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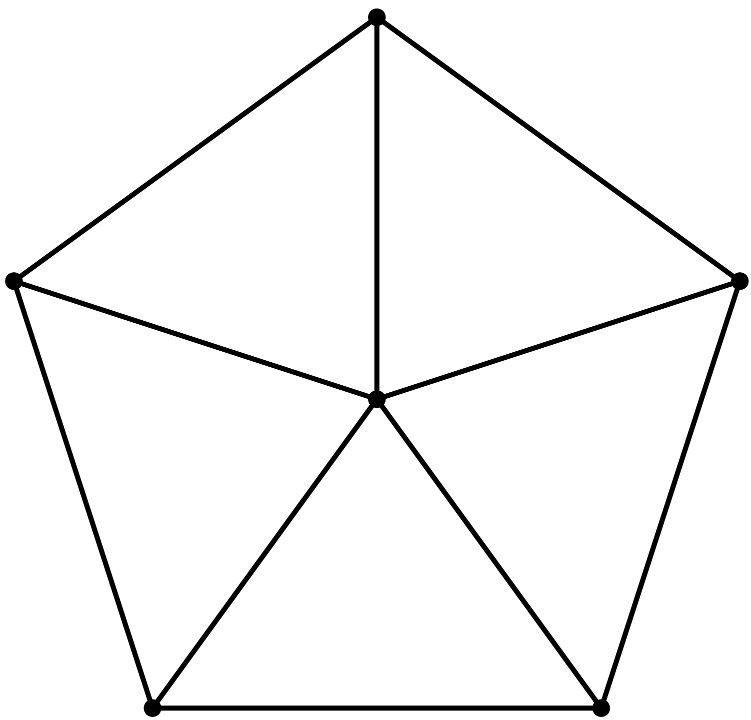
M500 291



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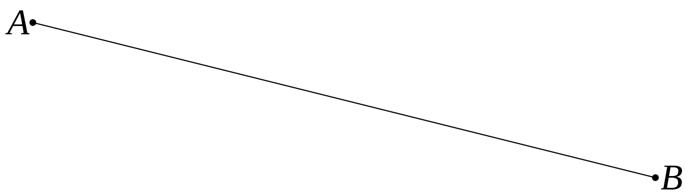
The Brachistochrone Problem

Sebastian Hayes

The so-called Brachistochrone Problem has perhaps attracted the attention of more great names in science and mathematics than any other mechanical problem. The term comes from two Greek words meaning 'shortest time' and the problem in modern terms may be posed thus:

> *What is the path of shortest time between two points A and B taken by a particle falling from rest under gravity?*

At first sight, this looks like a no-brainer—a straight line is obviously the path of shortest distance between two points and that is that. But, except in the trivial case when B lies directly below A, a straight line is by no means necessarily the path of shortest time. Since time = distance/speed we would also like to maximize the speed, for example by choosing a path with a large initial acceleration, i.e. a nearly vertical drop. The 'ideal' path is thus going to be the result of a compromise between these two requirements, i.e. a curved path that gives the particle a good initial velocity while not deviating *too* far from a straight line.



Galileo, the first person to address the problem, suggested that an arc of circle would be a better choice than a straight line which is indeed perfectly correct as timed experiments with actual chutes of different shapes demonstrate.

However, Jacob Bernoulli was convinced he could do much better than this and, in 1696, in an issue of the journal *Philosophical Transactions*, he challenged the mathematicians of the day to find the all-round curve of least time. The initial deadline was the end of the year but by then he had received only one answer—from Leibnitz. The latter persuaded Johann Bernoulli to extend the deadline to Easter 1697 and, when restating the problem, Bernoulli managed to make a sly dig at Leibnitz's great rival, Newton (without actually naming him)—for Johann Bernoulli was the great champion of Leibnitz in the so-called 'Calculus Wars' about priority. To make sure that Newton, who was now Master of the Mint, didn't miss the challenge, Bernoulli sent him a copy of the journal. The story goes that Newton came back at 4 in the afternoon from a hard day's work at the Mint, opened the package and stayed up until 4 in the morning by which time he had solved the problem.

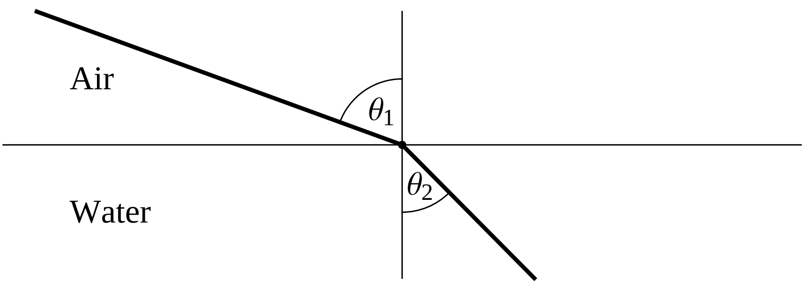
Come Easter and the journal gave the solution to the problem, namely the curve known as the *cycloid*, the path followed by a point on the circumference of a rolling wheel. The roll call of the successful puzzle solvers goes:

1. Johann Bernoulli, who posed it;
2. Leibnitz;
3. Jacob Bernoulli, Johann's brother (whom he detested);
4. the Marquis de l'Hôpital, author of the very first textbook on Calculus and known to students via 'l'Hôpital's Rule';
5. von Tschimhaus, inventor of the 'Tschimhaus Transformation';
6. an anonymous solution from England.

Regarding the last submission, Johann Bernoulli, not a generous man, allegedly said, "The lion is known by his footprint". A modern historian commented, "Newton was obviously getting old—in his youth he would have solved the problem by midnight".

So how did Johann Bernoulli reach his conclusion? Very neatly, if somewhat intuitively. Fermat had already shown that light, when moving from one medium to another (such as air to water), always follows the path of shortest time which is that dictated by Snell's Law of Refraction

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1}, \quad \text{or} \quad \frac{\sin\theta}{v} = \text{constant}.$$



Note that the angle we are interested in is the angle the path of light makes with the vertical.

