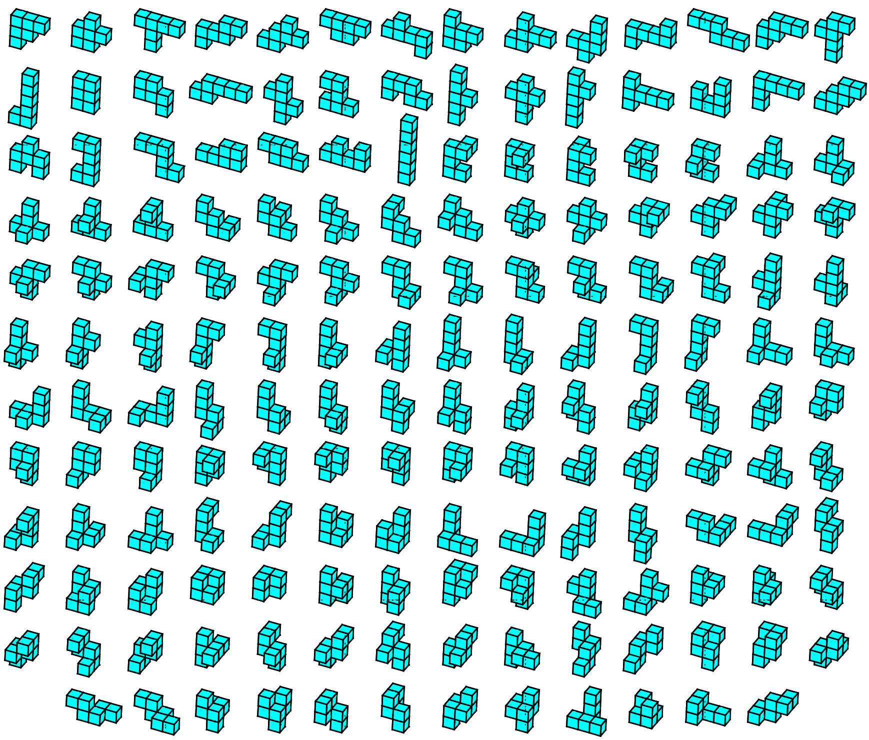
*

ISSN 1350-8539

# M500 290



# The M500 Society and Officers

**The M500 Society** is a mathematical society for students, staff and friends of the Open University. By publishing M500 and by organizing residential weekends, the Society aims to promote a better understanding of mathematics, its applications and its teaching. Web address: `m500.org.uk`.

**The magazine** M500 is published by the M500 Society six times a year. It provides a forum for its readers' mathematical interests. Neither the editors nor the Open University necessarily agree with the contents.

**The Revision Weekend** is a residential Friday to Sunday event providing revision and examination preparation for both undergraduate and postgraduate students. For details, please go to the Society's web site.

**The Winter Weekend** is a residential Friday to Sunday event held each January for mathematical recreation. For details, please go to the Society's web site.

---



---

**Advice to authors** We welcome contributions to M500 on virtually anything related to mathematics and at any level from trivia to serious research. Please send material for publication to the Editor, above. We prefer an informal style and we usually edit articles for clarity and mathematical presentation. For more information, go to `m500.org.uk/magazine/` from where a LaTeX template may be downloaded.

---

**M500 Mathematics Revision Weekend 2020** The M500 Revision Weekend 2020 will be held at

**Kents Hill Park Training and Conference Centre,**
**Milton Keynes MK7 6BZ**

**from Friday 15th to Sunday 17th May 2020.**

We expect to offer tutorials for most undergraduate and postgraduate mathematics Open University modules, subject to the availability of tutors and sufficient applications. Application forms will be sent via email to all members who supplied an email address. Contact the Revision Weekend Organizer, **Judith Furner**, at weekend@m500.org.uk if you have any queries about this event.

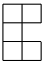
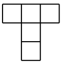
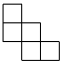
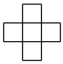
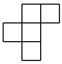
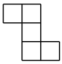
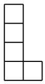
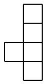
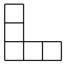
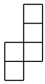
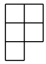
# Hexacubes

## Chris Pile

There are 166 ways of arranging six cubes in face-to-face contact to form a 'hexacube'. The hexacubes can be grouped according to the number of cubes that lie in a plane.

**(i) 6 cubes in a plane** There are 35 hexacubes with six coplanar cubes. Ten of these are symmetrical with respect to their mirror image. The other 25 have a different mirror image (or if turned over on to their opposite faces).

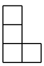
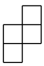
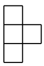
**(ii) 5 cubes in a plane** There are 72 hexacubes derived from the planar pentacubes by adding one cube above or below the plane of the other five. The resulting hexacube is either symmetrical or one of a mirror-image (chiral) pair.

| Pentacube |  |  |  |  |  |  |  |  |  |  |  | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Symmetrical | 1 | 3 | 1 | 2 | - | - | 1 | 1 | 1 | - | - | 10 |
| Chiral pairs | 2 | 1 | 2 | - | 5 | 3 | 2 | 4* | 2 | 5 | 5 | 62 |
| Total | 5 | 5 | 5 | 2 | 10 | 6 | 5 | 9 | 5 | 10 | 10 | 72 |

(* Some hexacubes have two pentacube planes.)

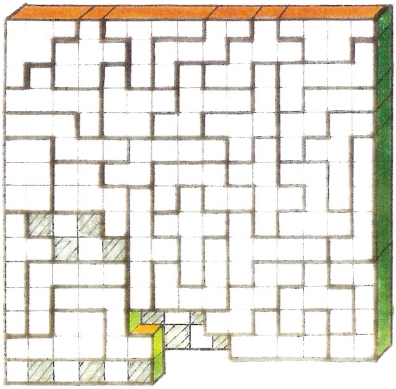
**(iii) 4 cubes in a plane** There are 54 hexacubes derived from the planar tetracubes by adding two cubes above, below or on either side of the plane of the other four. Again, the resulting hexacube is either symmetrical or one of a chiral pair.

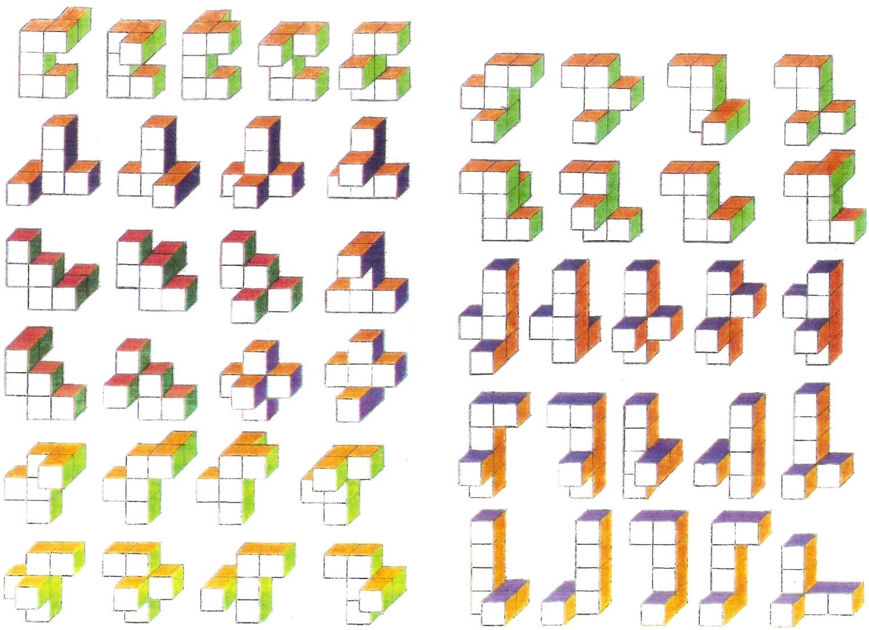
**(iv) 3 cubes in a plane** There are 5 hexacubes derived from adding three cubes to a tricube which is not in the same plane.

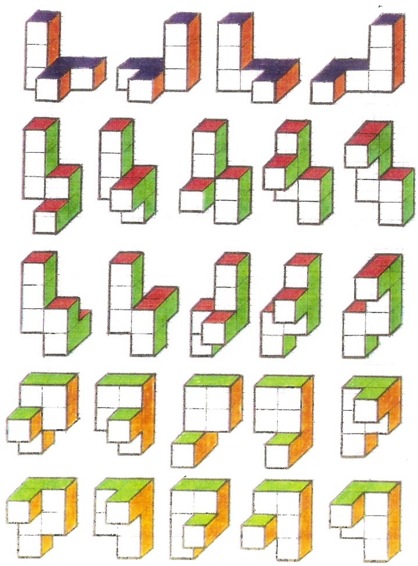
| Tetracube |  |  |  |  | Total | Tricube |  | Total |
|---|---|---|---|---|---|---|---|---|
| Symmetrical | - | 3 | 5 | 4 | 12 | | 1 | 1 |
| Chiral pairs | 9 | 8 | 3 | 1 | 42 | | 2 | 4 |
| Total | 18 | 19 | 11 | 6 | 54 | | 5 | 5 |

The 35 planar hexacubes cannot be arranged to cover any rectangle of 210 squares. If the rectangle is coloured alternate black and white, as on a chessboard, there will be 105 black and 105 white squares. If the same pattern is applied to the 35 planar hexacubes, regardless of which way up they are placed, there will always be an even number of black and white squares, for example, 106 black, 104 white. In fact, 24 of the hexacubes will be coloured 3 black, 3 white, giving 72 black, 72 white, and the other 11 will be 4 black, 2 white or 2 black, 4 white. So it is not possible to cover a board with 105 black, 105 white squares.

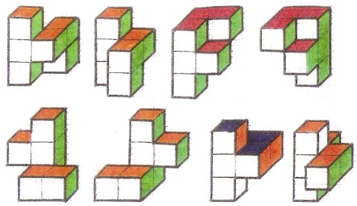
I do not have a set of hexacubes (yet!). To make such a set would require at least $41' \, 6''$ of half-inch square timber!