Johann Bernoulli's reasoning seems to have been something along the following lines. A particle descending under gravity traverses successive layers of space and may be considered to have a sort of 'spatial index of refraction' which governs its speed, this speed depending on the vertical distance from the highest point. Thus, by analogy with the behaviour of light, an equivalent rule whereby $(\sin\theta)/v$ remained constant should apply to descent under gravity and, taking the limit as the spatial layers become smaller and smaller, this feature should be implicit in the formula for a continuous curve.

Although Newton's Laws of Motion entail the conservation of *momentum*, the principle of the conservation of *energy* did not enter physics as such until very much later (Note 1). However, we can see Johann Bernoulli groping towards an early enunciation of the principle. Today, we would argue like this. If we make A's initial position $(0, h)$ and B's $(L, 0)$, then d, the particle's distance from the highest point at subsequent times will be given by $h - y$.

If the particle falls from rest

$$\text{K.E.} + \text{P.E.} \;=\; 0 + mgh.$$

At any later time, when the height above the origin is y,

$$\text{K.E.} + \text{P.E.} \;=\; \tfrac{1}{2}mv^2 + mgy.$$

Assuming no dissipation from heat, we equate the two formulae, giving

$$0 + mgh \;=\; \tfrac{1}{2}mv^2 + mgy, \quad \text{or} \quad v \;=\; \sqrt{2g(h-y)}.$$

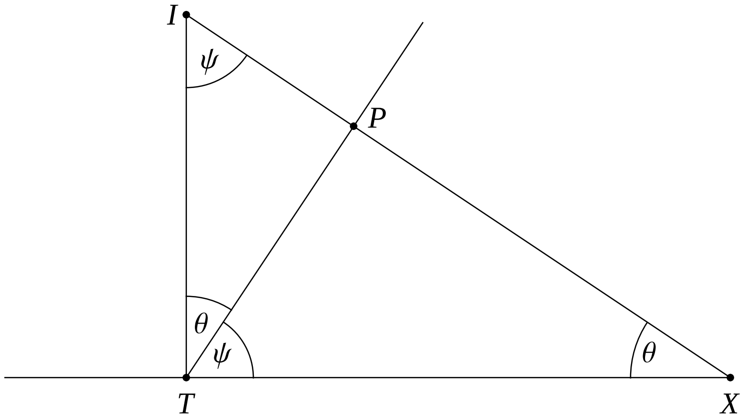
Since $2g$ is a constant, the velocity thus varies with the square root of d, the distance fallen,

$$v \;\propto\; \sqrt{d} \;=\; \sqrt{h-y}.$$

Bernoulli concluded that the required curve for a particle falling under gravity should vary directly with $\sin\theta$ and inversely with $\sqrt{d}$, i.e. $(\sin\theta)/\sqrt{d} = C$. Was there such a curve known to mathematics? To Johann Bernoulli's delight, it transpired that there was—the curve known as the cycloid that Galileo, Pascal and others had studied.

It is by no means obvious that the cycloid does have the desired property, and to show that this is the case we first of all need the following Lemma.

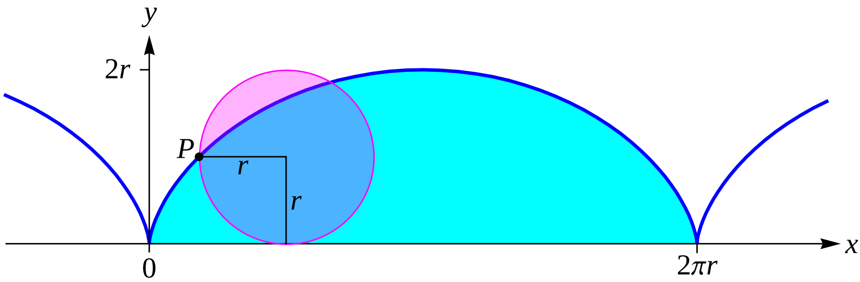
Lemma *If the tangent IPX to a curve at a point P makes an acute angle θ with a base line, touching it at X, then the normal through P makes the same angle with the base as the tangent makes with the perpendicular from I.*



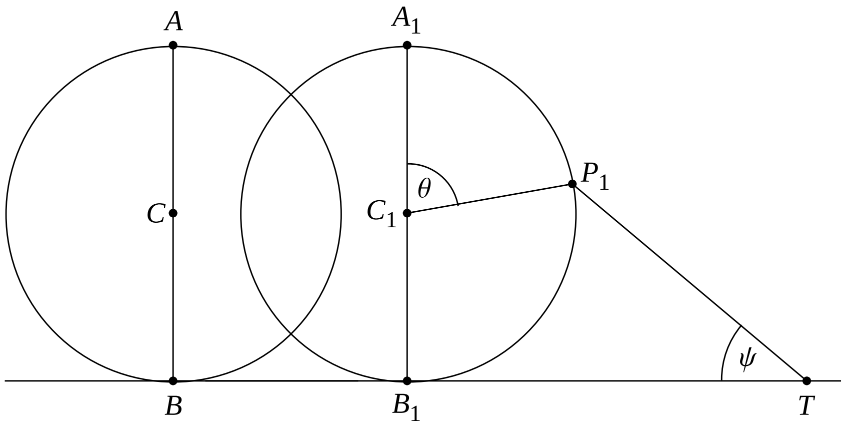
Proof Angle TPX is a right-angle since PT is the normal to the tangent at P. So, in $\triangle PTX$, $\angle PTX + \angle PXT = \pi/2$. So $\psi + \theta = \pi/2$. But, if IT is the perpendicular from I, $\triangle ITX$ is right-angled. Thus $\angle TIX = \angle PTX$.

Corollary If IT is the diameter of a circle passing through P, then the angle subtended at the centre by the arc PT is 2ψ (angle at centre = 2 (angle at circumference)), and by the arc IP is 2θ.