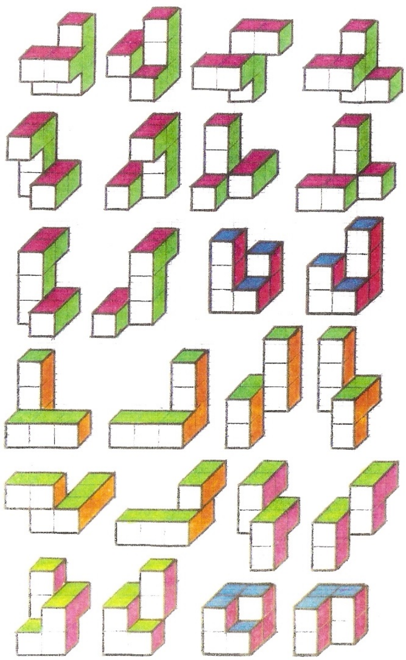


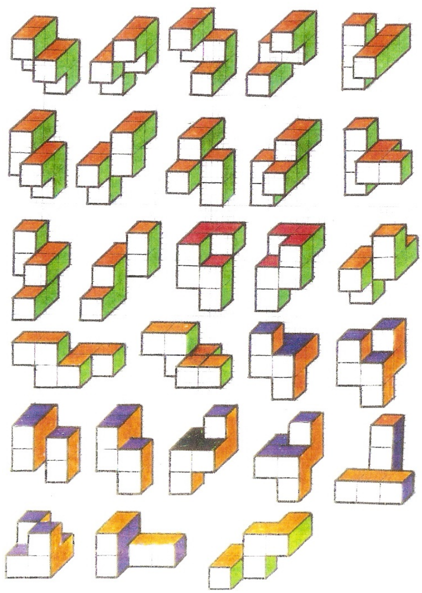




The two blocks at the bottom of the previous page and the block on the left contain the 72 hexacubes generated from the 11 coplanar pentacubes. In the bottom-left block on this page the first three in the last row and the last two in the row above are the 5 hexacubes with a 3-cube base. The 54 remaining hexacubes on this page have a tetracube base.







# Solution 285.1 – Roots

Suppose $r > 0$ and, for positive integer $m$, define

$$F(m,r,x) \;=\; \frac{\displaystyle\sum_{i=0}^{m/2} \binom{m}{2i} r^i x^{m-2i}}{\displaystyle\sum_{i=0}^{m/2} \binom{m}{2i+1} r^i x^{m-2i-1}}.$$

Show that

$$F(m,r,1) \;\to\; \sqrt{r} \quad \text{as} \quad m \;\to\; \infty.$$

For example, $F(20,5,1) = 2.2360679970\ldots$, giving $\sqrt{5}$ to 7 decimal places. Moreover, if you then compute $F(20,5,F(20,5,1))$, you get an even better approximation, 2.23606 79774 99789 69640 91736 68731 27623 54406 18359 61152 57242 70897 24541 05209 25637 80489 94144 14408 37878 22749 69508 17615 07737 83504 25326 77244 47073 86358 63601 21533 45270 88667 78173 . . . , correct to about 165 places.

## Tommy Moorhouse

**A useful expansion** The first step in my solution is actually a little puzzle. Given $t = \tanh u$ with $u > 0$, solve for $e^u$ or otherwise show that

$$\coth \beta u \;=\; \frac{(1+t)^\beta + (1-t)^\beta}{(1+t)^\beta - (1-t)^\beta}.$$

Deduce that if $\beta = m$ with $m$ an integer

$$\coth mu \;=\; \frac{\displaystyle\sum_{k=0}^{m/2} \binom{m}{2k} t^{2k}}{\displaystyle\sum_{k=0}^{m/2} \binom{m}{2k+1} t^{2k+1}}.$$

**Another expression for** $F$ Now taking $t = x/\sqrt{r}$ (or if $x > r$ taking $t = \sqrt{r}/x$ remembering that $t \in [-1,1]$) we can see, by multiplying and canceling terms in the numerator and denominator, that

$$F(m,r,x) \;=\; \sqrt{r}\coth mu.$$

Now, $u > 0$ so as $m \to \infty$ we have $\coth mu \to 1$ and so $F(m,r,x) \to \sqrt{r}$ in this limit, with $x$ fixed. With $x < \sqrt{r}$ we have $\tanh u = x/\sqrt{r}$ and intuitively we see that taking $x$ very close to $\sqrt{r}$ gives $u \gg 0$ and therefore takes us closer to the limit

$$F(m,r,\sqrt{r}) \;=\; \sqrt{r},$$

which holds for all $m > 0$. This means that iterating using successive approximations to $\sqrt{r}$ as the $x$ argument to $F$ improves the accuracy of the result.

---

# Solution 285.4 – Prime power divisors

> Given a number $d$, show that every sufficiently large integer has a prime power divisor $q \geq d$. For example, if $d = 6$, the only integers that fail are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60.

## Tommy Moorhouse

We have to show that, given a real number $d$, any sufficiently large integer has a prime power divisor $q \geq d$.

Given $d$, then, consider the integer $N_d = 2^{\alpha(2)}3^{\alpha(3)}5^{\alpha(5)}\cdots p^{\alpha(p)}$, where the product is over all the primes less than $d$, with $p$ the largest such prime. The exponent $\alpha(k)$ is chosen such that $k^{\alpha(k)}$ is the largest power of $k$ less than $d$. We assert that any integer $M > N_d$ has a prime power divisor $q \geq d$.

Let $M > N_d$ have prime factors $p_1, p_2, \ldots, p_m$. Then $M = \prod_{p|M} p^{\beta(p)}$ for some exponents $\beta(p)$. If any of these prime powers is greater than or equal to $d$ then we are done. Suppose, then, that each of the $p^{\beta(p)}$ is less than $d$. Since the largest power of $p$ less than $d$ is $p^{\alpha(p)}$ we must have $\beta(p) \leq \alpha(p)$ for each $p$ dividing $M$. But then

$$\frac{M}{N_d} \;\leq\; \prod_{p|M} p^{\beta(p)-\alpha(p)} \;\leq\; 1$$

and $M \leq N_d$, a contradiction. Thus $M$ has at least one prime power divisor greater than $d$.