Now, the cycloid is by definition the curve traced out by a point on the circumference of a circle that rolls, without slipping, along a straight line—I suggest the reader consults *Wikipedia* or a similar site to see a mobile representation. The cycloid is thus composed of a succession of arches, one for each complete revolution of the rolling circle, where the horizontal distance between the cusps is $2\pi r$. Normally, we assume a circle, such as the wheel of a carriage, rolling *above* a plane surface and moving from left to right. For our purposes, however, we require the circle to roll *underneath* a straight line. It still moves from left to right but the angle θ through which a radius vector turns now has the opposite sense, i.e. turns anti-clockwise instead of clockwise. This feature explains the minus sign in the Cartesian formula for the cycloid when it is 'upside down'.



As we are concerned only with the 'absolute' value of angles and line segments, we can derive the main result using the more customary 'above-the-line' cycloid. The maximum height of the cycloid curve occurs when P, the point that traces the curve, coincides with the top of the vertical diameter of the rolling circle, i.e. is at $(0, 2R)$ if we make the bottom end the origin. We compare this with a snapshot of P when the generating circle has turned through θ radians.



Thus AB = diameter $= 2R$, and P_1T is the tangent to the *cycloid* at P.

The point P of the cycloid is originally coincident with A at its maximum height $(0, 2R)$. When the rolling wheel has turned through θ radians, the tracing point has traversed the arc A_1P_1 of length $R\theta$. This is also the distance the entire wheel has moved to the right (since there is a one–one correspondence between points on the wheel perimeter and the ground covered). Thus the distance $AA_1 = BB' = CC' = R\theta$.

But P has *also* moved horizontally by an amount $R\sin\theta$. Thus, the x

coordinate of P_1 is $R\theta + R\sin\theta$. And P's y coordinate, originally set at $2R$, is now $R + R\cos\theta$. Thus,

$$P_1 \;=\; (R(\theta + \sin\theta),\; R(1 + \cos\theta)).$$

Now, suppose a tangent is drawn to the cycloid (not shown in diagram) at point P_1 striking the baseline at T and making an acute angle $\psi = P_1TB_1$. Then, $\tan\psi = dy/dx$, where x, y are the cycloidal coordinates at P_1. Differentiating with respect to θ,

$$\tan\psi \;=\; \frac{dy}{dx} \;=\; \frac{dy/d\theta}{dx/d\theta} \;=\; \frac{-R\sin\theta}{R + R\cos\theta} \;=\; \frac{-\sin\theta}{1+\cos\theta}.$$

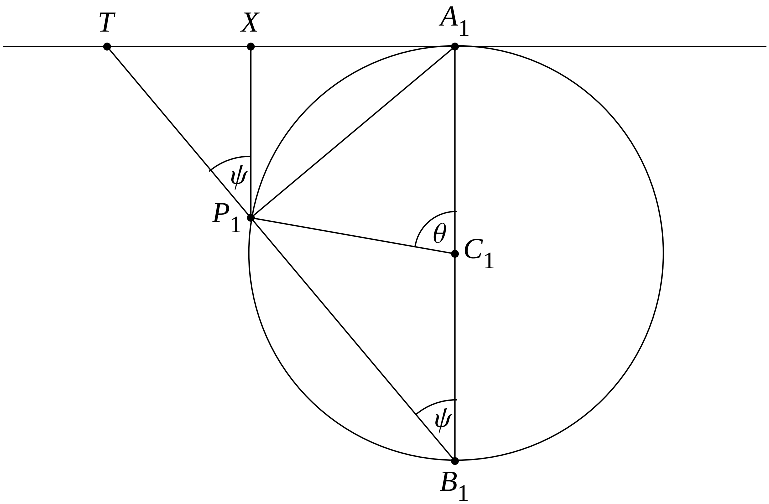
Since

$$\sin\theta \;=\; 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \quad\text{and}\quad 1+\cos\theta \;=\; 2\cos^2\frac{\theta}{2}$$

$\tan\psi$ reduces to $-\tan\frac{1}{2}\theta$, where the minus sign simply means that the angle is pointing back towards the origin. We are only interested in the 'absolute value' of the angle, so it transpires that $\psi = \theta/2$, or $\theta = 2\psi$. The conditions of the Lemma are thus met.

What this means is that the tangent to the cycloid at any point P passes through one end of the vertical diameter of the rolling circle and that the normal at P passes through the other end. This is a most remarkable property—but why is it relevant?

Since the property carries over to its mirror image, we can now pass to the examination of the 'upside-down' cycloid.



Here, TP_1B_1 is the tangent to the cycloid at P_1, and P_1A_1 is the normal. If $\angle P_1C_1A_1 = \theta$,

$$\angle P_1B_1A_1 \;=\; \angle XA_1P_1 = \psi = \theta/2.$$

Also, since $\angle P_1XT$ is a right angle, $\angle XP_1T = \psi = \theta/2$. Moreover, XP_1 is the perpendicular from TA_1 to P_1 and so, if P was originally coincident with the top of the vertical diameter, $XP_1 = d$, the distance fallen from rest—or $|y|$ if A is set at $(0, 0)$.

Now, in $\triangle A_1P_1B_1$, $A_1P_1 = 2R\sin\psi$. In $\triangle XP_1A_1$,

$$XP_1 \;=\; A_1P_1\sin(\angle XA_1P_1) \;=\; 2R\sin\psi\sin\psi \;=\; 2R\sin^2\psi.$$

Thus $d/(\sin^2\psi) = 2R$, or $(\sin\psi)/\sqrt{d} = 1/\sqrt{2R} =$ constant.

This is precisely the relation that was desired since the angle ψ is the angle between the tangent (representing the instantaneous trajectory of the particle) and the vertical analogous to the angle light makes when moving from one medium to another. This seems to have been the crux of Bernoulli's highly imaginative solution (Note 2), though one would need to show that the cycloid is the only curve with the desired property, and that this property is sufficient.

The Brachistochrone Problem was one of a number of 'extremal' problems that eventually gave rise to what we now know as the Variational Calculus. It can be solved algebraically by applying the Euler–Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\,\frac{\partial F}{\partial y'} \;=\; 0;$$

the somewhat tedious details are provided in 'The Brachistochrone Integral', page 8.