---

# Problem 290.1 – Prime powers

Suppose $\alpha$ is a number such that $p^\alpha$ is an integer for every prime $p$. Show that $\alpha$ must also be an integer, or find a counter-example.

---

# Solution 285.2 – Circulant graphs

The *circulant graph* $\mathrm{Ci}(n, S)$ has $n$ vertices labelled 0, 1, ..., $n-1$, and $S$ is a subset of $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. An edge exists between vertices $i$ and $j$ iff $i - j \equiv s \pmod{n}$ for some $s \in S$. For example, $\mathrm{Ci}(n, \{\})$ is the empty graph, $E_n$, $\mathrm{Ci}(n, \{1\})$ is the cycle graph, $C_n$, and $\mathrm{Ci}(n, \{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph, $K_n$. The elements of $S$ are usually called *step sizes*, or just *steps*. There are a few examples on the cover of M500 **285**.

Clearly, the number of possible pairwise non-isomorphic $\mathrm{Ci}(n, S)$ graphs is at most $2^{\lfloor n/2 \rfloor}$ and, moreover, this limit is actually attained when $n = 1$, 2, 3, 4, and 6. Not 5 because $\mathrm{Ci}(5, \{1\})$ is isomorphic to $\mathrm{Ci}(5, \{2\})$.

Prove that there are no other $n$ for which there exist $2^{\lfloor n/2 \rfloor}$ pairwise non-isomorphic $\mathrm{Ci}(n, S)$ graphs. Or find another one.

## Tommy Moorhouse

**An algebraic approach** This problem, although phrased in terms of graphs, can be expressed algebraically using the vertex labels. We note that the labeling integers $\{0, 1, 2, \ldots, n-1\}$ are permuted when multiplied by an integer relatively prime to $n$. The proof is straightforward: suppose that $(k, n) = 1$ and that $ka \equiv kb \bmod n$. Then $k(a - b) \equiv 0 \bmod n$ and since $a - b \not\equiv 0 \bmod n$ we must have $k \equiv 0 \bmod n$, a contradiction. Thus each label maps to a distinct label in the set. No label other than 0 maps to itself unless $k \equiv 1 \bmod n$ (easy proof). In terms of the graphs the multiplication-by-$k$ map reorders the vertices, leaving the connecting edges intact.

Thus $\mathrm{Ci}(n, \{k\}) \simeq \mathrm{Ci}(n, \{1\})$ whenever $(k, n) = 1$. For example, when $n = 5$ we can represent $\mathrm{Ci}(5, \{1\})$ with arrows representing undirected edges as

$$0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 0.$$

Multiplying the vertex labels by 2 mod 5 gives

$$0 \leftrightarrow 2 \leftrightarrow 4 \leftrightarrow 1 \leftrightarrow 3 \leftrightarrow 0,$$

which is $\mathrm{Ci}(5, \{2\})$. The problem is solved if we can show that for any $n > 6$ there is an integer $k \leq \lfloor n/2 \rfloor$ satisfying $(k, n) = 1$.

**The algebra** We attempt a proof by contradiction. Fix $n$ and consider its prime factors $p_1, p_2, \ldots, p_r$. If these factors omit any primes less than $\lfloor n/2 \rfloor$

then we are done, for we can take $k$ to be such a prime. So let us take

$$n = 2^{m_1} 3^{m_2} 5^{m_3} \cdots p_s^{m_s} N,$$

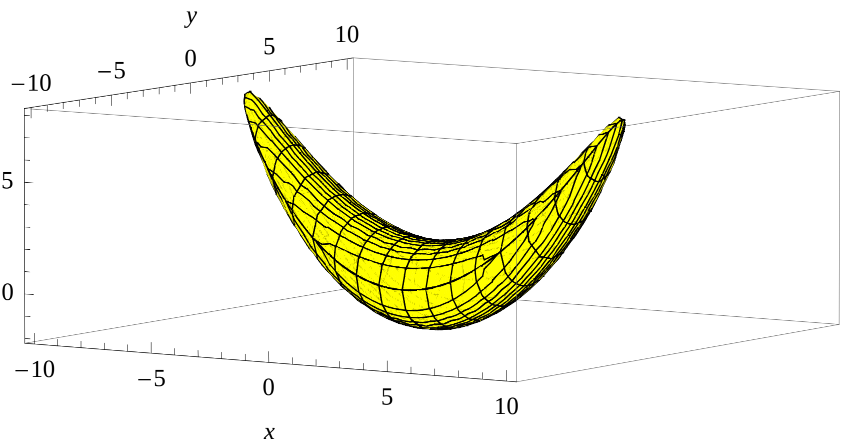
where $p_s$ is the largest prime less than or equal to $\lfloor n/2 \rfloor$ and the prime factors of $N$ are greater than $\lfloor n/2 \rfloor$. We also form the integer $\hat{n} = 2 \cdot 3 \cdot 5 \cdot \cdots \cdot p_s$. Then

$$\lfloor \hat{n}/2 \rfloor = 3 \cdot 5 \cdot 7 \cdot \cdots \cdot p_s \le \lfloor n/2 \rfloor.$$

Now we consider $k = 3 \cdot 5 \cdot 7 \cdot \cdots \cdot p_s - 2$. If $n > 6$ then $k$ is either a prime or is divisible by a prime less than $\hat{n}/2 - 2$ and not dividing $n$. But this contradicts our assertion that $n$ is divisible by every prime less than $\lfloor n/2 \rfloor$, and we conclude that the bound described in the problem is never saturated for $n > 6$.