Note 1 "It was [Thomson] who introduced the terms 'energy' and 'thermodynamics' into physics (in 1851) and then expanded 'energy' to cover all applications, not merely the interaction of heat and work." Jennifer Coopersmith, *Energy, the Subtle Concept*, p. 300.

Note 2. This updated version of Bernoulli's argument is based on a brief paper by Mark Levy, the author of *The Mathematical Mechanick*; the paper (with diagram) is given at the end of Steven Strogatz's illuminating discussion of the Brachistochrone Problem on the YouTube channel *3Blue1Brown*.

The Brachistochrone Integral

Sebastian Hayes

Making $A = (0, h)$ and $B = (L, 0)$ we wish to find the path of shortest time for a particle sliding without friction from A to B under the influence of gravity. In effect we require the path integral $S = \int ds$ that makes the time integral $T = \int dt$ a minimum. This is a problem of the Calculus of Variations and the answer is obtained by solving the Euler–Lagrange Equation

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad (1)$$

for the appropriate functional $F(x, y, y')$.

Taking ds as the basic element of distance we have by Pythagoras $ds^2 = dx^2 + dy^2$; so

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

As for v, it will (because of gravity) vary according to the particle's instantaneous vertical distance below A. However, if we situate B on the x axis and make A's initial position $(0, h)$, we obtain positive (diminishing) values for the particle's height at all subsequent times. If the particle starts by being at rest at A, from then onwards its height, henceforth given by the y coordinate of its position, will decrease as it falls. We obtain the speed from the principle of the conservation of energy. When $t = 0$, $v = 0$; so

$$\text{K.E.} + \text{P.E.} = 0 + mgh.$$

At any later time, when the height is y, with $0 \le y < h$,

$$\text{K.E.} + \text{P.E.} = \tfrac{1}{2}mv^2 + mgy.$$

Assuming no dissipation from heat, we can equate the two formulae, giving

$$0 + mgh = \tfrac{1}{2}mv^2 + mgy, \quad \text{or} \quad v = \sqrt{2g(h - y)}.$$

(Note that $h = h_{\text{max}} = \text{constant}$.) This allows us to determine the functional, or 'function of functions', $F(x, y, y')$, where $y' = dy/dt = v$ and F is the time integral = distance/speed which is to be minimized, i.e.

$$\int_0^L \frac{\sqrt{1 + (y')^2}}{\sqrt{2g(h - y)}}\, dx.$$

The Euler–Lagrange Equation (1) reduces to the more tractable Beltrami Identity,

$$F - y' \frac{\partial F}{\partial y'} \;=\; C \tag{2}$$

when, as here, x does not appear explicitly in $F(x, y, y')$.

First, we find the partial derivative of F with respect to y':

$$\frac{\partial F}{\partial y'} \;=\; \partial\left(\frac{\sqrt{1+(y')^2}}{\sqrt{2g(h-y)}}\right)\Big/\partial y' \;=\; \frac{y'}{\sqrt{2g(h-y)(1+(y')^2)}}$$

giving, when plugged into (2),

$$\frac{\sqrt{1+(y')^2}}{\sqrt{2g(h-y)}} - \frac{(y')^2}{\sqrt{2g(h-y)(1+(y')^2)}} \;=\; C.$$

We take $\sqrt{2g(h-y)}$ over to the other side and multiply both sides by $\sqrt{1=(y')^2}$ yielding

$$\begin{aligned} 1+(y')^2-(y')^2 \;&=\; C\sqrt{2g(h-y)}\,\sqrt{1+(y')^2}, \\ \frac{1}{C\sqrt{2g}} \;&=\; \sqrt{h-y}\,\sqrt{1+(y')^2}. \end{aligned}$$

We absorb the LHS into a new constant $\sqrt{K}$ and square both sides, giving

$$\begin{aligned} K \;&=\; (h-y)(1+(y')^2) \;=\; h-y+(h-y)(y')^2, \\ (y')^2 \;&=\; \frac{K-h+y}{h-y}, \quad y' \;=\; \frac{dy}{dx} \;=\; \frac{\sqrt{K-h+y}}{\sqrt{h-y}}. \end{aligned}$$

The RHS is a function in y only and, separating the variables, we obtain

$$\frac{\sqrt{h-y}}{\sqrt{K-h+y}}\,\frac{dy}{dx} \;=\; 1.$$

Integrating both sides with respect to x gives

$$\int \frac{\sqrt{h-y}}{\sqrt{K-h+y}}\,dy \;=\; x.$$

The LHS can be integrated using the substitution

$$y \;=\; h - K\sin^2\frac{\theta}{2}, \quad \frac{dy}{d\theta} \;=\; -K\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Plugging this into the LHS we obtain

$$
\begin{aligned}
x &= \int \frac{\sqrt{K \sin^2 \frac{1}{2}\theta}}{\sqrt{K - K \sin^2 \frac{1}{2}\theta}} \left(-K \sin \tfrac{1}{2}\theta \ \cos \tfrac{1}{2}\theta\right) d\theta \\
&= -K \int \frac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} \ \sin \tfrac{1}{2}\theta \ \cos \tfrac{1}{2}\theta \ d\theta \ = \ -K \int \sin^2 \tfrac{1}{2}\theta \ d\theta \\
&= -\frac{K}{2} \int (1 - \cos\theta) \ d\theta \ = \ -\frac{K}{2}(\theta - \sin\theta) + K_2.
\end{aligned}
$$

At $x = 0$, $\theta = 0$; so K_2 must be 0. We now have two parametric equations for x and y:

$$
\begin{aligned}
x &= -\tfrac{1}{2}K(\theta - \sin\theta), \\
y &= h - K \sin^2 \tfrac{1}{2}\theta \ = \ -\tfrac{1}{2}K(1 - \cos\theta) + h.
\end{aligned}
$$

The unknown constant $\frac{1}{2}K$, to be determined from the boundary conditions, may as well be turned into a constant, b. Thus, we end up with

$$
x = b(\theta - \sin\theta), \qquad y = b(1 - \cos\theta) + h.
$$

These are known to be parametric Cartesian equations of a cycloid—the h is present since we have situated the point A at $(0, h)$ rather than at $(0, 0)$ as is more customary.

Problem 291.1 – Treasure

There is some valuable stuff buried on an island and to your delight you have obtained precise instructions for digging it up.

> Locate a tall wooden post. From there, not far away you will see two tall stone pillars, one made of marble and the other of sandstone. Go to the wooden post, walk to the marble pillar, turn right, walk the same distance again and mark the spot. Go to the wooden post, walk to the sandstone pillar, turn left, walk the same distance again and mark the spot. The treasure is mid-way between the two marked spots.

You arrive at the island and find the stone structures, but alas! the wooden post has disappeared without trace. However, you had the foresight to bring a replacement with you. You erect your wooden post somewhere on the island, follow the instructions and successfully acquire the treasure.

How is this possible?

Solution 222.1 – Rectangle construction

Two parallel lines are tangent to a circle $\mathcal{C}$ at its North and South poles N and S. A segment of length l is constructed, starting from S and terminating on the same line at a point A.

A second line segment is constructed as follows: the line NA is drawn, intersecting the circle at a point E distinct from N. The line SE is extended to meet the line tangent to N at B. The line segment in question is NB which has length m.

Show that, whatever l we start with, a rectangle with sides of length l and m constructed in this way has the same area as the smallest square that completely encloses $\mathcal{C}$ (i.e. the square enclosing $\mathcal{C}$ which touches $\mathcal{C}$ at exactly four points).

Peter Fletcher

Since the lines NB and SA are parallel and tangential to $\mathcal{C}$ at N and S respectively, $B\hat{N}S = A\hat{S}N = 90^\circ$.

Let $NS = d$ and $A\hat{N}S = \theta$. Because NS is a diameter, $N\hat{E}S = 90^\circ$ and it follows that $N\hat{S}B = 90^\circ - \theta$ and $N\hat{B}S = \theta$. We also have $NA = \sqrt{d^2 + l^2}$ and $SB = \sqrt{d^2 + m^2}$.

Triangles NBS and NSA are both right-angled, so we have

$$\sin\theta = \frac{SA}{NA} = \frac{NS}{BS},$$

or

$$\frac{l}{\sqrt{d^2 + l^2}} = \frac{d}{\sqrt{d^2 + m^2}};$$

so

$$\begin{aligned} l^2(d^2 + m^2) &= d^2(d^2 + l^2), \\ l^2 d^2 + l^2 m^2 &= d^4 + d^2 l^2, \\ d^4 &= l^2 m^2 \end{aligned}$$

and

$$d^2 = lm.$$

Therefore the rectangle with sides l and m has the same area as the smallest square that completely encloses $\mathcal{C}$.

Solution 286.5 – Factorization

Given a positive integer n, denote by $\phi(n)$ the number of positive integers $m < n$ such that $\gcd(m, n) = 1$. If we know the complete factorization of n, say $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ with positive integers a_1, a_2, …, a_r and distinct primes p_1, p_2, …, p_r, we can easily compute

$$\phi(n) = (p_1 - 1)p_1^{a_1-1}(p_2 - 1)p_2^{a_2-1} \dots (p_r - 1)p_r^{a_r-1}.$$

Is this process reversible? Given n and $\phi(n)$, is it possible to construct the complete factorization of n without too much difficulty? If it is, try factorizing n = 1586 02481 31293 11974 04552 75968 73607 71145 55549 11334 22976 68001 07012 76942 47700 99421 02756 52867 12646 06754 07200 60245 86133 22978 29252 68997 66323 73467 11294 88572 28050 70734 96620 00789 92252 73781 given that $\phi(n)$ = 1586 02481 31293 11974 04552 75968 61357 52500 38117 66204 95814 65893 63834 77320 84117 64608 84450 72128 92453 11845 67688 17202 39729 99520 14675 73124 90380 04713 14884 75595 39954 75897 02033 54901 63971 74000.

Tommy Moorhouse

Given the expressions for n and $\phi(n)$ in terms of the prime factorizations an obvious first step is to find $\gcd(n, \phi(n))$. This can be done extremely efficiently, and MATHEMATICA has an inbuilt function GCD[ ] that returns the answer

$$\gcd(n, \phi(n)) = 217562429834685303748385019825119019861335147177168195894521$$

in a second or so. The factor turns out to be the square of the prime

$$p = 466435879660522367654413675211,$$

telling us that n has the form $p_1 p_2 \cdots p^3$.

We also get the bonus that since $p|n$ the prime factors of $\phi(n)$ include those of $p - 1$. Now we can divide $\phi(n)$ by p^2 and factorize, and cancel away the factors of $p - 1$ to get a fairly short list of prime factors of $\phi(n)/p^2$:

$$\{2, 2, 2, 2, 5, 5, 5, 7, 7, 11, 11, 11, 223, 379, 383, 503, 3877, 3967, 12689, 25999, 281663, 338159, 933782579083, 11096678434367, 136586140357561, 368382167483219, 136230490257812162183\}$$

reducing to

$$\{7, 7, 5, 5, 11, 379, 383, 503, 3877, 3967, 25999, 281663, 338159,\\ 933782579083, 11096678434367, 136586140357561, 368382167483219\}.$$

The factors of 2 can be set aside because we know that $2|p_k - 1$ for each prime factor of n and consequently we know that n must have four or fewer prime factors.

Now it is a matter of taking those combinations of the primes $q_1, q_2 \cdots q_N$ in this list such that $M = 2q_1q_2 \cdots q_r + 1$ is prime, and finding $\gcd(n, M)$. If this is not 1 then we have found a prime factor of n. In fact we take n/p^3 instead of n to streamline the process. This can be coded in your favourite computer language: I used MATHEMATICA running on a Raspberry Pi to quickly carry out the computation, finding that

$$n = p_1p_2p_3p_4^3,$$

where

$$\begin{aligned} p_1 &= 562324418721793120042174985351, \\ p_2 &= 494179332730633784520908832239, \\ p_3 &= 562422394447827908154562532159, \\ p_4 &= 466435879660522367654413675211. \end{aligned}$$

It seems it would be possible to produce an algorithm along these lines, but in this case a little intuition can go a long way.