## Problem 290.2 – Horn

This thing appeared on the front cover of M500 **21** ...




... accompanied by its defining formula,

$$64(z^2 + y^2) + x^4 - 16zx^2 + 4x^2 - 256 = 0.$$

It was also used for the cover of M500 **57** and again for M500 **173**. And now we have (at least) the fourth appearance in M500 of this interesting object. What is its volume?

# Solution 286.3 – Tritium oxide

Imagine you are foolish enough to take a bath in pure tritium oxide, $T_2O$. Would you be able to get out of the bath and survive? Whilst idly browsing online I (TF) was surprised to find several and diverse answers to this interesting question.

(i) No. The radioactivity of the tritium would kill you.
(ii) No. You would be cooked by boiling $T_2O$.
(iii) No. You would suffer chemical damage from radiolysis.
(iv) Yes. Get out and wash yourself with natural $H_2O$.

Which is correct? Assume the bath capacity is $2\,\mathrm{m}^3$, say $2\,\mathrm{m}$ long, $1\,\mathrm{m}$ wide and $1\,\mathrm{m}$ deep. Recall that tritium, hydrogen-3, is radioactive with half-life 12.3 years. A tritium nucleus beta-decays to helium-3 releasing 18.6 keV of which on average 5.7 keV is the kinetic energy of the electron. The rest of the energy is carried off by an electron antineutrino. As the He-3 nucleus is created in its ground state there is no gamma radiation.

## Colin Aldridge

The correct answers are

(i) The radioactivity would not kill you. At 5.7 keV the electron is not moving fast enough to get through the layer of dead skin so the radiation is harmless.

(ii) If you started with the bath at lukewarm you could perhaps enjoy it for about two minutes before you started to burn. The calculations which support this are below.

(iii) You would not suffer irreversible chemical damage due to radiolysis. The percentage of ionised water is determined by Ph of the water and any additional $OT^-$ and $T^+$ ions caused by radiolysis would immediately combine to form water until the equilibrium ionised ratio was reached.

(iv) You would be well advised to wash in pure water after the bath. Drying thoroughly with a towel would be just as effective but then you would have to wash the towel in the washing machine and then you would have a radioactive washing machine; so flushing $T_2O$ down the drains is probably best for you although it might well set off Geiger counters at the sewage farm.

The lightest atom is hydrogen. It consists of 1 proton and 1 electron and its atomic weight is about 1, which is about the weight of 1 proton. All other atoms are heavier, with oxygen being about 16 times as heavy. One of the largest numbers in the universe is $6.022 \times 10^{23}$. This is Avogadro's number and is the number of atoms of hydrogen it takes to weigh 1 gram. Avogadro, an early 19th century scientist, didn't actually calculate it but he did some ground-breaking work on gases which led to Avogadro's hypothesis, law, constant and later the number. Oxygen has an atomic weight of around 16 so water $H_2O$ has an atomic weight of around 18; 16 plus 1 for each hydrogen. So, if we have a gram of water, or 1 cubic centimetre, it will contain 1/18th of the number of molecules in Avogadro's number.

So, now you are in the bath, are you going to be badly burnt?

a. We need quite a few physical constants to work this out.

   i. The molecular weight of $T_2O$ is 22.
   ii. Density of $T_2O$ is 1.2 $\mathrm{g/cm^3}$.
   iii. Half-life of tritium is 12.3 years, or 389000000 seconds or so.
   iv. Energy of the electron released by the decay of tritium is 5.7 keV (kilo electron volts).
   v. $1\,\mathrm{eV} = 3.8293 \times 10^{-20}$ calories.
   vi. The specific heat of tritiated water is probably about 1, like normal water.
   vii. Avogadro's number, $N_{\mathrm{A}}$, is $6.022 \times 10^{23}$, the number of atoms in a gram molecule.

b. So, 1 cc of $T_2O$ weighs 1.2 g, which contains 1.2/(molecular weight of $T_2O$) $\times$ (Avogadro number) of molecules. This is about $3.28 \times 10^{22}$ molecules, which is $6.57 \times 10^{22}$ atoms of tritium.

c. Only 1 atom in $389000000/(\ln 2)$ decays every second assuming this tritium water is fresh! So, we have $6.57 \times 10^{22}(\ln 2)/(3.89 \times 10^{8})$ decays per second per cc, or about $1.17 \times 10^{14}$ decays per second and hence the same number of electrons to heat the water.

d. The energy produced is $5700 \times 3.8293 \times 10^{-20} = 2.18 \times 10^{-16}$ calories per electron, or 0.0256 calories per cc per second.

In summary then the bath proposed is totally safe provided you have a massive refrigeration system and a big pump. I was considering offering such a bath as a Winter Weekend prize but the price of pure $T_2O$ is around £100000.00 per cc and a bit out of range of the publicity budget for 1 cc let alone a bathful.

---

## Tony Forbes

There appear to be (at least) three distinct values for the specific gravity of $T_2O$: 1.2138, 1.215 and 1.85. The last one might be regarded as questionable; so let's go with the majority, say 1.21. Using the exact values for $e$, the electron charge, $1.602176634 \times 10^{-19}$ coulombs, and $N_{\mathrm{A}}$, Avogadro's constant, $6.02214076 \times 10^{23}$ mole$^{-1}$, from M500 **287**, p. 7, and recalling the stated values for the half-life of tritium, 12.32 (Julian) years, the mean decay energy of the beta particle, 5.7 keV, and the molecular weight of $\frac{1}{2}T_2O$, 11.0155 g/mol, we can easily compute the power generated from the beta particles emitted by decay of 2 cubic meters of $T_2O$:

$$\frac{2000000 \cdot 1.21}{11.0155} \cdot N_{\mathrm{A}} \cdot \frac{\ln 2}{12.32\ \mathrm{y}} \cdot 5700\, e \ \approx\ 215\,\mathrm{kW}. \tag{1}$$

This provides quite a lot of warmth in a normal bathroom environment.