Problem 291.2 – Chocolate

A chocolate bar consists of 42 'squares' arranged in a rectangular 14×3 array with deep grooves to allow easy breaking of any rectangular array of squares into two parts. For example you might want to snap the original bar along the 4th short groove from one end to create two slabs, 4×3 and 10×3.

It is required to disassemble the bar into its constituent 42 squares. How is this achieved with the minimum number of snappings?

What about an $n \times m$ slab?

Solution 287.2 – Magic cube

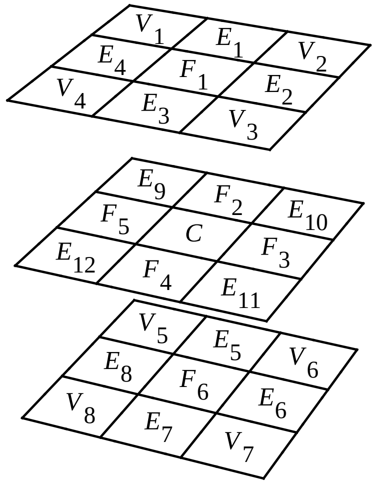
> Arrange the numbers 1, 2, ..., 27 in a $3 \times 3 \times 3$ cube such that each row, column and whatever the corresponding structure in the third dimension is called sums to 42.

Christopher Pile

In each direction there are 9 separate groups of three numbers. Since

$$\textstyle\sum_{n=1}^{27} n \;=\; 378$$

each group must sum to $378/9 = 42$. The $3 \times 3 \times 3$ cube has 8 vertices (V_1 to V_8), 12 edges (E_1 to E_{12}), 6 faces (F_1 to F_6) and one central position (C).



I recall this problem from the 'Dipole' column in *IEE News* in August 1980! An unfolded cube was shown with the squares on each face to be filled in with numbers 1 to 27 with the omission of one number, which turned out to be the central (hidden) number.

By dint of symmetry and having regard to the construction of magic squares, the central number is expected to be $C = 42/3 = 14$.

The opposite face cubes ($F_1 + F_6$, $F_2 + F_4$, $F_3 + F_5$) must therefore sum to 28. With the removal of number 14, the remaining 26 numbers give 13 possible pairs (1:27, 2:26, etc.). The 26 numbers remaining consist of 14 odd numbers and 12 even numbers. Each group of three numbers which sum to 42 must be either all even, or one even plus two odd. There are only 12 ways of choosing three even numbers which sum to 42 from the twelve available. These are 27 orthogonal sums in total, and each number appears three times:

$$3 \text{ sums } F + C + F, \qquad 12 \text{ sums } V + E + V, \qquad 12 \text{ sums } E + F + E.$$

Solutions can be characterized by the three smallest face numbers:

$$(1, 3, 9), \qquad (1, 3, 7), \qquad (3, 7, 9), \qquad (1, 7, 9).$$

In each case the central slice in each of the three places is a magic square (including diagonals), and the main 'space' diagonals also sum to 42 through the centre (e.g. $E_4 + C + E_6$ and $V_1 + C + V_7$).

12	22	8
23	9	10
7	11	24

10	23	9
24	7	11
8	12	22

18	22	2
20	9	13
4	11	27

16	20	6
23	9	10
3	13	26

26	3	13
1	14	27
15	25	2

26	3	13
1	14	27
15	25	2

23	3	16
7	14	21
12	25	5

24	7	11
1	14	27
17	21	4

4	17	21
18	19	5
20	6	16

6	16	20
17	21	4
19	5	18

1	17	24
15	19	8
26	6	10

2	15	25
18	19	5
22	8	12

Problem 291.3 – Mileage claims

Tommy Moorhouse

A uniform random variable X represents the event of choosing a number x in the interval $[0, 1]$ with the probability of x falling within a range of length δ being equal to δ. Here is an example for you to test.

My milometer displays the whole number of miles my car has travelled to date. If I make a mileage claim I can only claim full miles, and if I travel less than a mile the milometer reading may or may not change. Show that, under certain assumptions, the probability of the milometer changing from n to $n + 1$ (represented by event X) on a short trip (and leaving me in pocket) is equal to the fraction of a mile travelled (up to one mile). Deduce that X is a uniform random variable on $[0, 1]$.

How many journeys would I need to make between two fixed points (all starting with randomized mileage) to determine the distance between the points to an accuracy of 0.1 miles? What about 0.01 miles? Assume that we can take the milometer to consistently give an accurate reading.

Solution 289.5 – Cubic coefficients

A cubic x^3+ax^2+bx+c has a double root at $x=\alpha$ and another root at $x=\beta$. Show that the coefficients a, b and c are all real if and only if both α and β are real. Or find a counter-example.

Peter Fletcher

Using the well-known expressions for the sums and products of a polynomial's roots, we can write

$$a \;=\; -(2\alpha+\beta), \qquad b \;=\; \alpha^2+2\alpha\beta, \qquad c \;=\; -\alpha^2\beta.$$

Let

$$\alpha \;=\; \alpha_1+\alpha_2\mathrm{i} \qquad \text{and} \qquad \beta \;=\; \beta_1+\beta_2\mathrm{i}$$

where $\alpha_2 \neq 0$ and $\beta_2 \neq 0$.

Then we find

$$\begin{aligned}
a &= -2(\alpha_1+\alpha_2\mathrm{i})-(\beta_1+\beta_2\mathrm{i});\\
\Im(a) &= -(2\alpha_2+\beta_2);\\
b &= (\alpha_1+\alpha_2\mathrm{i})^2+2(\alpha_1+\alpha_2\mathrm{i})(\beta_1+\beta_2\mathrm{i})\\
&= \alpha_1^2-\alpha_2^2+2\alpha_1\alpha_2\mathrm{i}+2(\alpha_1\beta_1-\alpha_2\beta_2+\alpha_1\beta_2\mathrm{i}+\alpha_2\beta_1\mathrm{i});\\
\Im(b) &= 2(\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1);\\
c &= -(\alpha_1+\alpha_2\mathrm{i})^2(\beta_1+\beta_2\mathrm{i})\\
&= -(\alpha_1^2-\alpha_2^2+2\alpha_1\alpha_2\mathrm{i})(\beta_1+\beta_2\mathrm{i});\\
\Im(c) &= -(\alpha_1^2\beta_2-\alpha_2^2\beta_2+2\alpha_1\alpha_2\beta_1).
\end{aligned}$$

If a is real, then $\beta_2 \;=\; -2\alpha_2$ and

$$\begin{aligned}
\Im(b) &= 2(-\alpha_1\alpha_2+\alpha_2\beta_1);\\
\Im(c) &= -(-2\alpha_1^2\alpha_2+2\alpha_2^3+2\alpha_1\alpha_2\beta_1);
\end{aligned}$$

If b is also real, then $\beta_1 \;=\; \alpha_1$ and

$$\Im(c) \;=\; -2\alpha_2^3.$$

We have found that if α and β are complex, then a and b can be real but this then forces c to also be complex.

Therefore the coefficients a, b and c are all real if and only if both α and β are real.

Richard Gould

I wasn't sure what starting point the author expected for this solution but, using induction, it may readily be proved that, for complex numbers z_i,

$$\overline{\sum_{i=1}^{n} z_i} = \sum_{i=1}^{n} \overline{z_i},$$
$$\text{and} \quad \overline{\prod_{i=1}^{n} z_i} = \prod_{i=1}^{n} \overline{z_i}.$$

Hence, if the polynomial $\sum_{i=0}^{n} a_i\, z^i$ has a root $z = z_0$ then

$$\overline{\sum_{i=1}^{n} a_i\, z_0^i} = \overline{0} = 0,$$
$$\text{so} \quad \sum_{i=0}^{n} \overline{a_i}\, \overline{z_0}^{\,i} = 0.$$

If the coefficients are real then $\overline{a_i} = a_i$ and

$$\sum_{i=0}^{n} a_i\, \overline{z_0}^{\,i} = 0.$$

From the above it is clear that if a polynomial with real coefficients has a complex root then its conjugate is also a root. A cubic with real coefficients can therefore only have either three real roots or one real root and two complex conjugate roots. Conversely, a cubic with a single or repeated complex root cannot have real coefficients, proving the 'only if' part of the proposition.

Demonstrating the 'if' part of the proposition is trivial. If the roots are real and as given then the cubic may be written as

$$(z-\alpha)^2(z-\beta) = z^3 - (2\alpha+\beta)z^2 + \alpha(\alpha+2\beta)z - \alpha^2\beta,$$

the coefficients on the RHS being all real.

The Einstein–Pythagoras theorem: $E = mc^2 = m(a^2 + b^2)$.

Solution 289.2 – Tetrahedron

A regular tetrahedron with side length $2\sqrt{2}$ has its centre at the origin and has vertices $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$. Let $\mathbf{W}_i = (V_{i,x}, V_{i,y}, V_{i,z}, 1)$, where $\mathbf{V}_i = (V_{i,x}, V_{i,y}, V_{i,z})$, $i = 1, 2, 3, 4$. Show that $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$ and $\mathbf{W}_4$ have magnitude 2 and are mutually orthogonal.

Peter Fletcher

In the following, let $i, j = 1, 2, 3, 4$ and $i \neq j$.

The radius of a regular tetrahedron's circumsphere is, in terms of its edge-length a, $\sqrt{3/8}\,a$ per, *e.g.*,

`https://en.wikipedia.org/wiki/Tetrahedron`.

The length of each edge is $2\sqrt{2} = \sqrt{8}$, so we have $a = \sqrt{3}$. This means that the squared radius is, since the centre of the tetrahedron is the origin,

$$V_{i,x}^2 + V_{i,y}^2 + V_{i,z}^2 \;=\; 3.$$

We then have

$$|\mathbf{W}_i|^2 \;=\; \mathbf{W}_i{\cdot}\mathbf{W}_i \;=\; V_{i,x}^2 + V_{i,y}^2 + V_{i,z}^2 + 1 \;=\; 4$$

and $|\mathbf{W}_i| = 2$.

The length of each edge may also be written $|\mathbf{V}_i - \mathbf{V}_j|$, so we can write down

$$\begin{aligned} |\mathbf{V}_i - \mathbf{V}_j|^2 &= (V_{i,x} - V_{j,x})^2 + (V_{i,y} - V_{j,y})^2 + (V_{i,z} - V_{j,z})^2 \\ &= V_{i,x}^2 + V_{i,y}^2 + V_{i,z}^2 - 2(V_{i,x}V_{j,x} + V_{i,y}V_{j,y} + V_{i,z}V_{j,z}) \\ &\quad + V_{j,x}^2 + V_{j,y}^2 + V_{j,z}^2 \\ &= 3 - 2(V_{i,x}V_{j,x} + V_{i,y}V_{j,y} + V_{i,z}V_{j,z}) + 3 \\ &= 8 \end{aligned}$$

so that

$$(V_{i,x}V_{j,x} + V_{i,y}V_{j,y} + V_{i,z}V_{j,z}) \;=\; -1,$$

or

$$(V_{i,x}V_{j,x} + V_{i,y}V_{j,y} + V_{i,z}V_{j,z}) + 1 \;=\; 0.$$

But

$$(V_{i,x}V_{j,x} + V_{i,y}V_{j,y} + V_{i,z}V_{j,z}) + 1 \;=\; \mathbf{W}_i{\cdot}\mathbf{W}_j,$$

so that $\mathbf{W}_i{\cdot}\mathbf{W}_j = 0$ and $\mathbf{W}_i$ and $\mathbf{W}_j$ are orthogonal.

Stuart Walmsley

This problem concerns the representation of a regular tetrahedron (of side length $2\sqrt{2}$) in four dimensions.

It is conveniently solved using a property relating a regular tetrahedron to a cube. Explicitly, a cube has eight vertices. The four vertices of a suitable subset lie at the corners of a regular tetrahedron. The length of an edge of the tetrahedron is equal to the length of a diagonal of the square face of the cube.