But (1) does not represent everything. Recall that the total decay energy of a tritium nucleus is 18.6 keV, which is, on average, divided between the electron, 5.7 keV, and the antineutrino, 12.9 keV. Multiplying (1) by 12.9/5.7 gives 487 kW as the decay energy of the antineutrinos. And when an antineutrino interacts with you, it will probably change a proton to a neutron and a positron, which will certainly cause a lot of trouble if it happens too often.

Unfortunately I know very little about this subject. However, I am not going to let that minor deficiency interfere with what follows. So, until someone else provides a more convincing analysis, I shall now attempt to estimate the number of antineutrino interactions, albeit in a somewhat roundabout way.

In 1987 there was a supernova that created $10^{58}$ antineutrinos of which 11 were detected by 3000000 kg of water at the Kamioka Observatory of the Institute for Cosmic Ray Research, a neutrino and gravitational waves laboratory located deep underground in the Mozumi Mine, near the city of Hida, Japan. Therefore, assuming the detection rate was less than 100 percent, we would expect that a typical (male, 75 kg, say) human body would have interacted with at least $11 \cdot 75/3000000 = 0.000275$ of these particles.

Using the distance of SN 1987A, 168000 light years, we can compute its antineutrino flux at our hypothetical bathroom:

$$\frac{10^{58}}{4\pi(168000\text{ light years})^2} \;\approx\; 3.15\times 10^{14} \quad \text{antineutrinos per m}^2.$$

But the tritium in the bath is radiating antineutrinos at

$$\frac{2000000\cdot 1.21\,N_{\mathrm{A}}}{11.0155}\cdot\frac{\ln 2}{12.32\text{ y}}\cdot\frac{1}{2} \;\approx\; 1.18\times 10^{20} \quad \text{per second per m}^2,$$

assuming $2\,\mathrm{m}^2$ for the bath's cross-section area. So, by comparing with the supernova data, we conclude that (with a certain amount of arm-waving) at least

$$0.000275\times 1.18\times 10^{20}/(3.15\times 10^{14}) \;\approx\; 103$$

antineutrinos per second will interact with you. Hopefully your cell-repair mechanism can cope.

---

# Problem 290.3 – Votes

There are $p$ political parties, $P_1$, $P_2$, $\ldots$, $P_p$, and $s$ seats, $S_1$, $S_2$, $\ldots$, $S_s$, in a governing assembly of some kind. There is an election for which each party fields at least $s$ candidates.

To determine who gets seat $S_j$, $j = 1, 2, \ldots, s$, let

$$W_{i,j} \;=\; V_i/(N_{i,j}+1),$$

where $V_i$ is the total number of votes for $P_i$ and $N_{i,j}$ is the number of seats from $\{S_1, S_2, \ldots, S_{j-1}\}$ won by $P_i$, $i = 1, 2, \ldots, p$. Allocate $S_j$ to a candidate from the $P_i$ with the maximum $W_{i,j}$. (We leave the reader to decide how to resolve ties.)

You might recognize this as the *D'Hondt* highest-average method for proportional representation. For example, if there are 8 seats and 5 parties polling

$$(986977, 902534, 432084, 293217, 270242)$$

votes, the numbers of seats they get will be (3, 3, 1, 1, 0). Compare with

$$(2.7368, 2.50265, 1.19813, 0.813065, 0.749357).$$

Show that the seats are usually allocated to parties approximately in the same ratio as the votes.

---

# Solution 287.4 – Two games

The two-player game PickABead uses an even number $N$ of interlocking beads, numbered 1 to $N$. The first player puts the beads together in some order to form a necklace. The second player breaks the necklace in one place, then removes a bead from one end. The players then alternate removing beads from an end of their choice, until there are none left. The winning player is the one with the higher sum of bead numbers. What strategy can the second player use to ensure that the first player never wins?

A second game HighBead is identical to PickABead, except that each player must always remove the end bead with the higher number. What is the minimum $N$ ($M$, say) such that, by suitable construction of the necklace, the first player can always win. Is there a simple way of devising a winning construction for any $N \geq M$, ideally one that gives the highest possible winning margin?

## Roger Thompson

**PickABead**: The second player (B) breaks the necklace at some point, then adds up the values of the beads in odd positions, and those in even positions. In B's turn, there is always a bead which was originally in an even position at one end, and an odd one at the other. B can therefore always remove a bead that was in the winning set. For 9, 3, 10, 6, 2, 7, 11, 4, 12, 8, 1, 5, the odd numbered beads total 45, the even ones 33. Player B therefore removes 9, 10, 2, 11, 12, 1 (not necessarily in that order), with the first player (A) forced to pick from the losing set.

**HighBead**: $M = 12$. With the above example, B scores 37 $(9+3+6+7+4+8)$, and cannot score more by breaking at a different point. If B breaks optimally, no other arrangement allows A to win by more than 4 for $N = 12$.

I haven't succeeded in answering the other questions. The widest winning margin for $N = 14$ is 7 (56 to 49), for example 14, 8, 10, 9, 1, 5, 11, 3, 12, 6, 2, 7, 13, 4. This example also has the highest average winning margin over all 14 ways of breaking the necklace (415/7 to 320/7).

For $N = 16$, the best I've come up with so far is 75 to 61, for example 11, 9, 12, 3, 13, 8, 14, 10, 2, 6, 15, 4, 16, 7, 1, 5.

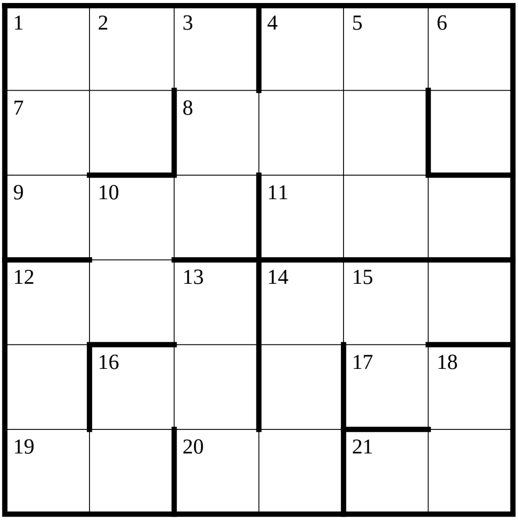
---

# IAN MURDOCH

Those who have been around a few years may remember Ian Murdoch. Sadly we have been informed recently of his death. He did not survive a heart operation. We at M500 are grateful for his friendship and help. Ian was a good companion. May he rest in peace.

# Crossnumber

Here is a crossnumber puzzle devised by **Michael Gregory** and which first appeared in M500 **10** (February 1974).




The solutions are 26 distinct values of the function $A(n)$ defined by the following algorithm.

(1) $n \leftarrow 1$; $y \leftarrow 3$; $A(1) \leftarrow 3$.

(2) If $y > 111$, then $y \leftarrow (y \bmod 111)$ and go to (5).

(3) If $y > 31$, then $y \leftarrow |2y - A(n-1) - A(n-2)|$ and go to (5).

(4) $y \leftarrow y^2 + n$.

(5) $n \leftarrow n + 1$; $A(n) \leftarrow y$.

(6) If $n > 35$, then STOP; otherwise go to (2).

We leave $A(n)$ undefined if $n$ is not in the set $\{1, 2, \ldots, 36\}$. As a check, you should get $A(1) = 3$, $A(18) = 25$ and $A(36) = 649$.

---

I (TF) ought to point out that in its original formulation, as printed in M500 **10**, there were also 13 across clues and 13 down clues corresponding to the little numbers in the diagram. However, these clues are actually unnecessary, and so they have been omitted.

# The acre and other units of land measure

## Jeremy Humphries

Here is some classy poetry combined with useful general knowledge.

*If the length of your field, stile to stile,*
*Is a furlong – one-eighth of a mile –*
*And the width, ditch to ditch,*
*Is a cricketing pitch,*
*That's an acre you've got – quite worthwhile.*

*Let's consider two big bovine bods,*
*Yoked up to a machine that turns sods.*
*Then the land that they say*
*They can plough in a day*
*Is one hundred and sixty square rods.*

The rod is five and a half yards, as is the pole and the perch. And the acre is traditionally defined as the area of land that a yoke of oxen can plough in a day. You remember at school when you had to learn the poem *Horatius* by Lord Macaulay. After Horatius, with his two companions, Herminius and Spurius Lartius, held the 90,000-strong Etruscan army at bay at the narrow Tiber bridge while the townsfolk demolished it, he was rewarded:

*They gave him of the corn-land,*
*That was of public right,*
*As much as two strong oxen*
*Could plough from morn till night;*

Now I don't know if that was a standard measure in Ancient Rome, but in Macaulay's terms it was an acre. And better than nothing, I suppose. But if you are Marlborough, and you give the French a kicking, they build Blenheim Palace for you, whereas if you are Horatius, and you save Rome, they give you a cornfield.

*You remember the railway, with porters,*
*And the Night Mail with all the mail sorters?*
*If an acre of mail*
*Was laid touching each rail,*
*It would stretch for a mile and three-quarters.*

The gauge of the standard track, as every schoolboy knows, is four feet eight and a half inches, or 56.5 inches. So it's a doddle to work out approximately how long an acre strip of that width would be. First, recall that an acre is a furlong by a chain, or one-eighth of a mile by 22 yards, which is 66 feet. So if an acre is a mile long then it's 66/8 feet wide, which is 8 feet 3 inches, or 99 inches. Now you spot immediately that three-quarters of 56.5 inches is about 42.4 inches, and that if you add those two you get 98.9 inches. That's near enough 99. So the acre strip that's as wide as the railway gauge is a mile and three-quarters long, give or take.

*Now the rod, a.k.a. perch or pole,*
*Plays a key mensurational role.*
*It is sixteen foot six*
*(About 22 bricks),*
*Or five yards and a half – bless my soul.*

The nominal length of a house brick plus mortar is nine inches.

*Here the rod, pole and perch are again,*
*Because four of them make up a chain,*
*And the chain used to serve*
*When they measured the curve*
*Of the rails that would carry the train.*

One use of the chain as a unit of measurement was for the radius of curvature of railway lines. The chain, 22 yards long and having 100 links of 7.92 inches, was introduced in 1620 as a surveying device by the mathematician and clergyman Edmund Gunter, for whom it is named. The acre is 10 square chains, or 100,000 square links.