In this problem, the tetrahedron has edges of length $2\sqrt{2}$, so that the corresponding cube has edges of length 2. Then if Cartesian coordinates are chosen to link the midpoints of opposite faces of the cube, the coordinates of the vertices of the cube are

$$\begin{array}{llll} (+1,+1,+1), & (+1,-1,+1), & (-1,-1,+1), & (-1,+1,+1), \\ (+1,-1,-1), & (+1,+1,-1), & (-1,+1,-1), & (-1,-1,-1), \end{array}$$

and the four vertices of the tetrahedron may be chosen to be:

$$(+1,+1,+1), \quad (+1,-1,-1), \quad (-1,+1,-1), \quad (-1,-1,+1).$$

In the notation of the problem:

$$\begin{array}{rclrcl} \mathbf{V}_1 & = & (+1,+1,+1), & \mathbf{V}_2 & = & (+1,-1,-1), \\ \mathbf{V}_3 & = & (-1,+1,-1), & \mathbf{V}_4 & = & (-1,-1,+1). \end{array}$$

The vectors are extended to four dimensions by adding to each a coordinate with value +1:

$$\begin{array}{rclrcl} \mathbf{W}_1 & = & (+1,+1,+1,+1), & \mathbf{W}_2 & = & (+1,-1,-1,+1), \\ \mathbf{W}_3 & = & (-1,+1,-1,+1), & \mathbf{W}_4 & = & (-1,-1,+1,+1). \end{array}$$

In each case, the scalar product

$$\mathbf{W}_j \cdot \mathbf{W}_j \ = \ 4$$

so that the magnitude of each vector is 2, proving the first result required by the problem.

Similarly

$$\mathbf{W}_j \cdot \mathbf{W}_k \ = \ 0, \qquad j \neq k.$$

So that the vectors are mutually orthogonal, proving the second result. The set of four directions from origin to the four vertices could therefore be chosen as Cartesian axes. With a suitable rotation of axes (and halving the scale factor), the vertex coordinates become

$$(1,0,0,0),\quad (0,1,0,0),\quad (0,0,1,0),\quad (0,0,0,1).$$

A similar result applies to the corresponding structure in two dimensions, the equilateral triangle. Its vertices can be represented in three dimensions by the set of coordinates

$$(1,0,0),\quad (0,1,0),\quad (0,0,1).$$

There is a corresponding structure for each dimension, known as the (n-dimensional) regular simplex.

Solution 286.4 – Evaporation

> A solution of sodium chloride contains 90 per cent water. After a while, due to loss by evaporation, the solution contains only 80 per cent water. What percentage of the water has evaporated?

Peter Fletcher

Let s be the mass of salt, w_1 the mass of water at time 1 and w_2 the mass of water at time 2. We are told that

$$\frac{w_1}{s+w_1} = \frac{9}{10} \qquad \text{and} \qquad \frac{w_2}{s+w_2} = \frac{4}{5}.$$

This means that

$$\frac{s+w_1}{w_1} = \frac{10}{9}, \qquad \text{so} \qquad \frac{s}{w_1} = \frac{1}{9} \qquad \text{and} \qquad w_1 = 9s,$$

and

$$\frac{s+w_2}{w_2} = \frac{5}{4}, \qquad \text{so} \qquad \frac{s}{w_2} = \frac{1}{4} \qquad \text{and} \qquad w_2 = 4s.$$

Then the proportion of water lost is

$$\frac{w_1-w_2}{w_1} = 1-\frac{w_2}{w_1} = 1-\frac{4s}{9s} = \frac{5}{9},$$

or 55.6%.

M500 Mathematics Revision Weekend 2020

The forty-sixth M500 Revision Weekend will be held at

Kents Hill Park Training and Conference Centre,

Milton Keynes, MK7 6BZ

from Friday 15th to Sunday 17th May 2020.

The standard cost, including accommodation (with en suite facilities) and all meals from dinner on Friday evening to lunch on Sunday is £275 for single occupancy, or £240 per person for two students sharing in either a double or twin bedded room. The standard cost for non-residents, including Saturday and Sunday lunch, is £160.

Members may make a reservation with a £25 deposit, with the balance payable at the end of February. Non-members must pay in full at the time of application and all applications received after 28th February 2020 must be paid in full before the booking is confirmed. Members will be entitled to a discount of £15 for all applications received before 14th April 2020. The Late Booking Fee for applications received after 14th April 2020 is £20, with no membership discount applicable.

There is free on-site parking for those travelling by private transport. For full details and an application form, see the Society's web site:

`www.m500.org.uk`.

The Weekend is open to all Open University students, and is designed to help with revision and exam preparation. We expect to offer tutorials for most undergraduate and postgraduate mathematics OU modules, subject to the availability of tutors and sufficient applications.

End of year quiz

Find the function that maps A onto B. Answer in the next issue.

A = {AIP, BRJ, BTLOTSM, C, CJ, DND, IAGF, IHTJ, ISYIMD, J, LCB, LM, LMOLM, LOB, MDIY, ML, MOD, OMB, PDETD, PT, ROTHS, S, SMNF, SW, TBOJ, TFT, TGBB, TMWKTM, TP, TPG, TTOIA, TTOL, TTOM, TWPS, TWT, WSYGE, WWYWTLWO, YAH, YMWAH},

B = {AA, AM, BB, CT, CT, CW, DD, EDJ, ES, EWA, GG, GLK, IP, JA, JB, JCM, JH, JJ, JK, JM, JM, JN, JO, JW, KBW, KP, KRM, KW, LR, LT, MG, MG, MH, MJC, MW, NC, PF, RE}.

Contents

M500 291 – December 2019

04.00

Problem 291.4 – Zero-free powers

Observe that the 57-digit number

$$19^{44} = 184144368549628275143663229532787625188711914273876985521$$

has no zeros. Show that it is the largest zero-free number of the form a^b with positive integers $a < b$. Or find a larger one.

Problem 291.5 – Points

A unit cube with the usual metric contains a set S of points such that each point $p \in S$ has an open neighbourhood that does not contain any point of S other than p. Show that S has measure zero, or find a counter-example.

Front cover Wheel graph with five spokes.