*Now George Everest's task was not small –*
*Survey India north to Nepal.*
*His precision is put*
*At one-sixth of a foot*
*Using just Gunter's chain. B⋆gger all!*

I read somewhere that they did indeed survey the meridian arc of India from the southernmost tip, Cape Comorin, to Nepal, 1500 miles, using the 22-yard Gunter's chain, to an accuracy of 2 inches. They had to make compensation for temperature changes affecting the length of the chain as they went about their measuring and triangulating task, and if any building or tree inconveniently stood in their path then they simply knocked it down.

I have also read, to the detriment of my scansion there, that Sir George pronounced his name with only two syllables, in the way that Adam would invite his partner to take a break from picking fruit in the Garden of Eden.

# Letter

Dear Eddie,

Many thanks for M500 **287**. I had a think about 287.4 – Two games.

I suppose that the strategy for the first game must be for the second player to break the necklace in a place between two adjacent low numbers with high numbers immediately beyond them, looking for the greatest mean difference between low and high numbers available on either side of the break; and thereafter to pick the bead which has the higher number unless, in so doing, he exposes a higher number still to be picked by his opponent. The second game is hardly a game at all, since it's won instantly by the second player breaking the necklace at the most profitable place—evidently the same place as in the first game, after which there is no choice of bead.

Problem 287.3 – Group determinant seems to relate to the British car number plate system. To simplify, London issues number plates where the first letter is L, Birmingham uses B and so on. The first letter is followed by any letter other than I or Q (because I looks like 1, and Q is reserved for temporary plates; I think Z, banned under the old system because it meant the Republic of Ireland, is now allowed). So there are 24 letters available in second place, and they simply use these letters in alphabetical order. There follows a year number for which there is no choice—the actual two last digits of the year for January to June issues, and these plus 50 for July to December issues. Then there are any three letters other than I or Q, with repeats allowed, which my feeble powers tell me gives 13,824 combinations before it is necessary to go on to a new second letter. Thus every authority can issue 331,776 number plates every six months. The system dies in the second half of 2049, since the year number for that is 99, and a new arrangement has to be devised, probably a reversal of the first one with the three letters at the beginning.

I had a limerick competition by text with a friend of my nephew's, who is a poet of a modern kind and thus unable to write anything that scans or rhymes, so I beat him effortlessly. In the process I wrote quite a lot to extend the original list of surname pronunciation lyrics that appeared on the old Electric Editors web site—now defunct after hanging around for several decades. Shortly afterwards, someone publishing a book of limericks asked for contributions, so I marshalled all these things into a text file, which I attach in the hope of producing a few groans.

Best wishes,

**Ralph Hancock**

Here's one of the limericks, which also appeared in M500 **160**. See the book for the others. — TF

*A colonel by name Leveson–Gower*$^\dagger$
*Mixed gin, rum, stout, lime jeveson–mower*
*In a drink for Miss Featherstonehaugh,*$^\ddagger$
*Who said to him, 'Meatherstonehaugh,*
*This stuff might have some ueveson–wower.'*

# Problem 290.4 – Sudoku with 3 empty boxes

## Tony Forbes

Construct a sudoku puzzle with the pattern on the left, below, where only the positions marked X may be occupied by starter-digits; i.e. the three $3 \times 3$ boxes at top-left must be empty. Or prove that no such puzzle exists.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | X | X | X |
| | | | | | | X | X | X |
| | | | | | | X | X | X |
| | | | X | X | X | X | X | X |
| | | | X | X | X | X | X | X |
| | | | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X |
| X | X | X | X | X | X | X | X | X |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | 9 | | | | | 1 | 2 | 8 |
| | | | | | | 4 | 7 | 9 |
| | | | | | | 3 | | |
| | | | | | 9 | | 1 | 6 |
| | | | | 1 | | 7 | | 5 |
| | 2 | 9 | | 8 | 6 | | | |
| | 1 | 4 | 5 | | 7 | | | |
| | 3 | 8 | 4 | | | | | |

On the other hand, if one of the empty $3 \times 3$ boxes is not quite empty, then there do exist puzzles, such as the one on the right, above.

# Problem 290.5 – Zero-free factorization

Let $n$ be a positive integer and suppose there exist positive integers $a$ and $b$ such that $10^n = ab$. Suppose also that the base-10 representations of $a$ and $b$ contain no zero digits. Show that $n \in \{1, 2, 3, 4, 5, 6, 7, 9, 18, 33\}$, or find another. For example, when $n = 33$ we have

$$10^{33} = 8589934592 \times 116415321826934814453125.$$

$^\dagger$Pronounced Loosen–Gore $^\ddagger$Fan sure

04.00

# M500 Winter Weekend 2020

The **thirty-ninth M500 Society Winter Weekend** will be held at

**Kents Hill Park Training and Conference Centre,**
**Milton Keynes MK7 6BZ,**

**Friday 10th – Sunday 12th January 2020**.

The cost for the Weekend will be confirmed when booking opens at the end of September 2019. The cost includes accommodation and all meals from dinner on Friday to lunch on Sunday.

The Winter Weekend provides you with an opportunity to do some non-module-based, recreational maths with a friendly group of like-minded people. The relaxed and social approach delivers maths for fun. And as well as a complete programme of mathematical entertainments, on Saturday we will be running a pub quiz with Valuable Prizes.

**Front cover** 166 hexacubes; see Chris Pile's article, page 1.